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The General Solution and Ulam Stability of Second Order Linear Dynamic Equations on Time Scales

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Abstract By using the variation of parameters, this paper deals with the general solution and Ulam stability of second order linear dynamic equations with variable coefficients on time scales. In particular, we also obtain the Ulam stability of second order linear dynamic equations with constant coefficients under different cases.

Keywords Ulam stability; second order linear dynamic equations; time scales

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1. Introduction

Questions concerning the Ulam stability of group homomorphisms were first proposed by Ulam [1] in 1940. a partial solution to the Ulam stability for additive mappings in Banach spaces was given by Hyers [2]. Several years later, Rassias [3] presented a generalization to the results of Hyers by considering an unbounded Cauchy difference. Since then, the Ulam stability of various types of functional equations has received extensive attention and has resulted in many systematic results in recent years.

In 1993, the Ulam stability of differential equations was first studied by Obloza [4]. Soon after, Alsina and Ger [5] showed that the differential equation $y' = y$ is Hyers-Ulam stable on any real interval. Subsequently, Miura and Takahasi et al. [6–8] deeply and systematically investigated the Ulam stability problem of the differential equation $y = \lambda y$ in various abstract spaces. As of now, many interesting and systematic results related to the Ulam stability of different types of differential equations, especially linear differential equations, have been established by various authors. For more details, please refer to [9–17] and the references therein.

In 2005, Popa [18] initiated the study of the Ulam stability problem of difference equations and proved the Hyers-Ulam-Rassias stability of the first order linear difference equation $x_{k+1} = a_k x_k + b_k$ in a Banach space. Meantime, Popa [19] further investigated the Hyers-Ulam stability of higher order linear difference equations with constant coefficients. The results showed that the Ulam stability of a linear difference equation with constant coefficients depends strongly on the roots of the corresponding characteristic equation. Several examples indicated that the

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difference equation is not Hyers-Ulam stable if the characteristic equation admits a root with modulus equal to 1. For this kind of problem, Brzdęk et al. [20] deeply studied the nonstability of linear difference equations with constant coefficients. Hereafter, Brzdęk et al. [21] also investigated the Ulam stability problem of nonlinear difference equation $x_{k+1} = a_k(x_k) + b_k$ in an Abel group with an invariant metric. Afterwards, Brzdęk et al. [22] proved the Hyers-Ulam stability of linear difference equations with constant coefficients in a normed space. In 2017, Brzdęk and Wójcik [23] obtained the Ulam stability of two kinds of difference equations in a metric space, which can be regarded as the most general result of the Ulam stability of difference equations so far. In addition, Jung [24] established the Hyers-Ulam stability of linear homogeneous matrix difference equations of first order. Soon after, Jung and Nam [25] further considered the Hyers-Ulam stability of the Pielou logistic difference equation. Recently, Onitsuka [26] studied the effect of stepsize on the HUS constant of linear homogenous difference equations of first order from a different perspective. Using the similar method, Onitsuka [27] considered the Hyers-Ulam stability and the best HUS constant of first order nonhomogeneous linear difference equations with constant stepsize. By using the z-transform method, Shen [28] established the Ulam stability of linear difference equations with constant coefficients, which can be viewed as an important complement to the existing results associated with the Ulam stability of linear difference equations with constant coefficients.

In 1988, Hilger [29] introduced the notion of time scale in order to unify continuous and discrete analysis. Correspondingly, the time scale calculus provides a unified framework for the study of differential equations and difference equations. Then, the theory of dynamic equations on time scales has gradually formed and developed in the past two decades, which can be regarded as a unification and extension of the theory of differential equations and difference equations. In 2011, Hamza and Yaseen [30] studied the Hyers-Ulam stability of abstract first order linear dynamic equations on time scales. Afterwards, Anderson et al. [31] established the Hyers-Ulam stability of second order nonhomogeneous linear dynamic equations on time scales. In 2013, András and Mészáros [32] studied the Hyers-Ulam stability of some linear, nonlinear dynamic equations and integral equations on time scales by using direct and operational methods. Meantime, they proposed a unified approach to the Ulam stability problem based on the theory of Picard operators. In 2014, Hamza and Yaseen [33] further investigated the Hyers-Ulam stability of abstract second order linear dynamic equations on time scales. At the same time, Anderson [34] also established the Hyers-Ulam stability of higher-order Cauchy-Euler dynamic equations on time scales. In 2017, Shen [35] considered the Ulam stability of first order linear dynamic equations on finite time scales by using the integrating factor method. Furthermore, Shen and He [36] also established the general solution and Ulam stability of inhomogeneous Cauchy-Euler dynamic equations on time scales. Lately, Anderson and Onitsuka [37] studied the Hyers-Ulam stability of certain first order linear dynamic equations with constant coefficients on time scales. Especially, they find the minimum HUS constant for certain parameter values in relation to the graininess of the time scale. Moreover, they clarified the Hyers-Ulam stability of first order linear dynamic equations with constant coefficients on the time scale, in which the time scale involves two altering step size [38]. Meantime, several problems related to the minimum HUS constant are also discussed. By using the variation of parameters, the aim of this paper is to establish the general solution and study the Ulam stability of second order linear dynamic equations with constant and variable coefficients on time scales.

2. Preliminaries

For the sake of completeness, in this section, we will review some basic concepts and fundamental results associated with the time scale which are derived from Refs. [39, 40].

Throughout this paper, let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}$ denote the set of all real numbers, the set of all nonnegative real numbers and the set of all integers, respectively. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Obviously, $\mathbb{R}$ and $\mathbb{Z}$ are two typical examples of time scales.

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ can be defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

for all $t \in \mathbb{T}$, where we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called left-scattered and right-scattered if $\rho(t) < t$ and $\sigma(t) > t$, respectively. Also, if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is said to be left-dense, and if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is said to be right-dense. The graininess function $\mu : \mathbb{T} \to \mathbb{R}_+$ is defined by $\mu(t) := \sigma(t) - t$. To match the differentiability of functions defined on the time scale $\mathbb{T}$, the set $\mathbb{T}^\kappa$ is introduced. Specifically, $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has a left-scattered maximum m. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Furthermore, $\mathbb{T}^{\kappa^2}$ can be defined similarly. In addition, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be Δ-differentiable (shortly, differentiable) at $t \in \mathbb{T}^\kappa$ provided there exists $f^\Delta(t)$ with the property that given any $\varepsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, the function f is said to be (Δ-)differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Accordingly, we say that f is twice (Δ-)differentiable on $\mathbb{T}^{\kappa^2}$ provided $f^\Delta(t)$ is (Δ-)differentiable on $\mathbb{T}^{\kappa^2}$. Meantime, the functions $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ and $f^{\Delta\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$ are called the (Δ-)derivative and second order (Δ-)derivative of f.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathbf{C_{rd}} = \mathbf{C_{rd}}(\mathbb{T}, \mathbb{R})$. Moreover, we denote by $\mathbf{C^1_{rd}} = \mathbf{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ ($\mathbf{C^2_{rd}} = \mathbf{C^2_{rd}}(\mathbb{T}, \mathbb{R})$) the set of functions $f : \mathbb{T} \to \mathbb{R}$ that are (twice) differentiable and whose (second order) derivative is rd-continuous. The following result is useful in this paper.

Theorem 2.1 ([39]) *If $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable on $\mathbb{T}^\kappa$, then* *(i)* $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$;

(ii) $(fg)^{\Delta}(t) = f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)$.

For an rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, the (Cauchy) integral can be defined by

$$\int_{t_0}^{t} f(\tau)\Delta\tau = F(t) - F(t_0),$$

where $t, t_0 \in \mathbb{T}$, $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, i.e., $F^{\Delta}(t) = f(t)$ for every $t \in \mathbb{T}^{\kappa}$. It is well known that every rd-continuous function possesses an antiderivative.

A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ the set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.

For $p, q \in \mathcal{R}$, the circle plus and circle minus are defined, respectively, by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t),$$
$$(p \ominus q)(t) := (p \oplus (\ominus q))(t)$$

for all $t \in \mathbb{T}^{\kappa}$, where $(\ominus p)(t) := -\frac{p(t)}{1+\mu(t)p(t)}$.

For $p \in \mathcal{R}$, the exponential function is introduced by

$$e_p(t, t_0) := \exp\Big(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big)$$

for $t, t_0 \in \mathbb{T}$, where $\xi_h(z)$ is the cylinder transformation. For more details on cylinder transformation, the readers can refer to the chapter 2 in [39]. The exponential function has the following properties:

Theorem 2.2 ([39]) *If $p, q \in \mathcal{R}$, then*

(i) $e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0)$;

(ii) $\frac{e_p(t,t_0)}{e_q(t,t_0)} = e_{p\ominus q}(t, t_0)$.

If $p \in \mathbf{C}_{rd}$ and $-\mu p^2 \in \mathcal{R}$, then the hyperbolic functions $\cos h_p(t, t_0)$ and $\sin h_p(t, t_0)$ can be defined by

$$\cos h_p(t, t_0) = \frac{e_p(t, t_0) + e_{-p}(t, t_0)}{2}, \quad \sin h_p(t, t_0) = \frac{e_p(t, t_0) - e_{-p}(t, t_0)}{2}$$

for $t, t_0 \in \mathbb{T}$. Analogously, if $p \in \mathbf{C}_{rd}$ and $\mu p^2 \in \mathcal{R}$, then the trigonometric functions $\cos_p(t, t_0)$ and $\sin_p(t, t_0)$ can be defined by

$$\cos_p(t, t_0) = \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2}, \quad \sin_p(t, t_0) = \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i}$$

for $t, t_0 \in \mathbb{T}$. The derivatives of these functions have the following properties:

(i) $\cos h_p^{\Delta}(t, t_0) = p \sin h_p(t, t_0)$ and $\sin h_p^{\Delta}(t, t_0) = p \cos h_p(t, t_0)$;

(ii) $\cos_p^{\Delta}(t, t_0) = -p \sin_p(t, t_0)$ and $\sin_p^{\Delta}(t, t_0) = p \cos_p(t, t_0)$.

3. Ulam stability of second order linear dynamic equations

In this section, we shall consider the general solution and Ulam stability of the second order nonhomogeneous linear dynamic equation

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t), \tag{3.1}$$

where $p, q, f \in \mathbf{C}_{rd}$. Correspondingly, the homogeneous linear dynamic equation is

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0. \tag{3.2}$$

Let y_1, y_2 be a fundamental system of solutions for (3.2). From [39], we know that $y_g(t) = \alpha y_1(t) + \beta y_2(t)$ is a general solution of (3.2), where α, β are constants. Here, we replace α and β with two undetermined functions $\alpha(t)$ and $\beta(t)$, respectively.

Set $y_p(t) = \alpha(t)y_1(t) + \beta(t)y_2(t)$. Assume that $y_p(t)$ is a particular solution of (3.1). Then, we have

$$y_p^{\Delta}(t) = \alpha^{\Delta}(t)y_1^{\sigma}(t) + \alpha(t)y_1^{\Delta}(t) + \beta^{\Delta}(t)y_2^{\sigma}(t) + \beta(t)y_2^{\Delta}(t). \tag{3.3}$$

Without loss of generality, we assume that $\alpha(t)$ and $\beta(t)$ can be picked so that

$$\alpha^{\Delta}(t)y_1^{\sigma}(t) + \beta^{\Delta}(t)y_2^{\sigma}(t) = 0. \tag{3.4}$$

Furthermore, we can infer from (3.3) that

$$y_p^{\Delta\Delta}(t) = \alpha^{\Delta}(t)y_1^{\Delta^{\sigma}}(t) + \alpha(t)y_1^{\Delta\Delta}(t) + \beta^{\Delta}(t)y_2^{\Delta^{\sigma}}(t) + \beta(t)y_2^{\Delta\Delta}(t). \tag{3.5}$$

Substituting (3.3)–(3.5) into (3.1), we can obtain

$$\begin{aligned} y_p^{\Delta\Delta} + p(t)y_p^{\Delta} + q(t)y_p &= \alpha(t)(y_1^{\Delta\Delta}(t) + p(t)y_1^{\Delta}(t) + q(t)y_1(t)) + \\ &\quad \beta(t)(y_2^{\Delta\Delta}(t) + p(t)y_2^{\Delta}(t) + q(t)y_2(t)) + \alpha^{\Delta}(t)y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta}(t)y_2^{\Delta^{\sigma}}(t) \\ &= \alpha^{\Delta}(t)y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta}(t)y_2^{\Delta^{\sigma}}(t), \end{aligned} \tag{3.6}$$

where we have used the fact that y_1 and y_2 solve (3.2) for the second equality. To ensure that y_p is a solution of (3.1), $\alpha(t)$ and $\beta(t)$ need to satisfy the following equation

$$\alpha^{\Delta}(t)y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta}(t)y_2^{\Delta^{\sigma}}(t) = f(t). \tag{3.7}$$

By combining equations (3.4) and (3.7), we can obtain

$$\begin{cases} \alpha^{\Delta}(t)y_1^{\sigma}(t) + \beta^{\Delta}(t)y_2^{\sigma}(t) = 0, \\ \alpha^{\Delta}(t)y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta}(t)y_2^{\Delta^{\sigma}}(t) = f(t). \end{cases}$$

Note that y_1 and y_2 form a fundamental system for (3.2), so the Wronskian

$$W^{\sigma}(y_1, y_2)(t) = \begin{vmatrix} y_1^{\sigma}(t) & y_2^{\sigma}(t) \\ y_1^{\Delta^{\sigma}}(t) & y_2^{\Delta^{\sigma}}(t) \end{vmatrix} \neq 0$$

for all $t \in \mathbb{T}^{\kappa}$. Using the Cramer's rule, we can infer that

$$\alpha^{\Delta}(t) = \frac{W_{1,2}(t)}{W^{\sigma}(y_1, y_2)(t)}, \quad \beta^{\Delta}(t) = \frac{W_{2,2}(t)}{W^{\sigma}(y_1, y_2)(t)}, \tag{3.8}$$

where

$$W_{1,2}(t) = \begin{vmatrix} 0 & y_2^{\sigma}(t) \\ f(t) & y_2^{\Delta^{\sigma}}(t) \end{vmatrix} = -y_2^{\sigma}(t)f(t), \quad W_{2,2}(t) = \begin{vmatrix} y_1^{\sigma}(t) & 0 \\ y_1^{\Delta^{\sigma}}(t) & f(t) \end{vmatrix} = y_1^{\sigma}(t)f(t).$$

By integrating both sides of (3.8) from t_0 to t with respect to τ, we get

$$\alpha(t) = \int_{t_0}^{t} \frac{W_{1,2}(\tau)}{W^{\sigma}(y_1, y_2)(\tau)}\Delta\tau, \quad \beta(t) = \int_{t_0}^{t} \frac{W_{2,2}(\tau)}{W^{\sigma}(y_1, y_2)(\tau)}\Delta\tau.$$

From the statement above, we can obtain the following result related to the general solution of (3.1).

Theorem 3.1 *Let $p, q, f \in \mathbf{C}_{rd}$ and let y_1 and y_2 be a fundamental system of solutions of the homogeneous equation (3.2). Then the general solution $y(t)$ of the nonhomogeneous equation (3.1) can be given by*

$$\begin{aligned} y(t) &= y_g(t) + y_p(t) \\ &= c_1 y_1(t) + c_2 y_2(t) + y_1(t) \int_{t_0}^{t} \frac{W_{1,2}(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau + y_2(t) \int_{t_0}^{t} \frac{W_{2,2}(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau, \end{aligned}$$

where c_1 and c_2 are constants, $t_0 \in \mathbb{T}^{\kappa}$ is an arbitrary fixed point.

With the help of Theorem 3.1, we will study the Ulam stability of the nonhomogeneous second order linear dynamic equation (3.1).

Theorem 3.2 *Let $p, q, f \in \mathbf{C}_{rd}$ and let y_1 and y_2 be a fundamental system of solutions of the homogeneous equation (3.2). Assume that $\varphi : \mathbb{T} \to \mathbb{R}_+$ is an rd-continuous function. If $y_\varphi \in \mathbf{C}^2_{rd}$ satisfies the following inequality*

$$|y_\varphi^{\Delta\Delta}(t) + p(t) y_\varphi^{\Delta}(t) + q(t) y_\varphi(t) - f(t)| \le \varphi(t) \tag{3.9}$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}^2_{rd}$ of (3.1) such that

$$|y_\varphi(t) - y(t)| \le \Big| \int_{t_0}^{t} \Big| \frac{(y_1^{\Delta^\sigma}(\tau) y_2(t) - y_1(t) y_2^{\Delta^\sigma}(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Big| \varphi(\tau) \Delta\tau \Big|$$

for all $t \in \mathbb{T}$.

Proof For convenience, we set

$$f_\varphi(t) := y_\varphi^{\Delta\Delta}(t) + p(t) y_\varphi^{\Delta}(t) + q(t) y_\varphi(t). \tag{3.10}$$

It is easy to see that $y_\varphi(t)$ is a solution of (3.10). According to Theorem 3.1, there are two constants c_1 and c_2 such that

$$y_\varphi(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t) \int_{t_0}^{t} \frac{-y_2^{\Delta^\sigma}(\tau) f_\varphi(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau + y_2(t) \int_{t_0}^{t} \frac{y_1^{\Delta^\sigma}(\tau) f_\varphi(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau,$$

where $t_0 \in \mathbb{T}^{\kappa}$ is an arbitrary fixed point.

Define

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t) \int_{t_0}^{t} \frac{-y_2^{\Delta^\sigma}(\tau) f(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau + y_2(t) \int_{t_0}^{t} \frac{y_1^{\Delta^\sigma}(\tau) f(\tau)}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau.$$

By Theorem 3.1, we know that $y(t)$ is a solution of (3.1). Then, we can infer that

$$\begin{aligned} |y_\varphi(t) - y(t)| &= \Big| y_1(t) \int_{t_0}^{t} \frac{-y_2^{\Delta^\sigma}(\tau)(f_\varphi(\tau) - f(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau + y_2(t) \int_{t_0}^{t} \frac{y_1^{\Delta^\sigma}(\tau)(f_\varphi(\tau) - f(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau \Big| \\ &= \Big| \int_{t_0}^{t} \frac{-y_1(t) y_2^{\Delta^\sigma}(\tau)(f_\varphi(\tau) - f(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau + \int_{t_0}^{t} \frac{y_1^{\Delta^\sigma}(\tau) y_2(t)(f_\varphi(\tau) - f(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau \Big| \\ &= \Big| \int_{t_0}^{t} \frac{(y_1^{\Delta^\sigma}(\tau) y_2(t) - y_1(t) y_2^{\Delta^\sigma}(\tau))(f_\varphi(\tau) - f(\tau))}{W^{\sigma}(y_1, y_2)(\tau)} \Delta\tau \Big| \end{aligned}$$

$$\leq \Big| \int_{t_0}^{t} \Big| \frac{(y_1^{\Delta^\sigma}(\tau) y_2(t) - y_1(t) y_2^{\Delta^\sigma}(\tau))}{W^\sigma(y_1, y_2)(\tau)} \Big| \varphi(\tau) \Delta\tau \Big|.$$

This completes the proof. □

As a direct consequence, we can obtain the Hyers-Ulam stability of the nonhomogeneous linear dynamic equation (3.1) on a compact interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Corollary 3.3 *Let $p, q, f \in \mathbf{C}_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ and let y_1 and y_2 be a fundamental system of solutions of the homogeneous equation (3.2). For a given $\varepsilon > 0$, if $y_\varepsilon \in \mathbf{C}_{rd}^2([a, b]_{\mathbb{T}}, \mathbb{R})$ satisfies the following inequality*

$$|y_\varepsilon^{\Delta\Delta}(t) + p(t) y_\varepsilon^{\Delta}(t) + q(t) y_\varepsilon(t) - f(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}_{rd}^2$ of (3.1) such that

$$|y_\varepsilon(t) - y(t)| \leq K\varepsilon$$

for all $t \in [a, b]_{\mathbb{T}}$, where

$$K = \max_{t \in [a,b]_{\mathbb{T}}} \int_a^t \Big| \frac{(y_1^{\Delta^\sigma}(\tau) y_2(t) - y_1(t) y_2^{\Delta^\sigma}(\tau))}{W^\sigma(y_1, y_2)(\tau)} \Big| \Delta\tau.$$

From the statement above, in particular, we can establish the Ulam stability of the nonhomogeneous second order linear dynamic equation with constant coefficients. Now, we consider the dynamic equation

$$y^{\Delta\Delta}(t) + m y^{\Delta}(t) + n y(t) = f(t), \tag{3.11}$$

where $m, n \in \mathbb{R}$, $f \in \mathbf{C}_{rd}$. The associated characteristic equation is

$$\lambda^2 + m\lambda + n = 0. \tag{3.12}$$

The roots λ_1, λ_2 of the characteristic equation (3.12) are given by

$$\lambda_{1,2} = \frac{-m \pm \sqrt{m^2 - 4n}}{2}.$$

Theorem 3.4 *Let $\varphi : \mathbb{T} \to \mathbb{R}_+$ be an rd-continuous function and let $m^2 - 4n > 0$. Define*

$$r = -\frac{m}{2} \quad \text{and} \quad s = \frac{\sqrt{m^2 - 4n}}{2}.$$

Assume that r and $\mu n - m$ are regressive. If $y_\varphi \in \mathbf{C}_{rd}^2$ satisfies the following inequality

$$|y_\varphi^{\Delta\Delta}(t) + m y_\varphi^{\Delta}(t) + n y_\varphi(t) - f(t)| \leq \varphi(t)$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}_{rd}^2$ of (3.11) such that

$$|y_\varphi(t) - y(t)| \leq \frac{1}{2} |e_r(t, t_0)| \Big| \int_{t_0}^{t} \Big| \frac{e_{r \ominus (\mu n - m)}(\sigma(\tau), t_0)[(r - s)(1 - \mu) e_{\frac{s}{1+\mu r}}(t, t_0) e_{-\frac{s}{1+\mu r}}(\tau, t_0)}{(1 + \mu r)}$$
$$\frac{-(r + s)(1 + \mu) e_{\frac{s}{1+\mu r}}(\tau, t_0) e_{-\frac{s}{1+\mu r}}(t, t_0)]}{(1 + \mu r)} |\varphi(\tau) \Delta\tau|$$

for all $t \in \mathbb{T}$.

Proof By Theorem 3.24 in [39], we know that

$$y_1(t)=\cosh_{s/(1+\mu r)}(t,t_0)e_r(t,t_0),\quad y_2(t)=\sinh_{s/(1+\mu r)}(t,t_0)e_r(t,t_0)$$

form a fundamental system of solutions of the corresponding homogeneous equation of (3.11), where $t_0\in\mathbb{T}$, and the Wronskian of these two solutions is $W(y_1,y_2)(t)=se_{\mu n-m}(t,t_0)$.

For convenience, we write $\rho=\frac{s}{1+\mu r}$. Then, we can obtain

$$y_1^{\Delta^\sigma}(\tau)y_2(t)-y_1(t)y_2^{\Delta^\sigma}(\tau)=\begin{vmatrix}y_1^{\Delta^\sigma}(\tau) & y_1(t)\\ y_2^{\Delta^\sigma}(\tau) & y_2(t)\end{vmatrix}$$

$$=e_r(\sigma(\tau),t_0)e_r(t,t_0)\begin{vmatrix}s\sinh_\rho(\sigma(\tau),t_0)+r\cosh_\rho(\sigma(\tau),t_0) & \cosh_\rho(t,t_0)\\ s\cosh_\rho(\sigma(\tau),t_0)+r\sinh_\rho(\sigma(\tau),t_0) & \sinh_\rho(t,t_0)\end{vmatrix}$$

$$=e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big(s\begin{vmatrix}\sinh_\rho(\sigma(\tau),t_0) & \cosh_\rho(t,t_0)\\ \cosh_\rho(\sigma(\tau),t_0) & \sinh_\rho(t,t_0)\end{vmatrix}+r\begin{vmatrix}\cosh_\rho(\sigma(\tau),t_0) & \cosh_\rho(t,t_0)\\ \sinh_\rho(\sigma(\tau),t_0) & \sinh_\rho(t,t_0)\end{vmatrix}\Big)$$

$$=e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big(s\begin{vmatrix}\sinh_\rho(\tau,t_0)+\mu\cosh_\rho(\tau,t_0) & \cosh_\rho(t,t_0)\\ \cosh_\rho(\tau,t_0)+\mu\sinh_\rho(\tau,t_0) & \sinh_\rho(t,t_0)\end{vmatrix}+$$

$$r\begin{vmatrix}\cosh_\rho(\tau,t_0)+\mu\sinh_\rho(\tau,t_0) & \cosh_\rho(t,t_0)\\ \sinh_\rho(\tau,t_0)+\mu\cosh_\rho(\tau,t_0) & \sinh_\rho(t,t_0)\end{vmatrix}\Big)$$

$$=e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big((s+\mu\rho r)\begin{vmatrix}\sinh_\rho(\tau,t_0) & \cosh_\rho(t,t_0)\\ \cosh_\rho(\tau,t_0) & \sinh_\rho(t,t_0)\end{vmatrix}+$$

$$(r+\mu\rho s)\begin{vmatrix}\cosh_\rho(\tau,t_0) & \cosh_\rho(t,t_0)\\ \sinh_\rho(\tau,t_0) & \sinh_\rho(t,t_0)\end{vmatrix}\Big)$$

$$=\frac{1}{4}e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big((s+\mu\rho r)\begin{vmatrix}e_\rho(\tau,t_0)-e_{-\rho}(\tau,t_0) & e_\rho(t,t_0)+e_{-\rho}(t,t_0)\\ e_\rho(\tau,t_0)+e_{-\rho}(\tau,t_0) & e_\rho(t,t_0)-e_{-\rho}(t,t_0)\end{vmatrix}+$$

$$(r+\mu\rho s)\begin{vmatrix}e_\rho(\tau,t_0)+e_{-\rho}(\tau,t_0) & e_\rho(t,t_0)+e_{-\rho}(t,t_0)\\ e_\rho(\tau,t_0)-e_{-\rho}(\tau,t_0) & e_\rho(t,t_0)-e_{-\rho}(t,t_0)\end{vmatrix}\Big)$$

$$=\frac{1}{2}e_r(\sigma(\tau),t_0)e_r(t,t_0)\big((s+\mu\rho r)(-e_\rho(\tau,t_0)e_{-\rho}(t,t_0)-e_\rho(t,t_0)e_{-\rho}(\tau,t_0)+$$

$$(r+\mu\rho s)(e_\rho(t,t_0)e_{-\rho}(\tau,t_0)-e_\rho(\tau,t_0)e_{-\rho}(t,t_0))\big)$$

$$=\frac{1}{2}e_r(\sigma(\tau),t_0)e_r(t,t_0)\frac{(r-s)(1-\mu)e_\rho(t,t_0)e_{-\rho}(\tau,t_0)-(r+s)(1+\mu)e_\rho(\tau,t_0)e_{-\rho}(t,t_0)}{1+\mu r}.$$

From Theorem 3.2, it follows that

$$|y_\varphi(t)-y(t)|\le\frac{1}{2}|e_r(t,t_0)|\Big|\int_{t_0}^t\Big|\frac{e_{r\ominus(\mu n-m)}(\sigma(\tau),t_0)[(r-s)(1-\mu)e_\rho(t,t_0)e_{-\rho}(\tau,t_0)}{(1+\mu r)}$$

$$\frac{-(r+s)(1+\mu)e_\rho(\tau,t_0)e_{-\rho}(t,t_0)]}{(1+\mu r)}|\varphi(\tau)\Delta\tau|.$$

We have thus proved the theorem. $\square$

When the characteristic equation (3.12) has two different real eigenvalues, in view of Theorem 3.4, we can obtain the Hyers-Ulam stability of the second order nonhomogeneous linear dynamic equation (3.11) with constant coefficients on a finite time scale.

Corollary 3.5 *Let $m^2 - 4n > 0$ and let r, s be given as in Theorem 3.4. Assume that r and $\mu n - m$ are regressive. For a given $\varepsilon > 0$, if $y_\varepsilon \in \mathbf{C}_{rd}^2([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfies the following inequality*

$$|y_\varepsilon^{\Delta\Delta}(t) + my_\varepsilon^{\Delta}(t) + ny_\varepsilon(t) - f(t)| \le \varepsilon$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}_{rd}^2([a,b]_{\mathbb{T}}, \mathbb{R})$ of (3.11) such that

$$|y_\varepsilon(t) - y(t)| \le K\varepsilon$$

for all $t \in \mathbb{T}$, where

$$K = \frac{1}{2}\max_{t\in[a,b]_{\mathbb{T}}} |e_r(t,t_0)| \int_a^t \Big| \frac{e_{r\ominus(\mu n - m)}(\sigma(\tau), t_0)[(r-s)(1-\mu)e_{\frac{s}{1+\mu r}}(t,t_0)e_{-\frac{s}{1+\mu r}}(\tau,t_0)}{(1+\mu r)}$$
$$\frac{-(r+s)(1+\mu)e_{\frac{s}{1+\mu r}}(\tau,t_0)e_{-\frac{s}{1+\mu r}}(t,t_0)]}{(1+\mu r)}\Big|\Delta\tau.$$

Theorem 3.6 *Let $\varphi : \mathbb{T} \to \mathbb{R}_+$ be an rd-continuous function and let $m^2 - 4n < 0$. Define*

$$r = -\frac{m}{2}, \quad s = \frac{\sqrt{m^2 - 4n}}{2}.$$

Assume that r and $\mu n - m$ are regressive. If $y_\varphi \in \mathbf{C}_{rd}^2$ satisfies the following inequality

$$|y_\varphi^{\Delta\Delta}(t) + my_\varphi^{\Delta}(t) + ny_\varphi(t) - f(t)| \le \varphi(t)$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}_{rd}^2$ of (3.11) such that

$$\begin{aligned}
&|y_\varphi(t) - y(t)| \\
&\le \frac{1}{2}|e_r(t,t_0)|\Big|\int_{t_0}^t \Big|e_{r\ominus(\mu n-m)}(\sigma(\tau),t_0)\big[i(r - \frac{\mu s^2}{1+\mu r})(e_{-\frac{s}{1+\mu r}i}(t,t_0) - e_{\frac{s}{1+\mu r}i}(t,t_0))e_{-\frac{s}{1+\mu r}i}(\tau,t_0) - \\
&(\frac{s(1+2\mu s)}{1+\mu r})(e_{\frac{s}{1+\mu r}i}(t,t_0)e_{-\frac{s}{1+\mu r}i}(\tau,t_0) + e_{\frac{s}{1+\mu r}i}(\tau,t_0)e_{-\frac{s}{1+\mu r}i}(t,t_0))\big]|\varphi(\tau)\Delta\tau|
\end{aligned}$$

for all $t \in \mathbb{T}$.

Proof From Theorem 3.32 in [39], we know that

$$y_1(t) = \cos_{s/(1+\mu r)}(t,t_0)e_r(t,t_0), \quad y_2(t) = \sin_{s/(1+\mu r)}(t,t_0)e_r(t,t_0)$$

form a fundamental system of solutions of the corresponding homogeneous equation of (3.11), where $t_0 \in \mathbb{T}$. By calculation, the Wronskian of these two solutions is $W(y_1,y_2)(t) = se_{\mu n - m}(t,t_0)$.

Similar to Theorem 3.5, we write $\rho = \frac{s}{1+\mu r}$, and then we can get

$$\begin{aligned}
&y_1^{\Delta^\sigma}(\tau)y_2(t) - y_1(t)y_2^{\Delta^\sigma}(\tau) \\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0)\begin{vmatrix} -s\sin_\rho(\sigma(\tau),t_0) + r\cos_\rho(\sigma(\tau),t_0) & \cos_\rho(t,t_0) \\ s\cos_\rho(\sigma(\tau),t_0) + r\sin_\rho(\sigma(\tau),t_0) & \sin_\rho(t,t_0)\end{vmatrix} \\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big(s\begin{vmatrix} -\sin_\rho(\sigma(\tau),t_0) & \cos_\rho(t,t_0) \\ \cos_\rho(\sigma(\tau),t_0) & \sin_\rho(t,t_0)\end{vmatrix} + r\begin{vmatrix} \cos_\rho(\sigma(\tau),t_0) & \cos_\rho(t,t_0) \\ \sin_\rho(\sigma(\tau),t_0) & \sin_\rho(t,t_0)\end{vmatrix}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big(s\begin{vmatrix}-\sin_\rho(\tau,t_0)-\mu\rho\cos_\rho(\tau,t_0) & \cos_\rho(t,t_0)\\ \cos_\rho(\tau,t_0)-\mu\rho\sin_\rho(\tau,t_0) & \sin_\rho(t,t_0)\end{vmatrix}+\\
&\qquad r\begin{vmatrix}\cos_\rho(\tau,t_0)-\mu\rho\sin_\rho(\tau,t_0) & \cos_\rho(t,t_0)\\ \sin_\rho(\tau,t_0)+\mu\rho\cos_\rho(\tau,t_0) & \sin_\rho(t,t_0)\end{vmatrix}\Big)\\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big((s+\mu r\rho)\begin{vmatrix}-\sin_\rho(\tau,t_0) & \cos_\rho(t,t_0)\\ \cos_\rho(\tau,t_0) & \sin_\rho(t,t_0)\end{vmatrix}+\\
&\qquad (r-\mu s\rho)\begin{vmatrix}\cos_\rho(\tau,t_0) & \cos_\rho(t,t_0)\\ \sin_\rho(\tau,t_0) & \sin_\rho(t,t_0)\end{vmatrix}\Big)\\
&= \frac{1}{4}e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big((s+\mu r\rho)\begin{vmatrix}i(e_{i\rho}(\tau,t_0)-e_{-i\rho}(\tau,t_0)) & e_{i\rho}(t,t_0)+e_{-i\rho}(t,t_0)\\ e_{i\rho}(\tau,t_0)+e_{-i\rho}(\tau,t_0) & -i(e_{i\rho}(t,t_0)-e_{-i\rho}(t,t_0))\end{vmatrix}+\\
&\qquad (r-\mu s\rho)\begin{vmatrix}e_{i\rho}(\tau,t_0)+e_{-i\rho}(\tau,t_0) & e_{i\rho}(t,t_0)-e_{-i\rho}(t,t_0)\\ -i(e_{i\rho}(\tau,t_0)-e_{-i\rho}(\tau,t_0)) & -i(e_{i\rho}(t,t_0)-e_{-i\rho}(t,t_0))\end{vmatrix}\Big)\\
&= \frac{1}{2}e_r(\sigma(\tau),t_0)e_r(t,t_0)\Big(i(r-\mu s\rho)(e_{-i\rho}(t,t_0)-e_{i\rho}(t,t_0))e_{-i\rho}(\tau,t_0)-\\
&\qquad (s+\mu r\rho)(e_{i\rho}(t,t_0)e_{-i\rho}(\tau,t_0)+e_{i\rho}(\tau,t_0)e_{-i\rho}(t,t_0))\Big).
\end{aligned}
$$

According to Theorem 3.2, we can infer that

$$
\begin{aligned}
|y_\varphi(t)-y(t)| \le &\frac{1}{2}|e_r(t,t_0)|\Big|\int_{t_0}^{t}\Big|e_{r\ominus(\mu n-m)}(\sigma(\tau),t_0)\Big[e_{-i\rho}(t,t_0)-e_{i\rho}(t,t_0))e_{-i\rho}(\tau,t_0)-\\
&(s+\mu r\rho)(e_{i\rho}(t,t_0)e_{-i\rho}(\tau,t_0)+e_{i\rho}(\tau,t_0)e_{-i\rho}(t,t_0))\Big]\Big|\varphi(\tau)\Delta\tau\Big|.
\end{aligned}
$$

This completes the proof. □

Furthermore, if the characteristic equation (3.12) has a pair of complex eigenvalues, we then obtain the Hyers-Ulam stability of the second order nonhomogeneous linear dynamic equation (2.11) with constant coefficients on a finite time scale.

Corollary 3.7 *Let $m^2-4n<0$ and let r,s be given as in Theorem 3.6. Assume that r and $\mu n-m$ are regressive. For a given $\varepsilon>0$, if $y_\varepsilon\in \mathbf{C}^2_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ satisfies the following inequality*

$$|y_\varepsilon^{\Delta\Delta}(t)+my_\varepsilon^{\Delta}(t)+ny_\varepsilon(t)-f(t)|\le\varepsilon$$

for all $t\in\mathbb{T}^{\kappa^2}$, then there exists a solution $y\in\mathbf{C}^2_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ of (3.11) such that

$$|y_\varepsilon(t)-y(t)|\le K\varepsilon$$

for all $t\in\mathbb{T}$, where

$$
\begin{aligned}
K=&\frac{1}{2}\max_{t\in[a,b]_{\mathbb{T}}}|e_r(t,t_0)||\\
&\int_a^t\Big|e_{r\ominus(\mu n-m)}(\sigma(\tau),t_0)[i(r-\frac{\mu s^2}{1+\mu r})(e_{-\frac{s}{1+\mu r}i}(t,t_0)-e_{\frac{s}{1+\mu r}i}(t,t_0))e_{-\frac{s}{1+\mu r}i}(\tau,t_0)-\\
&(\frac{s(1+2\mu s)}{1+\mu r})(e_{\frac{s}{1+\mu r}i}(t,t_0)e_{-\frac{s}{1+\mu r}i}(\tau,t_0)+e_{\frac{s}{1+\mu r}i}(\tau,t_0)e_{-\frac{s}{1+\mu r}i}(t,t_0))]\Big|\Delta\tau.
\end{aligned}
$$

Theorem 3.8 *Let $\varphi : \mathbb{T} \to \mathbb{R}_+$ be an rd-continuous function and let $m^2 - 4n = 0$. Let $r = -\frac{m}{2}$ with $r \in \mathcal{R}$. If $y_\varphi \in \mathbf{C}^2_{rd}$ satisfies the following inequality*

$$|y_\varphi^{\Delta\Delta}(t) + my_\varphi^{\Delta}(t) + ny_\varphi(t) - f(t)| \leq \varphi(t)$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}^2_{rd}$ of (3.11) such that

$$|y_\varphi(t) - y(t)| \leq |e_r(t,t_0)| \Big| \int_{t_0}^t \Big| e_{r \ominus \frac{\mu\alpha^2}{4}}(\sigma(\tau), t_0) \Big(r \int_\tau^t \frac{1}{1 + r\mu(\omega)} \Delta\omega - (r+1) \Big) |\varphi(\tau) \Delta\tau|$$

for all $t \in \mathbb{T}$.

Proof In view of Theorem 3.34 in [39], we know that $y_1(t) = e_r(t, t_0)$ and

$$y_2(t) = e_r(t,t_0) \int_{t_0}^t \frac{1}{1 + r\mu(\omega)} \Delta\omega$$

form a fundamental system of solutions of the corresponding homogeneous equation of (3.11), where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is $e_{\frac{\mu\alpha^2}{4}}(t, t_0)$. Using the same procedure as Theorem 3.5, we can obtain

$$\begin{aligned}
& y_1^{\Delta^\sigma}(\tau) y_2(t) - y_1(t) y_2^{\Delta^\sigma}(\tau) \\
&= \begin{vmatrix} re_r(\sigma(\tau), t_0) & e_r(t,t_0) \\ e_r(\sigma(\tau),t_0)(1 + r\int_{t_0}^{\sigma(\tau)} \frac{1}{1+r\mu(\omega)}\Delta\omega) & e_r(t,t_0)\int_{t_0}^t \frac{1}{1+r\mu(\omega)}\Delta\omega \end{vmatrix} \\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0) \begin{vmatrix} r & 1 \\ 1 + r\int_{t_0}^{\sigma(\tau)} \frac{1}{1+r\mu(\omega)}\Delta\omega & \int_{t_0}^t \frac{1}{1+r\mu(\omega)}\Delta\omega \end{vmatrix} \\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0) \begin{vmatrix} r & 1 \\ 1 + r(1 + \int_{t_0}^{\tau} \frac{1}{1+r\mu(\omega)}\Delta\omega) & \int_{t_0}^t \frac{1}{1+r\mu(\omega)}\Delta\omega \end{vmatrix} \\
&= e_r(\sigma(\tau),t_0)e_r(t,t_0) \Big(r \int_\tau^t \frac{1}{1 + r\mu(\omega)} \Delta\omega - (r+1) \Big).
\end{aligned}$$

Then, it follows from Theorem 3.2 that

$$|y_\varphi(t) - y(t)| \leq |e_r(t,t_0)| \Big| \int_{t_0}^t \Big| \frac{e_r(\sigma(\tau),t_0)(r\int_\tau^t \frac{1}{1+r\mu(\omega)}\Delta\omega - (r+1))}{e_{\frac{\mu\alpha^2}{4}}(\sigma(\tau),t_0)} |\varphi(\tau)\Delta\tau|.$$

This completes the proof. □

Based on the previous theorem, if the characteristic equation (3.12) has two identical eigenvalues, we then obtain the Hyers-Ulam stability of the second order nonhomogeneous linear dynamic equation (3.11) with constant coefficients on a finite time scale.

Corollary 3.9 *Let $m^2 - 4n = 0$ and let r be given as in Theorem 3.8. For a given $\varepsilon > 0$, if $y_\varepsilon \in \mathbf{C}^2_{rd}([a,b]_\mathbb{T}, \mathbb{R})$ satisfies the following inequality*

$$|y_\varepsilon^{\Delta\Delta}(t) + my_\varepsilon^{\Delta}(t) + ny_\varepsilon(t) - f(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y \in \mathbf{C}^2_{rd}([a,b]_\mathbb{T}, \mathbb{R})$ of (3.11) such that

$$|y_\varepsilon(t) - y(t)| \leq K\varepsilon$$

for all $t \in \mathbb{T}$, *where*

$$K = \max_{t\in[a,b]_{\mathbb{T}}} |e_r(t,t_0)| \int_a^t \Big| e_{r\ominus\frac{\mu\alpha^2}{4}}(\sigma(\tau),t_0)\Big(r\int_\tau^t \frac{1}{1+r\mu(\omega)}\Delta\omega - (r+1)\Big)\Big|\Delta\tau.$$

Finally, by Theorem 3.2, we can consider the Ulam stability of the second order linear dynamic equation of the form

$$x^{\Delta\Delta} + p(t)x^{\Delta^\sigma} + q(t)x^\sigma = f(t), \tag{3.13}$$

where $t \in \mathbb{T}$. If $p \in \mathcal{R}$ and $q, f \in \mathbf{C}_{rd}$, then the equation (3.13) is equivalent to the following equation

$$(1+\mu p)(x^{\Delta\Delta} + \widetilde{p}(t)x^\Delta + \widetilde{q}(t)x) = f(t), \tag{3.14}$$

where

$$\widetilde{p}(t) = \frac{p+\mu q}{1+\mu p}, \quad \widetilde{q}(t) = \frac{q}{1+\mu p}.$$

Theorem 3.10 *Let* $p \in \mathcal{R}$, $q, f \in \mathbf{C}_{rd}$ *and let* x_1 *and* x_2 *be a fundamental system of solutions of the corresponding homogeneous equation of (3.13). Assume that* $\varphi : \mathbb{T} \to \mathbb{R}_+$ *is an rd-continuous function. If* $x_\varphi \in \mathbf{C}_{rd}^2$ *satisfies the following inequality*

$$|x_\varphi^{\Delta\Delta}(t) + p(t)x_\varphi^{\Delta^\sigma}(t) + q(t)x_\varphi^\sigma(t) - f(t)| \le \varphi(t) \tag{3.15}$$

for all $t \in \mathbb{T}^{\kappa^2}$, *then there exists a solution* $x \in \mathbf{C}_{rd}^2$ *of (3.1) such that*

$$|x_\varphi(t) - x(t)| \le \Big| \int_{t_0}^t \Big| \frac{(x_1^{\Delta^\sigma}(\tau)x_2(t) - x_1(t)x_2^{\Delta^\sigma}(\tau))}{(1+\mu(\tau)p(\tau))W^\sigma(x_1,x_2)(\tau)} |\varphi(\tau)\Delta\tau| \tag{3.16}$$

for all $t \in \mathbb{T}$.

Proof Since $p \in \mathcal{R}$, we know that $1+\mu(t)p(t) \ne 0$ for $t \in \mathbb{T}^\kappa$. Using the equivalence of equations (3.13) and (3.14), it is easy to verify that x_1 and x_2 are also a fundamental system of solutions of the corresponding homogeneous equation of (3.14). Furthermore, it follows that the inequality (3.15) is equivalent to

$$|x_\varphi^{\Delta\Delta}(t) + \widetilde{p}(t)x_\varphi^\Delta(t) + \widetilde{q}(t)x_\varphi(t) - \frac{f(t)}{1+\mu p}| \le \frac{\varphi(t)}{|1+\mu p|}. \tag{3.17}$$

Therefore, we can infer from Theorem 3.2 that the inequality (3.16) is valid. □

In particular, we can show the Hyers-Ulam stability of the nonhomogeneous linear dynamic equation (3.13) on a compact interval $[a,b]_{\mathbb{T}}$.

Corollary 3.11 *Let* $p \in \mathcal{R}$, $q, f \in \mathbf{C}_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *and let* x_1 *and* x_2 *be a fundamental system of solutions of the homogeneous equation (3.13). For a given* $\varepsilon > 0$, *if* $x_\varepsilon \in \mathbf{C}_{rd}^2([a,b]_{\mathbb{T}}, \mathbb{R})$ *satisfies the following inequality*

$$|x_\varepsilon^{\Delta\Delta}(t) + p(t)y_\varepsilon^{\Delta^\sigma}(t) + q(t)y_\varepsilon^\sigma(t) - f(t)| \le \varepsilon$$

for all $t \in \mathbb{T}^{\kappa^2}$, *then there exists a solution* $x \in \mathbf{C}_{rd}^2$ *of (3.13) such that*

$$|x_\varepsilon(t) - x(t)| \le K\varepsilon$$

for all $t \in [a,b]_{\mathbb{T}}$, *where*

$$K = \max_{t\in[a,b]_{\mathbb{T}}} \int_a^t \Big|\frac{(x_1^{\Delta^\sigma}(\tau)x_2(t) - x_1(t)x_2^{\Delta^\sigma}(\tau))}{(1+\mu(\tau)p(\tau))W^\sigma(x_1,x_2)(\tau)}\Big|\Delta\tau.$$

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References

[1] S. M. ULAM. *Problems in Modern Mathematics*. Wiley, New York, 1960.

[2] D. H. HYERS. *On the stability of the linear functional equation*. Proc. Nat. Acad. Sci. U.S.A., 1941, **27**: 222–224.

[3] T. M. RASSIAS. *On the stability of the linear mapping in Banach spaces*. Proc. Amer. Math. Soc., 1978, **72**(2): 297–300.

[4] M. OBLOZA. *Hyers stability of the linear differential equation*. Rocznik Nauk.-Dydakt. Prace Mat., 1993, **13**: 259–270.

[5] C. ALSINA, R. GER. *On some inequalities and stability results related to the exponential function*. J. Inequal. Appl., 1998, **2**(4): 373–380.

[6] T. MIURA. *On the Hyers-Ulam stability of a differentiable map*. Sci. Math. Jpn., 2002, **55**(1): 17–24.

[7] T. MIURA, S. E. TAKAHASI, H. CHODA. *On the Hyers-Ulam stability of real continuous function valued differentiable map*. Tokyo J. Math., 2001, **24**(2): 467–476.

[8] S. E. TAKAHASI, T. MIURA, S. MIYAJIMA. *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y = \lambda y$. Bull. Korean Math. Soc., 2002, **39**(2): 309–315.

[9] J. BRZDĘK, D. POPA, I. RAŞA, et al. *Ulam Stability of Operators*. Elsevier, San Diego, 2018.

[10] S. S. CîMPEAN, D. POPA. *On the stability of the linear differential equation of higer order with constant coefficients*. Appl. Math. Comput., 2010, **27**: 4141–4146.

[11] S. M. JUNG. *Hyers-Ulam stability of linear differential equations of first order*. Appl. Math. Lett., 2004, **17**(10): 1135–1140.

[12] S. M. JUNG. *Hyers-Ulam stability of linear differential equations of first order (III)*. J. Math. Anal. Appl., 2005, **311**(1): 139–146.

[13] S. M. JUNG. *Hyers-Ulam stability of linear differential equations of first order (II)*. Appl. Math. Lett., 2006, **19**(9): 854–858.

[14] Yongjin LI, Yan SHEN. *Hyers-Ulam stability of linear differential equations of second order*. Appl. Math. Lett., 2010, **23**(3): 306–309.

[15] D. POPA, G. PUGUA. *Hyers-Ulam stability of Euler's differential equation*. Results Math., 2015, **69**(3-4): 317–325.

[16] D. POPA, I. RAŞA. *On the Hyers-Ulam stability of the linear differential equation*. J. Math. Anal. Appl., 2011, **381**(2): 530–537.

[17] D. POPA, I. RAŞA. *Hyers-Ulam stability of the linear differential operator with non-constant coefficients*. Appl. Math. Comput., 2012, **219**: 1562–1568.

[18] D. POPA. *Heyers-Ulam-Rassias stability of a linear recurrence*. J. Math. Anal. Appl., 2005, **309**: 591–597.

[19] D. POPA. *Hyers-Ulam stability of the linear recurrence with constant coefficients*. Adv. Difference Equ., 2005, **2**: 101–107.

[20] J. BRZDĘK, D. POPA, Bing XU. *Note on nonstability of the linear recurrence*. Abh. Math. Sem. Univ. Hamburg, 2006, **76**: 183C-189.

[21] J. BRZDĘK, D. POPA, Bing XU. *The Hyers-Ulam stability of nonliear recurrences*. J. Math. Anal. Appl., 2007, **335**: 443–449.

[22] J. BRZDĘK, D. POPA, Bing XU. *Remarks on stability of linear recurrence of higher order*. Appl. Math. Lett., 2010, **23**(12): 1459–1463.

[23] J. BRZDĘK, P. WÓJCIK. *On approximate solutions of some difference equations*. Bull. Aust. Math. Soc., 2017, **95**(3): 476–481.

[24] S. M. JUNG. *Hyers-Ulam stability of the first-order matrix difference equations*. Adv. Difference Equ. 2015, 2015:170, 13 pp.

[25] S. M. JUNG, Y. W. NAM. *Hyers-Ulam stability of Pielou logistic difference equation*. J. Nonlinear Sci. Appl., 2017, **10**(6): 3115–3122.

[26] M. ONITSUKA. *Influence of the stepsize on Hyers-Ulam stability of first-order homogeneous linear difference equations*. Int. J. Difference Equ., 2017, **12**(2): 281–302.

[27] M. ONITSUKA. *Hyers-Ulam stability of first-order nonhomogeneous linear difference equations with a constant stepsize*. Appl. Math. Comput., 2018, **330**: 143–151.

[28] Yonghong SHEN, Yongjin LI. *The z-transform method for the Ulam stability of linear difference equations with constant coefficients*. Adv. Difference Equ., 2018, Paper No. 396, 15 pp.

[29] S. HILGER. *Ein Makettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*. Ph. D. Thesis, Universität Würzburg, 1988.

[30] A. E. HAMZA, N. A. YASEEN. *Hyers-Ulam stability of abstract first-order dynamic equations on time scales*. J. Adv. Math., 2011, **7**: 1095–1104.

[31] D. R. ANDERSON, B. GATES, D. HEUER. *Hyers-Ulam stability of second-order linear dynamic equations on time scales*. Commun. Appl. Math., 2012, **16**(3): 281–292.

[32] S. ANDRÁS, A. R. MÉSZÁROS. *Ulam-Hyers stability of dynamic equations on time scales via Picard operators*. Appl. Math. Comput., 2013, **219**(9): 4853–4864.

[33] A. E. HAMZA, N. A. YASEEN. *Hyers-Ulam stability of abstract second-order linear dynamic equations on time scales*. Int. J. Math. Anal., 2014, **8**: 1421–1432.

[34] D. R. ANDERSON. *Hyers-Ulam stability of higher-order Cauchy-Euler dynamic equations on time scales*. Dynam. Syst. Appl., 2014, **23**: 653–664.

[35] Yonghong SHEN. *The Ulam stability of first order linear dynamic equations on time scales*. Results Math., 2017, **72**(4): 1881–1895.

[36] Yonghong SHEN, Deming HE. *The general solution and Ulam stability of inhomogeneous Euler-Cauchy dynamic equations on time scales*. J. Comput. Anal. Appl., 2019, **26**(2): 234–241.

[37] D. R. ANDERSON, M. ONITSUKA. *Hyers-Ulam stability of first-order homogeneous linear dynamic equations on time scales*. Demonstr. Math., 2018, **51**(1): 198–210.

[38] D. R. ANDERSON, M. ONITSUKA. *Hyers-Ulam stability for a discrete time scale with two step sizes*. Appl. Math. Comput., 2019, **344-345**: 128–140.

[39] M. BOHNER, A. PETERSON. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston Inc., Boston, MA, 2001.

[40] M. BOHNER, A. PETERSON. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.