Journal of Mathematical Research with Applications
Sept., 2020, Vol. 40, No. 5, pp. 507–518
DOI:10.3770/j.issn:2095-2651.2020.05.007
Http://jmre.dlut.edu.cn

Some Properties of Solutions for Some Types of Q-Difference Equations Originated from Q-Difference Painlevé Equation

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Abstract In this paper, we mainly investigate some properties of meromorphic solutions for several q-difference equations, which can be seen as the q-difference analogues of Painlevé equations. Some results about the existence and the estimates of growth of meromorphic solution f for q-difference equations are obtained, especially for some estimates for the exponent of convergence of poles of $\Delta_q f(z) := f(qz) - f(z)$, which extends some previous results by Qi and Yang.

Keywords meromorphic function; q-difference equation; zero order

MR(2010) Subject Classification 39A50; 30D35

1. Introduction

About a hundred years ago, Painlevé and his colleagues [1] studied the equation

$$w''(z) = F(z; w; w'),$$

where F is rational in w and w' and (locally) analytic in z. They singled out a list of 50 equations, six of which could not be integrated in terms. Differential Painlevé equations have been an important research subject in the field of the mathematics and physics since the beginning of last century. They occur in many physical situations–plasma physics, statistical mechanics, nonlinear waves, and so on. Therefore, Painlevé equations have attracted much interest as the reduction of solution equations which are solvable by inverse scattering transformations, and so on.

The discrete Painlevé equations in the 1990s have become important research problems [2,3]. For example, the following equations

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c, \quad y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + \frac{c}{y_n^2},$$

Received July 22, 2019; Accepted April 23, 2020
Supported by the National Natural Science Foundation of China (Grant Nos. 11561033; 11761035), the Natural Science Foundation of Jiangxi Province (Grant No. 20181BAB201001) and the Foundation of Education Department of Jiangxi Province (Grant Nos. GJJ190876; GJJ191042; GJJ190895).
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are called the special discretization of discrete P_I, and the equation

$$y_{n+1}+y_{n-1}=\frac{(an+b)y_n+c}{1-y_n^2},$$

is called the special discretization of the discrete P_{II}, where a,b,c are constants, $n\in\mathbb{N}_+$.

Of late, due to those results about the difference analog of the lemma on the logarithmic derivative given by Chiang and Feng in [4], and Halburd and Korhonen in [5] independently, there has been an increasing interest in studying complex difference equations, a number of papers [4, 6, 7] focused on complex difference equations and difference analogues of Nevanlinna's theory. For example, Halburd and Korhonen [5, 8, 9] around 2006s used Nevanlinna value distribution theory to single out the difference Painlevé I and II equations from the following form

$$w(z+1)+w(z-1)=R(z,w), \tag{1.1}$$

where $R(z,w)$ is rational in w and meromorphic in z. They obtained that if (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed by a linear change in w to some difference equations, which include difference Painlevé I equations

$$w(z+1)+w(z-1)=\frac{az+b}{w(z)}+c, \tag{1.2}$$

$$w(z+1)+w(z-1)=\frac{(az+b)}{w(z)}+\frac{c}{w(z)^2}, \tag{1.3}$$

and difference Painlevé II equation

$$w(z+1)+w(z-1)=\frac{(az+b)w(z)+c}{1-w(z)^2}. \tag{1.4}$$

Recently, there were many interest results on some properties of finite order transcendental meromorphic solutions of (1.2)–(1.4). In 2007, Barnett, Halburd, Korhonen and Morgan [10] firstly established an analogue of the Logarithmic Derivative Lemma on q-difference operators. In 2007 and 2010, Laine and Yang, Zhang and Korhonen, Zheng and Chen further studied some properties on the value distribution of q-difference operator of meromorphic function. Moreover, in the past 15 years, there were also lots of results about q-difference operators, q-difference equations, and so on [11–21], by replacing the q-difference $f(qz)$, $q\in\mathbb{C}-\{0,1\}$ with the usual shift $f(z+c)$ of a meromorphic function in some expression concerning complex difference equations and complex difference operators. To state their results, we first introduce some notations and some assumptions as follows.

Throughout this article, a term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. In what follows, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [22–24]. Further, for a meromorphic function f, let $\sigma(f),\lambda(f)$ and $\lambda(\frac{1}{f})$ be the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$, respectively, and let $\tau(f)$ be the exponent of convergence of fixed points of $f(z)$, which is defined by

$$\tau(f)=\limsup_{r\to+\infty}\frac{\log N(r,\frac{1}{f(z)-z})}{\log r}.$$

In addition, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure

$$\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{\mathrm{d}t}{t}<\infty,$$

and a meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a(z))=S(r,f)$, and we use $\mathcal{S}(f)$ to denote the field of small functions relative to $f(z)$.

In 2010, Chen and Shon [25] considered the difference Painlevé I equation (1.2) and obtained the following theorem.

Theorem 1.1 ([25, Theorem 4]) *Let a,b,c be constants, where a,b are not both equal to zero. Then*

(i) If $a\neq 0$, then (1.2) has no rational solution;

(ii) If $a=0$, and $b\neq 0$, then (1.2) has a nonzero constant solution $w(z)=A$, where A satisfies $2A^2-cA-b=0$.

The other rational solution $w(z)$ satisfies $w(z)=\frac{P(z)}{Q(z)}+A$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P<\deg Q$.

In 2015, Li ang Huang [26] further investigated the properties of solutions of a certain type of difference equation, and obtained some results which are an improvement of Theorem 1.1.

Theorem 1.2 ([26, Theorem 2.1]) *Let c be a nonzero constant, and $A(z)=\frac{m(z)}{n(z)}$ be an irreducible rational function, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z)=m$ and $\deg n(z)=n$.*

(i) Suppose that $m\geq n$ and $m-n$ is an even number or zero. If the difference equation

$$w(z+1)+w(z-1)=\frac{A(z)}{w(z)}+c \tag{1.5}$$

has an irreducible rational solution $w(z)=\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials with $\deg P(z)=p$ and $\deg Q(z)=q$, then $p-q=\frac{m-n}{2}$.

(ii) Suppose that $m<n$. If the difference equation (1.5) has an irreducible rational solution $w(z)=\frac{P(z)}{Q(z)}$, then $q-p=n-m\geq 1$ or $q-p=0$.

(iii) Suppose that $m>n$ and $m-n$ is an odd number. Then the difference equation (1.5) has no rational solution.

Theorem 1.3 ([26, Theorem 3.1]) *Suppose that the equation*

$$w(z+1)+w(z-1)=\frac{A(z)}{w(z)}+C(z),$$

where $A(z),C(z)\in\mathcal{S}(w)$, admits a finite order transcendental meromorphic solution $w(z)$. Then

(i) $\lambda(w)=\lambda(\frac{1}{w})=\sigma(w)$;

(ii) $w(z)$ has no Borel exceptional value;

(iii) If $A(z) \not\equiv 2z^2 - zC(z)$, then the exponent of convergence of fixed points of $w(z)$ satisfies $\tau(w) = \sigma(w)$.

In the same year, Qi and Yang [27] investigated the following equation

$$w(qz) + w(\frac{z}{q}) = \frac{az+b}{w(z)} + c, \tag{1.6}$$

which can be seen as q-difference analogues of (1.2), and obtained the following theorem.

Theorem 1.4 ([27, Theorem 1.1]) *Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (1.6), and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

(i) $f(z)$ has infinitely many poles.

(ii) If $a \neq 0$ and any $d \in \mathbb{C}$, then $f(z) - d$ has infinitely many zeros.

(iii) If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

In 2017, Xu, Liu and Zheng [28] further investigated some properties of transcendental meromorphic solutions of the equations (1.6), and obtained the following theorem, which extends Theorem 1.4.

Theorem 1.5 ([28, Theorem 1.3]) *Let a, b, c be constants with $|a| + |b| \neq 0$. Suppose that $w(z)$ is a zero order transcendental meromorphic solution of (1.6). Then*

(i) If $a \neq 0$, $p(z)$ is a polynomial of degree $k \geq 0$ and $|q| \neq 1$, then $w(z) - p(z)$ has infinitely many zeros and $\lambda(w - p) = \sigma(w)$;

If $a = 0$, then Borel exceptional values of $w(z)$ can only come from the set $E = \{z | 2z^2 - cz - b = 0\}$;

(ii) $\lambda(\frac{1}{w}) = \lambda(\frac{1}{\Delta_q w}) = \sigma(\Delta_q w) = \sigma(w)$.

Inspired by the above results, we further investigate some properties of meromorphic solutions of some types of q-difference equations which are different from (1.5) and (1.6) to some extent, and obtain the following theorems.

Theorem 1.6 *Let $q \in \mathbb{C} - \{0\}$ and $|q| \neq 1$, c be a nonzero constant, and $A(z) = \frac{m(z)}{n(z)}$ be an irreducible rational function, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$.*

(i) Suppose that $m \geq n$ and $m - n$ is an even number or zero. If the difference equation

$$f(qz) + f(z) + f(\frac{z}{q}) = \frac{A(z)}{f(z)} + c \tag{1.7}$$

has an irreducible rational solution $f(z) = \frac{P(z)}{R(z)}$, where $P(z)$ and $R(z)$ are polynomials with $\deg P(z) = p$ and $\deg R(z) = r$, then $p - r = \frac{m-n}{2}$.

(ii) Suppose that $m < n$. If the difference equation (1.7) has an irreducible rational solution $f(z) = \frac{P(z)}{R(z)}$, then $f(z)$ satisfies one of the following two cases

(a) $r - p = n - m \geq 1$;

(b) $r-p=0$ *and* $f(z)=\frac{P(z)}{R(z)}=\eta+\frac{S(z)}{H(z)}$, *where* $\eta=\frac{c}{3}$, $S(z)$ *and* $H(z)$ *are polynomials with* $\deg S(z)=s$ *and* $\deg H(z)=h$, *and* $s-h=m-n$;

(iii) Suppose that $m>n$ *and* $m-n$ *is an odd number. Then the difference equation (1.7) has no rational solution.*

Theorem 1.7 *Let* $q\in\mathbb{C}-\{0,1\}$. *Suppose that the equation*

$$f(qz)+f(z)+f(\frac{z}{q})=\frac{A(z)}{f(z)}+C(z), \tag{1.8}$$

where $A(z)(\not\equiv 0), C(z)\in\mathcal{S}(f)$, *admits a zero order transcendental meromorphic solution* $f(z)$. *Then*

(i) $f(z)$ *has infinitely many poles and zeros,* $\Delta_q f$ *also has infinitely many poles, and*

$$\lambda(f)=\lambda(\frac{1}{f})=\lambda(\frac{1}{\Delta_q f}),$$

and further, if $C(z)\not\equiv 0$, *we have that* $\Delta_q^2 f, \frac{\Delta_q f}{f}, \frac{\Delta_q^2 f}{f}$ *have infinitely many poles, respectively, and*

$$\lambda(\frac{1}{\Delta_q^2 f})=\lambda(\frac{1}{\frac{\Delta_q f}{f}})=\lambda(\frac{1}{\frac{\Delta_q^2 f}{f}}).$$

(ii) If $A(z)\not\equiv(q+1+\frac{1}{q})z^2-zC(z)$, *then* f *has infinitely many fixed points and the exponent of convergence of fixed points of* f *satisfies* $\tau(f)=\sigma(f)$.

2. Some lemmas

The following result can be called an analogue of q-difference Clunie lemma, recently proved by Barnett et al. [10, Theorem 2.1]. Here a q-difference polynomial of f for $q\in\mathbb{C}\backslash\{0,1\}$ is a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz),\dots,f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set F of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}\mathrm{d}t.$$

Lemma 2.1 ([14, Theorem 2.5]) *Let* f *be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z,f)P_q(z,f)=Q_q(z,f),$$

where $U_q(z,f), P_q(z,f)$ *and* $Q_q(z,f)$ *are q-difference polynomials such that the total degree* deg $U_q(z,f)=n$ *in* $f(z)$ *and its q-shifts, whereas* $\deg Q_q(z,f)\le n$. *Moreover, we assume that* $U_q(z,f)$ *contains just one term of maximal total degree in* $f(z)$ *and its q-shifts. Then*

$$m(r,P_q(z,f))=o(T(r,f)),$$

on a set of logarithmic density 1.

Lemma 2.2 ([10, Theorem 2.5]) *Let* f *be a nonconstant zero-order meromorphic solution* *of $P_q(z,f)=0$, where $P_q(z,f)$ is a q-difference polynomial in $f(z)$. If $P_q(z,a)\not\equiv 0$ for slowly moving target $a(z)$, then*

$$m(r,\frac{1}{f-a})=o(T(r,f)),$$

on a set of logarithmic density 1.

Lemma 2.3 ([19, Theorems 1.1 and 1.3]) *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q\in\mathbb{C}\setminus\{0\}$. Then*

$$T(r,f(qz))=(1+o(1))T(r,f(z)),\quad N(r,f(qz))=(1+o(1))N(r,f(z)),$$

on a set of lower logarithmic density 1.

Lemma 2.4 ([29]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z),b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies that

$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),$$

where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Lemma 2.5 ([10, Theorem 1.1]) *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q\in\mathbb{C}\setminus\{0\}$. Then*

$$m(r,\frac{f(qz)}{f(z)})=S(r,f)$$

on a set of logarithmic density 1.

3. The Proof of Theorem 1.6

Assume that (1.7) has a rational solution $f(z)=\frac{P(z)}{R(z)}$ and has poles $z_1,z_2,\ldots,z_k$. Then $f(z)$ can be represented in the following form

$$f(z)=\frac{P(z)}{R(z)}=\sum_{j=1}^{k}[\frac{c_{j\lambda_j}}{(z-z_j)^{\lambda_j}}+\cdots+\frac{c_{j1}}{(z-z_j)}]+a_0+a_1z+\cdots+a_vz^v, \tag{3.1}$$

where $c_{j\lambda_j}(\neq 0),\ldots,c_{j1}$ $(j=1,2,\ldots,k)$, and $a_0,a_1,\ldots,a_v$ are constants, z_j $(j=1,2,\ldots,k)$ are poles of $f(z)$ with multiplicity λ_j, respectively.

(i) Suppose that $m>n$ and $m-n$ is an even number. Then by (1.7) and (3.1), we deduce

$$\frac{P(z)}{R(z)}[\frac{P(qz)}{R(qz)}+\frac{P(z)}{R(z)}+\frac{P(\frac{z}{q})}{R(\frac{z}{q})}]-c\frac{P(z)}{R(z)}=\frac{m(z)}{n(z)}. \tag{3.2}$$

If $\deg P(z)=p<r=\deg R(z)$, then it yields

$$\frac{P(qz)}{R(qz)}\to 0,\quad \frac{P(z)}{R(z)}\to 0,\quad \frac{P(\frac{z}{q})}{R(\frac{z}{q})}\to 0,$$

as $z \to \infty$. However, $\frac{m(z)}{n(z)} \to \infty$ as $z \to \infty$, thus, a contradiction.

If $p = r$, then

$$\frac{P(qz)}{R(qz)} \to \alpha, \quad \frac{P(z)}{R(z)} \to \alpha, \quad \frac{P(\frac{z}{q})}{R(\frac{z}{q})} \to \alpha,$$

as $z \to \infty$, where α is a non-zero constant. From (3.2), we also get a contradiction. Thus, we have $p > r$. Then, we can assume that $a_v \neq 0$ $(v \geq 1)$. As $z \to \infty$, it follows

$$f(z) = a_v z^v(1 + o(1)), \quad f(qz) = a_v q^v z^v(1 + o(1)),$$
$$f(\frac{z}{q}) = a_v q^{-v} z^v(1 + o(1)), \quad \frac{m(z)}{n(z)} = \beta z^{m-n}(1 + o(1)),$$

where $\beta(\neq 0)$ is a constant, and it follows now in view of (3.2) that

$$[q^v + 1 + q^{-v}]a_v^2 z^{2v} - k a_v z^v(1 + o(1)) = \beta z^{m-n}(1 + o(1)), \tag{3.3}$$

as $z \to \infty$. Since $|q| \neq 1$, we have $q^v + 1 + q^{-v} \neq 0$. Hence it follows from (3.3) that

$$p - r = v = \frac{m - n}{2}.$$

Now, we assume that $m = n$. As $z \to \infty$, it follows

$$\frac{m(z)}{n(z)} = \beta(1 + o(1)),$$

where $\beta(\neq 0)$ is a constant. If $p < r$, then by using the same argument as above, we get a contradiction. If $p > r$, then we can assume that $a_v \neq 0$ $(v \geq 1)$. By using the same argument as above, we conclude

$$[q^v + 1 + q^{-v}]a_v^2 z^{2v} - k a_v z^v(1 + o(1)) = \beta(1 + o(1)), \tag{3.4}$$

as $z \to \infty$, a contradiction. Therefore, it yields $p - r = 0 = \frac{m-n}{2}$.

(ii) Suppose that $m < n$. If $p > r$, then we can assume that $a_v \neq 0$ $(v \geq 1)$. Thus by using the same argument as above, we obtain $[q^v + 1 + q^{-v}]a_v^2 z^{2v}(1 + o(1)) = 0$, a contradiction.

If $p = r$, then we can assume that $a_0 \neq 0$ and $a_j = 0, j = 1, 2, \ldots, v$. Thus, we have

$$3a_0(1 + o(1)) = \frac{o(1)}{a_0(1 + o(1))} + c, \tag{3.5}$$

which implies $a_0 = \frac{c}{3}$. Hence, $f(z)$ can be represented as

$$f(z) = \eta + \frac{S(z)}{H(z)}, \tag{3.6}$$

where $\eta = \frac{c}{3}$, $S(z)$ and $H(z)$ are polynomials, and $\deg S(z) = s < \deg H(z) = h$. Substituting (3.6) into (1.7) yields

$$\begin{aligned}
&\eta\big(n(z)S(z)H(qz)H(z)H(\frac{z}{q}) + n(z)S(qz)H^2(z)H(\frac{z}{q}) + n(z)S(\frac{z}{q})H^2(z)H(qz)\big) + \\
&n(z)S(z)S(qz)H(\frac{z}{q})H(z) + n(z)S(z)S(\frac{z}{q})H(qz)H(z) + \\
&n(z)S(z)^2 H(qz)H(\frac{z}{q}) = m(z)H(z)^2 H(qz)H(\frac{z}{q}).
\end{aligned} \tag{3.7}$$

Then, we conclude

$$\begin{aligned}&\deg[n(z)S(z)H(qz)H(z)H(\frac{z}{q})+n(z)S(qz)H^2(z)H(\frac{z}{q})+n(z)S(\frac{z}{q})H^2(z)H(qz)]\\&\quad=3h+n+s,\\&\deg[n(z)S(z)S(qz)H(\frac{z}{q})H(z)+n(z)S(z)S(\frac{z}{q})H(qz)H(z)+n(z)S(z)^2H(qz)H(\frac{z}{q})]\\&\quad=2h+n+2s,\\&\deg[m(z)H(z)^2H(qz)H(\frac{z}{q})]=4h+m.\end{aligned}$$

Since $h>s$, we have $3h+n+s>2h+n+2s$ and $3h+n+s=4h+m$, that is, $n-m=h-s$.

If $p<r$, by $f(z)=\frac{P(z)}{R(z)}$ and (1.7), we deduce

$$\begin{aligned}&n(z)P(z)^2R(qz)R(\frac{z}{q})+n(z)P(z)P(qz)R(z)R(\frac{z}{q})+\\&\quad n(z)P(z)P(\frac{z}{q})R(z)R(qz)-cn(z)P(z)R(z)R(qz)R(\frac{z}{q})\\&\quad=m(z)R(z)^2R(qz)R(\frac{z}{q}).\end{aligned}$$

So, it follows

$$\begin{aligned}&\deg[n(z)P(z)^2R(qz)R(\frac{z}{q})+n(z)P(z)P(qz)R(z)R(\frac{z}{q})+n(z)P(z)P(\frac{z}{q})R(z)R(qz)]\\&\quad=n+2p+2r,\\&\deg[n(z)P(z)R(z)R(qz)R(\frac{z}{q})]=n+p+3r,\\&\deg[m(z)R(z)^2R(qz)R(\frac{z}{q})]=m+4r.\end{aligned}$$

Since $m<n$ and $p>r$, it yields $n+2p+2r<n+p+3r=m+4r$, that is, $1\le n-m=r-p$.

(iii) Suppose that $m>n$ and $m-n$ is an odd number. If (1.7) has a rational solution $f(z)=\frac{P(z)}{R(z)}$. By using the same argument as in (i), we also get $p-r=\frac{m-n}{2}$, a contradiction. Hence, the equation (1.7) has no rational solution.

Therefore, we complete the proof of Theorem 1.6. □

4. The Proof of Theorem 1.7

Suppose that $f(z)$ is a zero order transcendental meromorphic solution of (1.8).

(i) From (1.8), it yields

$$f(z)[f(qz)+f(z)+f(\frac{z}{q})]=C(z)f(z)+A(z). \tag{4.1}$$

In view of (4.1) and Lemma 2.1, it follows

$$m(r,f(qz)+f(z)+f(\frac{z}{q}))=S(r,f), \tag{4.2}$$

on a set F of logarithmic density 1. Since $f(z)$ is of zero-order, we can conclude from Lemma 2.2 that

$$N(r,f(qz)+f(z)+f(\frac{z}{q}))\le N(r,f(qz))+N(r,f)+N(r,f(\frac{z}{q}))$$

$$= 3(1 + o(1))N(r, f), \tag{4.3}$$

on a set of lower logarithmic density 1.

Since $A(z) \not\equiv 0$ and $A(z), C(z) \in \mathcal{S}(f)$, applying Lemma 2.4 for (1.8) gives

$$T(r, f(qz) + f(z) + f(\frac{z}{q})) = T(r, f) + S(r, f). \tag{4.4}$$

Thus, it follows from (4.2)–(4.4) that

$$T(r, f) \leq 3(1 + o(1))N(r, f) + S(r, f), \tag{4.5}$$

on a set F of logarithmic density 1. Thus, $f(z)$ has infinite many poles and

$$\lambda(\frac{1}{f}) \geq \sigma(f). \tag{4.6}$$

On the other hand, we can rewrite from (1.8) as

$$P_1(z, f(z)) = f(z)[f(qz) + f(z) + f(\frac{z}{q})] - C(z)f(z) - A(z) = 0. \tag{4.7}$$

Since $A(z) \not\equiv 0$, we have $P_1(z, 0) = -A(z) \not\equiv 0$. Thus, by Lemma 2.2, it follows $m(r, \frac{1}{f}) = S(r, f)$, on a set F of logarithmic density 1. Hence

$$N(r, \frac{1}{f}) = T(r, f) + S(r, f),$$

on a set F of logarithmic density 1. Therefore, $f(z)$ has infinite many zeros and $\lambda(f) = \sigma(f)$.

Next, we will prove that $\lambda(\frac{1}{\Delta_q f}) \geq \lambda(\frac{1}{f})$. Set $z = qw$, then we can rewrite (1.8) as the form

$$f(q^2w) + f(qw) + f(w) = \frac{A(qw)}{f(qw)} + C(qw). \tag{4.8}$$

Then it follows from (4.8) that

$$f(qw)[f(q^2w) + f(qw) + f(w)] = A(qw) + C(qw)f(qw). \tag{4.9}$$

Since $\Delta_q f(w) = f(qw) - f(w)$, it follows $f(qw) = \Delta_q f(w) + f(w)$ and $f(q^2w) = \Delta_q f(qw) + \Delta_q f(w) + f(w)$. Substituting them into (4.9) yields

$$[\Delta_q f(w) + f(w)][\Delta_q f(qw) + 2\Delta_q f(w) + 3f(w)] = A(qw) + C(qw)[\Delta_q f(w) + f(w)],$$

i.e.,

$$\begin{aligned} -3f(w)^2 =& [\Delta_q f(qw) + 5\Delta_q f(w) - C(qw)]f(w) - A(qw) + \\ & [\Delta_q f(qw) + 2\Delta_q f(w) - C(qw)]\Delta_q f(w). \end{aligned} \tag{4.10}$$

Since $f(z)$ is a zero order transcendental meromorphic function and $z = qw$, $A(z), C(z) \in \mathcal{S}(f)$, we can conclude that $f(w), \Delta_q f(w), \Delta_q f(qw)$ are of zero order, and $A(qw)$, $C(qw) \in \mathcal{S}(f)$. Set $\Delta_q^2 f(w) := \Delta_q(\Delta_q f(w))$, so it follows $\Delta_q f(qw) = \Delta_q^2 f(w) + \Delta_q f(w)$. Thus, by Lemma 2.3 it yields

$$N(r, \Delta_q^2 f(w)) \leq 2N(r, \Delta_q f(w)) + S(r, f), \tag{4.11}$$

on a set F of logarithmic density 1. Thus, from (4.11) we conclude

$$N(r, \Delta_q f(qw)) \leq 3N(r, \Delta_q f(w)) + S(r, f), \tag{4.12}$$

on a set F of logarithmic density 1. Thus, from (4.11) and (4.12), it follows

$$\begin{aligned}2N(r,f(w)) =&N(r,[\Delta_q f(qw)+5\Delta_q f(w)-C(qw)]f(w)-A(qw)+\\&[\Delta_q f(qw)+\Delta_q f(w)-C(qw)]\Delta_q f(w)\\ \le& N(r,f(w))+9N(r,\Delta_q f(w))+S(r,f),\end{aligned}$$

on a set F of logarithmic density 1, that is,

$$N(r,f(w))\le 9N(r,\Delta_q f(w))+S(r,f), \tag{4.13}$$

on a set F of logarithmic density 1. Then, it follows from (4.13) that $\Delta_q f(w)$ has infinite many poles and

$$\lambda(\frac{1}{\Delta_q f})\ge\lambda(\frac{1}{f}). \tag{4.14}$$

So, we can conclude from Lemma 2.3 that $T(r,\Delta_q f)\le 2T(r,f)+S(r,f)$ on a set F of logarithmic density 1, that is, $\sigma(f)\ge\sigma(\Delta_q f)$. Thus, combining this and (4.6) yields

$$\lambda(f)=\lambda(\frac{1}{f})=\lambda(\frac{1}{\Delta_q f}).$$

Here, we will prove that

$$\lambda(\frac{1}{\Delta_q^2 f})=\lambda(\frac{1}{\frac{\Delta_q f}{f}})=\lambda(\frac{1}{\frac{\Delta_q^2 f}{f}}).$$

At first, it can be seen that $A(z)+C(z)f(z)$ and $f(z)^2$ are mutually prime polynomials in $f(z)$, where $A(z),C(z)$ are nonzero small functions with respect to $f(z)$. In fact, we can take $u(z,f)=A(z)-C(z)f(z)$ and $v(z,f)=C(z)^2$, it follows $u(z,f)(A(z)+C(z)f(z))+v(z,f)f(z)^2=A(z)^2$. Thus, from (1.7) and by Lemma 2.4, we have

$$\begin{aligned}2T(r,f)&=T(r,\frac{A(z)+C(z)f(z)}{f(z)^2})+S(r,f)\\&=T(r,\frac{f(qz)+f(z)+f(\frac{z}{q})}{f(z)})+S(r,f)\\&\le 2T(r,\frac{f(qz)}{f(z)})+S(r,f)=2T(r,\frac{\Delta_q f}{f})+S(r,f),\end{aligned}$$

that is,

$$T(r,f)\le T(r,\frac{\Delta_q f}{f})+S(r,f).$$

Hence we can conclude from Lemma 2.5 that

$$N(r,\frac{\Delta_q f}{f})=T(r,\frac{\Delta_q f}{f})-m(r,\frac{\Delta_q f}{f})\ge T(r,f)+S(r,f), \tag{4.15}$$

and by combining (4.15), it follows

$$N(r,\frac{\Delta_q f}{f})\le T(r,\frac{\Delta_q f}{f})+S(r,f)\le 3T(r,f)+S(r,f),$$

which means that $\frac{\Delta_q f}{f}$ has infinitely many poles and $\lambda(\frac{1}{\frac{\Delta_q f}{f}})=\sigma(f)$.

Besides, in view of (1.8), it yields

$$\Delta_q^2 f(\frac{z}{q}) = \Delta_q f(z) - \Delta_q f(\frac{z}{q}) = \frac{-3f(z)^2 + C(z)f(z) + A(z)}{f(z)},$$

then by applying Lemmas 2.3 and 2.4 for the above equality, we conclude

$$\begin{aligned} 2T(r, f(z)) &= T(r, \frac{-3f(z)^2 + C(z)f(z) + A(z)}{f(z)}) + S(r,f) \\ &= T(r, \Delta^2 f(\frac{z}{q})) + S(r,f) = T(r, \Delta_q^2 f(z)) + S(r,f). \end{aligned} \tag{4.16}$$

Thus, by Lemma 2.3 and (4.16), it yields

$$\begin{aligned} 3T(r,f) + S(r,f) &\geq N(r, \Delta_q^2 f) = T(r, \Delta_q^2 f) - m(r, \Delta_q^2 f) \\ &\geq 2T(r,f) - T(r,f) + S(r,f) = T(r,f) + S(r,f), \end{aligned}$$

which implies that $\Delta_q^2 f$ has infinitely many poles and $\lambda(\frac{1}{\Delta_q^2 f}) = \sigma(f)$.

Finally, from the above argument, we can deduce

$$\begin{aligned} 3T(r,f) + S(r,f) &\geq N(r, \frac{\Delta_q^2 f}{f}) = T(r, \frac{\Delta_q^2 f}{f}) - m(r, \frac{\Delta_q^2 f}{f}) \\ &\geq T(r, \Delta_q^2 f) - T(r,f) + S(r,f) = T(r,f) + S(r,f), \end{aligned}$$

which implies that $\frac{\Delta_q^2 f}{f}$ has infinitely many poles and $\lambda(\frac{1}{\frac{\Delta_q^2 f}{f}}) = \sigma(f)$. Therefore, (i) is proved.

(ii) Set $g(z) = f(z) - z$. Then $g(z)$ is a zero-order transcendental meromorphic function with $\sigma(g) = \sigma(f)$ and $\tau(f) = \lambda(g)$. Substituting $f(z) = g(z) + z$ into (1.8), we can deduce

$$\begin{aligned} P_2(z, g(z)) =& (g(z) + z)[g(qz) + g(z) + g(\frac{z}{q})] + [(q + 1 + \frac{1}{q})z - C(z)]g(z) + \\ & (q + 1 + \frac{1}{q})z^2 - zC(z) - A(z) = 0. \end{aligned}$$

Since $P_2(z, 0) = (q + 1 + \frac{1}{q})z^2 - zC(z) - A(z) \not\equiv 0$, by Lemma 2.2, it follows $m(r, \frac{1}{g}) = S(r,f)$ on a set F of logarithmic density 1. By using the same argument as in the proof of Theorem 1.7 (i), we conclude

$$N(r, \frac{1}{g}) = T(r,f) + S(r,f)$$

on a set F of logarithmic density 1, which implies $\tau(f) = \lambda(g) = \sigma(f)$.

Therefore, we complete the proof of Theorem 1.7. □

Acknowledgements We thank the referees for their time and comments.

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