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Complete Classification of Flag-Transitive Point-Primitive 2-Designs with Socle M_{11}

Delu TIAN
Department of Mathematics, Guangdong University of Education, Guangdong 510303*, P. R. China*

Abstract In recent years several authors have determined some 2-designs with flag-transitive point-primitive automorphism group G of almost simple type with sporadic socle. Here we obtain the complete classification of flag-transitive point-primitive non-trivial 2-designs with socle M_{11}, and find that there are precisely 14 such nonisomorphic 2-designs.

Keywords 2-design; flag-transitive; point-primitive; Mathieu group

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1. Introduction

For positive integers $2 \le k \le v$ and λ, we define a 2-(v, k, λ) design to be a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ denotes a set of points, $|\mathcal{P}| = v$, and $\mathcal{B}$ a set of blocks consisting of k-subsets of $\mathcal{P}$, $|\mathcal{B}| = b$, with the properties that each element of $\mathcal{P}$ is incident with r blocks, and each 2-subset of $\mathcal{P}$ is incident with λ blocks. A flag in a design is an incident point-block pair. If $2 < k < v - 2$ holds, then we speak of a non-trivial 2-design [1]. If $b = v$, the design $\mathcal{D}$ is called a symmetric design, otherwise called a non-symmetric design. The complement of a design with parameters (v, k, λ, b, r) is a design with parameters $(v, v - k, b - 2r + \lambda, b, b - r)$.

An automorphism g of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which induces a permutation on the blocks. The full automorphism group of $\mathcal{D}$ consists of all automorphisms of $\mathcal{D}$ and is denoted by $\mathrm{Aut}(\mathcal{D})$. Any subgroup of $\mathrm{Aut}(\mathcal{D})$ is called an automorphism group of $\mathcal{D}$. For $G \le \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is called point-primitive (or block-primitive) if G is primitive on $\mathcal{P}$ (or $\mathcal{B}$), and flag-transitive if G is transitive on the set of flags.

From [2], we know that if a non-trivial 2-(v, k, λ) symmetric design $\mathcal{D}$ with $\lambda \le 100$ admits a flag-transitive point-primitive automorphism group G, then G must be an affine or almost simple type. Furthermore, the authors conjectured that the result holds for general λ. In [3, 4], Liang and Zhou considered the non-symmetric cases and proved that if $\mathcal{D}$ is a 2-(v, k, λ) ($\lambda = 2$ or 3) non-symmetric design admitting a flag-transitive point-primitive automorphism group G, then G is an affine or almost simple group. Hence it is interesting to consider the case when the automorphism group G is of almost simple type.

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E-mail address: tiandelu@gdei.edu.cn

In 1999, Camina et al. [5] proved that if G acts line-transitively on a 2-$(v,k,1)$ design, then the socle of G is not a sporadic simple group. In 2015, Tian and Zhou [6] completely classified all flag-transitive point-primitive 2-(v,k,λ) symmetric designs with sporadic socle. In [7], Zhan and Zhou considered the non-symmetric cases and classified the non-symmetric 2-designs with $\lambda \geq (r,\lambda)^2$ admitting a flag-transitive automorphism group of almost simple type with sporadic socle. In this paper, we study the non-symmetric 2-designs with $\mathrm{Soc}(G) = M_{11}$ for general λ, and obtain the complete classification of flag-transitive point-primitive 2-designs with socle M_{11}.

Our main result is the following.

Theorem 1.1 *Let $\mathcal{D}$ be a non-trivial 2-(v,k,λ) design with a flag-transitive, point-primitive automorphism group G of almost simple type. If $\mathrm{Soc}(G) = M_{11}$, then one of the following holds:*

(i) $\mathcal{D}$ has one of the following parameters:

$$(11,3,9),(11,4,36),(11,5,12),(11,5,72),(11,6,18),(11,8,84),$$

and the point stabilizer $G_x = M_{10}$;

(ii) $\mathcal{D}$ has one of the following parameters:

$$(12,3,10),(12,4,15),(12,4,30),(12,5,20),(12,6,5),(12,6,25),(12,8,70),$$

and the point stabilizer $G_x = L_2(11)$;

(iii) $\mathcal{D}$ has parameters $(55,4,8)$, and the point stabilizer $G_x = M_9 : 2$.

Remark 1.2 (i) There are 14 different 2-designs up to isomorphism.

(ii) There are 5 pairs 2-designs which are mutually complement designs, the parameters are listed as follows:

$(11,5,12)$ and $(11,6,18)$; $(11,3,9)$ and $(11,8,84)$; $(12,6,5)$ and itself;

$(12,6,25)$ and itself; $(12,4,15)$ and $(12,8,70)$.

(iii) 2-designs are full designs, in which the block set B contains all the k-subsets of $\mathcal{P}$: $(11,3,9)$, $(11,4,36)$, $(11,8,84)$ and $(12,3,10)$.

2. Some preliminary results

In this section we give some preliminary results which are used throughout the paper.

The following lemmas are well-known.

Lemma 2.1 *Let $\mathcal{D}$ be a 2-(v,k,λ) design. Then the following hold:*

(i) $bk = vr$;

(ii) $r = \frac{\lambda(v-1)}{k-1}$; $b = \frac{\lambda v(v-1)}{k(k-1)}$;

(iii) $b \geq v$ (Fisher's inequality).

Lemma 2.2 *Let $\mathcal{D} = (\mathcal{P},\mathcal{B})$ be a non-trivial 2-(v,k,λ) design and G be an automorphism group of $\mathcal{D}$. For any point $x \in \mathcal{P}$ and block $B \in \mathcal{B}$, the following three statements are equivalent:*

(i) G acts flag-transitively on $\mathcal{D}$;

(ii) G acts point-transitively on $\mathcal{D}$ and G_x acts transitively on $B(x)$, where $B(x)$ denotes the set of all blocks which are incident with a point x;

(iii) G acts block-transitively on $\mathcal{D}$ and G_B acts transitively on the points of B.

Combining [6, Theorem 1] and [7, Theorem 1], we obtain the following lemma:

Lemma 2.3 *Let $\mathcal{D}$ be a non-trivial 2-(v,k,λ) design with a flag-transitive, point-primitive automorphism group G of almost simple type and with $\mathrm{Soc}(G) = M_{11}$. Then*

(i) There is no such symmetric 2-(v,k,λ) design;

(ii) If $\mathcal{D}$ is a non-symmetric 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$, then $\mathcal{D}$ has the parameters $(12,6,5)$ or $(12,6,25)$.

3. Proof of Theorem 1.1

We prove Theorem 1.1 in the following three subsections.

By Lemma 2.3, we only need to consider the case when $b \neq v$ and $1 < \lambda < (r,\lambda)^2$.

3.1. Potential parameters of 2-designs

In this subsection we describe briefly our approach to search for potential parameters of 2-(v,k,λ) designs.

Suppose that there is a non-trivial 2-design $\mathcal{D}$ admitting a flag-transitive, point-primitive almost simple automorphism group G with socle M_{11}. Then $G = M_{11}$, since $\mathrm{Aut}(M_{11}) = M_{11}$. It is well-known that G is point-primitive on $\mathcal{P}$ if and only if the stabilizer G_x is a maximal subgroup of G, where $x \in \mathcal{P}$. Thus $v = |G : G_x|$. The list of the maximal subgroups of G is given in the ATLAS [8].

No.	G_x	v	No.	G_x	v
1	M_{10}	11	2	$L_2(11)$	12
3	$M_9:2$	55	4	S_5	66
5	$2S_4$	165			

Table 1 The stabilizer G_x and the parameter v

We compute all possible parameters (v,k,λ,b,r) satisfying the following conditions:

(i) $G = M_{11}$ and G_x is one of the groups which are listed in Table 1;

(ii) $v \in \{11, 12, 55, 66, 165\}$;

(iii) $2 < k < v-2$, $1 < \lambda < (r,\lambda)^2$;

(iv) $r \big| |G_x|$, $r > \lambda$ and $r^2 > \lambda v$ (see [7, Lemma 2.4]);

(v) $bk = vr$, $r = \frac{\lambda(v-1)}{k-1}$, $b = \frac{\lambda v(v-1)}{k(k-1)}$;

(vi) $b > v$, and $b \big| |G|$ (see [Lemma 2.2]).

By using the computer algebra system GAP [9], we get 106 5-tuples of parameters (v,k,λ,b,r). By Lemma 2.2, G acts block-transitively on $\mathcal{D}$. Thus G has a subgroup G_B with index $b = |G : G_B|$. Using the Magma-command Subgroups (G: OrderEqual:=n) (see [10]) where $n = |G|/b$, it is easily checked that whether there exists at least one subgroup with index b in G. Then 27 parameters can be ruled out. By the fact that G_B acts transitively on the points of B, there exists at least one orbit O of G_B with size k and $|O^G| = b$. There are 62 parameters (v, k, λ, b, r) that do not satisfy this fact.

The remaining 17 parameters (v, k, λ, b, r) are listed in Table 2. Note that in the 4-th column, "Lengths" means the orbit lengths of subgroup with index b, and the notation s^t means that the degree s appears with multiplicity t. In the following, "Case i" denotes the i-th line of Table 2.

Case	G_x	(v, k, λ, b, r)	Lengths	Reference
1	M_{10}	(11, 3, 9, 165, 45)	3, 8	$\mathcal{D}_1$
2		(11, 4, 36, 330, 120)	1, 4, 6	$\mathcal{D}_2$
3		(11, 5, 12, 66, 30)	5, 6	$\mathcal{D}_3$
4		(11, 5, 72, 396, 180)	1, 5, 5	$\mathcal{D}_4$
5		(11, 6, 18, 66, 36)	5, 6	$\mathcal{D}_5$
6		(11, 8, 84, 165, 120)	3, 8	$\mathcal{D}_6$
7	$L_2(11)$	(12, 3, 10, 220, 55)	3^2, 6	$\mathcal{D}_7$
8		(12, 4, 15, 165, 55)	4, 8	$\mathcal{D}_8$
9		(12, 4, 30, 330, 110)	2, 4, 6	$\mathcal{D}_9$
10		(12, 5, 20, 132, 55)	1, 5, 6	$\mathcal{D}_{10}$
11		(12, 8, 70, 165, 110)	4, 8	$\mathcal{D}_{11}$
12	$M_9 : 2$	(55, 4, 8, 1980, 144)		$\mathcal{D}_{12}$
13		(55, 4, 4, 990, 72)		3.2.2
14		(55, 10, 2, 66, 12)	10, 15, 30	3.2.1
15		(55, 10, 12, 396, 72)	5^3, 10^2, 20	3.2.1
16		(55, 10, 24, 792, 144)	5^7, 10^2	3.2.1
17		(55, 16, 40, 495, 144)	1, 2, 4, 8^2, 16^2	3.2.1

Table 2 Potential 2-designs

3.2. Rule out 5 potential parameters

Now, we will rule out 5 potential Cases listed in Table 2.

3.2.1. Rule out Cases 14–17

For each case, there exists at least one orbit O of G_B with size k and $|O^G| = b$. But each 2-subset of $\mathcal{P}$ is not incident with λ blocks.

For example, in Case 17, there exist two orbits O_1 and O_2 of G_B with size 16 and $|O_i^G| = 495$ $(i = 1, 2)$. For $1 \leq j < k \leq 55$, 2-subset $\{j, k\}$ is incident with 28 or 64 blocks in O_1^G, while 36 or 48 blocks in O_2^G.

Similarly, we can rule out Cases 14–16.

The number of blocks which contain arbitrary 2-subset $\{j, k\}$ can be listed as follows:

Cases 14. One orbit O with size 10, the number is 1 or 4;

Cases 15. Two orbits O_1 and O_2 with size 10, for each orbit, the number is 6 or 24;

Cases 16. Two orbits O_1 and O_2 with size 10, for each orbit, the number is 16 or 28.

3.2.2. Rule out Cases 13

For Cases 13, G contains three conjugacy class of subgroups with index 990, denoted by H, K and L as representatives.

The orbit lengths of H are $1^3, 4, 8^6$, but $|O_1^G| = 165 \neq 990$, where O_1 is the orbit with size 4. The orbit lengths of K are $1, 2, 4, 8^6$, but $|O_2^G| = 165 \neq 990$, where O_2 is the orbit with size 4. The orbit lengths of L are $1, 2^3, 4^4, 8^4$, and the orbits with size 4 are denoted by O_3, O_4,O_5 and O_6. It is not hard to check that $|O_i^G| = 330 \neq 990$ $(i = 3, 4)$, $|O_t^G| = 990$ $(t = 5, 6)$. For $1 \leq j < k \leq 55$, 2-subset $\{j, k\}$ is incident with 2 or 8 blocks in O_t^G $(t = 5, 6)$, a contradiction.

3.3. Twelve 2-designs

3.3.1. Designs with 11 points

For Case 1, we can get the primitive permutation representations of the Mathieu group $G = M_{11}$ acting on the set $\Omega = \{1, 2, \ldots, 11\}$ by using Magma command PrimitiveGroup (11,6). G contains only one conjugacy classes of the subgroups with index 165, denoted by H. We list the generators of H:

$$g_1 = (1, 4)(3, 10)(6, 11)(7, 9), \quad g_2 = (1, 10)(2, 5)(3, 4)(6, 11),$$
$$g_3 = (1, 3, 10, 4)(2, 6, 5, 11), \quad g_4 = (1, 11, 3)(4, 10, 6)(7, 9, 8),$$
$$g_5 = (1, 5, 10, 2)(3, 6, 4, 11).$$

The command Orbits (H) returns the orbits of H acting on Ω. There are 2 orbits with lengths 3 and 8, respectively.

$$O_1 = \{7, 8, 9\}, \quad O_2 = \{1, 2, 3, 4, 5, 6, 10, 11\}.$$

It follows that $|O_1^G| = 165$ and each 2-subset $\{j, k\}$ is incident with 9 blocks in O_1^G (for any $1 \leq j < k \leq 11$). Thus this is really a non-trivial flag-transitive point-primitive 2-design, denoted by $\mathcal{D}_1$. The basic block of $\mathcal{D}_1$ is O_1.

The analysis of Case 2 to Case 6 is the same as Case 1.

3.3.2. Designs with 12 points

For Case 7, the group is PrimitiveGroup (12,3), and G contains three conjugacy classes of subgroups with index $b = 220$. Only one conjugacy class satisfies the fact that there exists at least one orbit O of G_B with size 3 and $|O^G| = 220$, denoted by H as a representative. The orbits of H acting on $\Omega = \{1, 2, \ldots, 12\}$ are:

$$O_1 = \{2, 6, 8\}, \quad O_2 = \{9, 10, 11\}, \quad O_3 = \{1, 3, 4, 5, 7, 12\}.$$

Thus the lengths of the orbits of H are 3^2, 6. In fact, $O_1^G = O_2^G$, $|O_1^G| = |O_2^G| = 220$ and each 2-subset $\{j,k\}$ is incident with 10 blocks in O_i^G (for any $1 \le j < k \le 12$ and $i = 1,2$). Thus there are two same non-trivial flag-transitive point-primitive 2-designs, denoted by D_7. The basic block of $\mathcal{D}_7$ is O_1 or O_2.

The analysis of Case 8 to Case 11 is the same as Case 7.

3.3.3. Design with 55 points

For Case 12, the group is PrimitiveGroup (55,4), and G contains two conjugacy classes of subgroups with index $b = 1980$, denoted by H and K as representatives.

We list the generators of H:

$$\begin{aligned}g_1 =&(1,23)(2,29)(3,54)(4,10)(5,43)(6,51)(7,45)(8,49)(9,22)(11,16)\\&(12,15)(13,46)(14,42)(17,20)(18,41)(19,50)(25,47)(26,37)(30,38)\\&(32,52)(33,55)(35,36)(39,44)(40,48),\\g_2 =&(1,26,23,37)(2,44,29,39)(3,50,54,19)(4,48,10,40)(5,30,43,38)\\&(6,7,51,45)(8,16,49,11)(9,52,22,32)(12,35,15,36)(13,14,46,42)\\&(17,33,20,55)(18,25,41,47)(21,28)(27,53).\end{aligned}$$

The orbits of H acting on $\Omega = \{1,2,\ldots,55\}$ are:

$$\begin{aligned}&O_1 = \{24\},\quad O_2 = \{31\},\quad O_3 = \{34\},\quad O_4 = \{21,28\},\quad O_5 = \{27,53\},\\&O_6 = \{1,23,26,37\},\quad O_7 = \{2,29,39,44\},\quad O_8 = \{3,19,50,54\},\\&O_9 = \{4,10,40,48\},\quad O_{10} = \{5,30,38,43\},\quad O_{11} = \{6,7,45,51\},\\&O_{12} = \{8,11,16,49\},\quad O_{13} = \{9,22,32,52\},\quad O_{14} = \{12,15,35,36\},\\&O_{15} = \{13,14,42,46\},\quad O_{16} = \{17,20,33,55\},\quad O_{17} = \{18,25,41,47\}.\end{aligned}$$

After calculation, $|O_8^G| = |O_{12}^G| = 330 \neq 1980$, $|O_{13}^G| = |O_{15}^G| = 990 \neq 1980$. Although $|O_i^G| = 1980$ ($i = 6,7,9,10,11,14,16,17$), 2-subset $\{j,k\}$ ($1 \le j < k \le 55$) is incident with 0 or 24 blocks in O_i^G ($i = 6,7,14,17$), while 0 or 12 blocks in O_i^G ($i = 9,10,11,16$). Thus there is no design that meets the requirements with $G_B = H$.

Now, we consider the generators of K:

$$\begin{aligned}g_1 =&(1,4)(2,13)(3,34)(5,51)(6,40)(7,35)(9,27)(10,15)(12,30)(14,29)\\&(16,49)(17,48)(18,44)(19,50)(20,43)(21,31)(22,53)(23,38)(24,54)\\&(25,42)(26,45)(33,37)(36,55)(46,47),\\g_2 =&(1,25)(2,30)(4,42)(5,9)(7,15)(8,11)(10,35)(12,13)(14,36)(16,19)\\&(17,22)(18,31)(20,43)(21,44)(23,37)(24,54)(26,47)(27,51)(28,32)\\&(29,55)(33,38)(45,46)(48,53)(49,50).\end{aligned}$$

The orbits of K acting on $\Omega = \{1,2,\ldots,55\}$ are:

$$O_1 = \{39\},\ O_2 = \{41\},\ O_3 = \{52\},\ O_4 = \{3,34\},\ O_5 = \{6,40\},\ O_6 = \{8,11\},$$

$$O_7 = \{20, 43\},\ O_8 = \{24, 54\},\ O_9 = \{28, 32\},\ O_{10} = \{1, 4, 25, 42\},$$
$$O_{11} = \{2, 12, 13, 30\},\ O_{12} = \{5, 9, 27, 51\},\ O_{13} = \{7, 10, 15, 35\},$$
$$O_{14} = \{14, 29, 36, 55\},\ O_{15}\{16, 19, 49, 50\},\ O_{16} = \{17, 22, 48, 53\},$$
$$O_{17} = \{18, 21, 31, 44\},\ O_{18} = \{23, 33, 37, 38\},\ O_{19} = \{26, 45, 46, 47\}.$$

After calculation, $|O_{15}^G| = 330 \neq 1980$, $|O_i^G| = 990 \neq 1980$ $(i = 13, 17, 18)$. For $i = 10, 11, 12, 14, 16, 19$, the sets O_i^G are the same, denoted by S. It is easy to calculate that $|S| = 1980$, and 2-subset $\{j, k\}$ $(1 \leq j < k \leq 55)$ is incident with 8 blocks in S. So there is a non-trivial flag-transitive point-primitive 2-designs, denoted by D_{12}. The basic block of $\mathcal{D}_{12}$ is O_{10}.

This completes the proof of Theorem 1.1. □

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