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Costar Subcategories and Cotilting Subcategories with Respect to Cotorsion Triples

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Abstract Let $\mathcal{A}$ be an abelian category, and $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ be a complete hereditary cotorsion triple. We introduce the definition of n-$\mathcal{Y}$-cotilting subcategories of $\mathcal{A}$, and give a characterization of n-$\mathcal{Y}$-cotilting subcategories, which is similar to Bazzoni characterization of n-cotilting modules. As an application, we prove that if $\mathcal{GP}$ is n-$\mathcal{GI}$-cotilting over a virtually Gorenstein ring R, then R is an n-Gorenstein ring, where $\mathcal{GP}$ denotes the subcategory of Gorenstein projective R-modules and $\mathcal{GI}$ denotes the subcategory of Gorenstein injective R-modules. Furthermore, we investigate n-costar subcategories over arbitrary ring R, and the relationship between n-$\mathcal{I}$-cotilting subcategories with respect to cotorsion triple $(\mathcal{P}, R\text{-Mod}, \mathcal{I})$ and n-costar subcategories, where $\mathcal{P}$ denotes the subcategory of projective left R-modules and $\mathcal{I}$ denotes the subcategory of injective left R-modules.

Keywords cotorsion triple; n-$\mathcal{Y}$-cotilting subcategories; self-orthogonal-$\mathcal{Y}$; n-quasi-injective; n-costar subcategories

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1. Introduction

Tilting theory plays an important role in the representation of Artin algebra. The classical tilting modules were first considerd in the early eighties by Brenner-Bulter [1], Bongartz [2] and Happle and Ringel [3] etc. Begining with Miyashita [4], tilting modules over arbitrary rings were investigated by many authors [5–8]. In 1999, Colpi [9] gave the definition of tilting objects in any Grothedieck category and proved some basic facts of tilting theory in it. In 2007, Colpi and Fuller [10] investigated tilting objects in arbitrary abelian category. Recently, Di et al [11] introduced the notion of n-$\mathcal{X}$-tilting subcategories with respect to a complete hereditary cotorsion triple $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ in abelian category $\mathcal{A}$, and proved that a virtually Gorenstein ring R was n-Gorenstein if and only if $\mathcal{GI}$ is n-$\mathcal{GP}$-tilting, where $\mathcal{GP}$ denotes the subcategory of Gorenstein projective R-modules and $\mathcal{GI}$ denotes the subcategory of Gorenstein injective R-modules. Wei [12] studied n-star modules, and proved that n-tilting modules are n-star modules n-presenting all injectives. Cotilting modules are also important part of tilting theory. In this paper, we give the definition of n-$\mathcal{Y}$-cotilting subcategories with respect to a complete hereditary

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cotorsion triple $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ in abelian category $\mathcal{A}$, and investigate properties and characterizations of self-orthogonal-$\mathcal{Y}$ and n-$\mathcal{Y}$-cotilting subcategories. As an application, we obtain a sufficient condition for R to be n-Gorenstein ring over a virtually Gorenstein ring. Furthermore, we give a characterization of n-$\mathcal{Y}$-cotilting subcategories, which is similar to Bazzoni characterization of n-cotilting modules. Then we introduce n-costar subcategories over an arbitrary ring R, and we obtain that $\mathcal{M}$ is an n-$\mathcal{I}$-cotilting subcategory with respect to cotorsion triple ($\mathcal{P}$, R-Mod, $\mathcal{I}$), if and only if $\mathcal{M}$ is an n-costar subcategory with $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M})$, where $\mathcal{P}$ denotes the subcategory of projective left R-modules and $\mathcal{I}$ denotes the subcategory of injective left R-modules.

We now state the main results of this paper.

Theorem 1.1 *Let R be a virtually Gorenstein ring. If $\mathcal{GP}$ is an n-$\mathcal{GI}$-cotilting subcategory, then R is an n-Gorenstein ring. Moreover, $\mathcal{GI}$ is an n-$\mathcal{GP}$-tilting subcategory.*

Theorem 1.2 *Let $\mathcal{N}$ be a subcategory of $\mathcal{A}$ which is closed under summands. If every object in ${}^{\mathcal{Y}\perp}\mathcal{N}$ admits a left $\mathcal{N}$-approximation, then $\mathcal{N}$ is n-$\mathcal{Y}$-cotilting (with respect to $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ if and only if* $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N}) = {}^{\mathcal{Y}\perp}\mathcal{N}$.

Theorem 1.3 *Let n be a non-negative integer and $\mathcal{M}$ be a subcategory of R-Mod closed under direct summands and direct products. $\mathcal{M}$ has a class of representatives. Then the following conditions are equivalent.*

(1) $\mathcal{M}$ is an n-$\mathcal{I}$-cotilting subcategory with respect to cotorsion triple ($\mathcal{P}$, R-Mod, $\mathcal{I}$);

(2) $\mathrm{Copres}^n(\mathcal{M}) = {}^{\perp_{1 \leqslant i \leqslant n}} \mathcal{M}$;

(3) $\mathcal{M}$ is an n-costar subcategory with $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M})$;

(4) $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M}) \subseteq {}^{\perp} \mathcal{M}$.

The contents of this paper are summarized as follows. In Section 2, we collect some known notions and results. In Section 3, we introduce self-orthogonal-$\mathcal{Y}$ subcategories of $\mathcal{A}$ and discuss properties of them. In Section 4, we investigate n-$\mathcal{Y}$-cotilting subcategories with respect to a complete hereditary cotorsion tirple $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ in abelian category. Section 5 is devoted to n-costar subcategories and n-$\mathcal{I}$-cotilting subcategory with respect to cotorsion triple ($\mathcal{P}$, R-Mod, $\mathcal{I}$).

2. Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category with enough projective objects and injective objects. Subcategories are all full additive subcategory of $\mathcal{A}$ closed under isomorphisms. $\mathcal{P}$ (respectively, $\mathcal{I}$) is the subcategory of projectives (respectively, injectives). We denote $\mathcal{X}^{\perp} = \{Y \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0 \text{ for any } X \in \mathcal{X}\}$, ${}^{\perp}\mathcal{Y} = \{X \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0 \text{ for any } Y \in \mathcal{Y}\}$. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{A}$ is said to be a cotorsion pair if $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$ (see [13]). Obviously, $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$ are cotorsion pairs. The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is complete if for any $N \in \mathcal{A}$ there are short exact sequences $0 \to Y \to X \to N \to 0$ and $0 \to N \to Y^{\to} X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$. A subcategory $\mathcal{Y}$ is coresolving if it contains all injective objects, and for any short exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{A}$ with $Y' \in \mathcal{Y}$, we have $Y \in \mathcal{Y}$ if and only if $Y'' \in \mathcal{Y}$. And dually the notion of resolving subcategory is defined. The cotorsion pair $(\mathcal{X},\mathcal{Y})$ is hereditary if $\mathcal{Y}$ is coresolving, i.e., $\mathcal{X}$ is resolving (more details see [14]).

Let $\mathcal{X},\mathcal{Y},\mathcal{Z}$ be subcategory of $\mathcal{A}$. Following [15], the triple $(\mathcal{X},\mathcal{Z},\mathcal{Y})$ is called a cotorsion triple provided that both $(\mathcal{X},\mathcal{Z})$ and $(\mathcal{Z},\mathcal{Y})$ are cotorsion pair. Moreover, if both $(\mathcal{X},\mathcal{Z})$ and $(\mathcal{Z},\mathcal{Y})$ are complete (hereditary) cotorsion pair, we say $(\mathcal{X},\mathcal{Z},\mathcal{Y})$ is a complete (hereditary) cotorsion triple. $(\mathcal{P}, R\text{-Mod},\mathcal{I})$ is a complete hereditary cotorsion triple over a ring R. If R is a virtually Gorenstein ring, then $(\mathcal{GP},\mathcal{GP}^{\perp} ={}^{\perp}\mathcal{GI},\mathcal{GI})$ is also a complete hereditary cotorsion triple, where $\mathcal{GP}$ denotes the subcategory of Gorenstein projective R-modules and $\mathcal{GI}$ denotes the subcategory of Gorenstein injective R-modules [11].

Following [16], a complex $\mathbf{Y} = \cdots \to Y^{-2} \to Y^{-1} \to Y^0 \to Y^1 \to Y^2 \to \cdots$ is called a $\mathcal{Y}$-coresolution of N if $Y^i \in \mathcal{Y}$ for $i \geqslant 0$, $Y^i = 0$ for all $i < 0$, $H_i(\mathbf{Y}) = 0$ for $i > 0$, and $H_0(\mathbf{Y}) \cong N$. The exact sequence ${}^+\mathbf{Y} = 0 \to Y^0 \to Y^1 \to Y^2 \to \cdots$ is the argumented $\mathcal{Y}$-coresolution of N. If ${}^+\mathbf{Y}$ is $\mathrm{Hom}_{\mathbf{A}}(-,\mathcal{Y})$-exact, $\mathbf{Y}$ is called proper $\mathcal{Y}$-coresolution. We denote $\mathcal{Y}$-id N=inf$\{\sup\{n \geqslant 0 \mid Y^n \neq 0\} | \mathbf{Y}$ is $\mathcal{Y}$-coresolution of $N\}$. If N admits a proper $\mathcal{Y}$-coresolution, then such a proper coresolution is unique up to homotopy equivalence. Hence, it derived the relative $\mathcal{AY}$ cohomology group $\mathrm{Ext}^k_{\mathcal{AY}}(M,N) = H_k(M,\mathbf{Y})$ for every $k \in \mathbb{Z}$ and every object $M \in \mathbf{A}$. Dually, we can define $\mathcal{X}$-resolution, proper $\mathcal{X}$-resolution, $\mathcal{X}$-pdN and derived relative $\mathcal{XA}$ cohomology group $\mathrm{Ext}^k_{\mathcal{XA}}(M,N) = H_k(\mathbf{X},N)$ for every $k \in \mathbb{Z}$ and every object $N \in \mathcal{A}$. Obviously, every object in $\mathcal{A}$ admits a proper $\mathcal{Y}$-coresolution and a proper $\mathcal{X}$-resolution provided that $(\mathcal{X},\mathcal{Y})$ is a complete hereditary cotorsion pair.

Some results are spread out as follows.

Lemma 2.1 ([16, Lemma 4.3,4.4]) *Assume that the short exact sequence* $\mathbf{L} = 0 \to L' \to L \to L'' \to 0$ *is* $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$*-exact and* $N \in \mathcal{A}$.

(1) If N admits a proper $\mathcal{Y}$-coresolution, then $\mathbf{L}$ *induces a long exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(L'',N) \to \mathrm{Hom}_{\mathcal{A}}(L,N) \to \mathrm{Hom}_{\mathcal{A}}(L',N) \to \mathrm{Ext}^1_{\mathcal{AY}}(L'',N) \to \cdots$$
$$\to \mathrm{Ext}^k_{\mathcal{AY}}(L'',N) \to \mathrm{Ext}^k_{\mathcal{AY}}(L,N) \to \mathrm{Ext}^k_{\mathcal{AY}}(L',N) \to \mathrm{Ext}^{k+1}_{\mathcal{AY}}(L'',N) \to \cdots$$

(2) If both L' and L'' admit proper $\mathcal{Y}$-coresolution, then $\mathbf{L}$ *induces a long exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(N,L') \to \mathrm{Hom}_{\mathcal{A}}(N,L) \to \mathrm{Hom}_{\mathcal{A}}(N,L'') \to \mathrm{Ext}^1_{\mathcal{AY}}(N,L') \to \cdots$$
$$\to \mathrm{Ext}^k_{\mathcal{AY}}(N,L') \to \mathrm{Ext}^k_{\mathcal{AY}}(N,L) \to \mathrm{Ext}^k_{\mathcal{AY}}(N,L'') \to \mathrm{Ext}^{k+1}_{\mathcal{AY}}(N,L') \to \cdots.$$

Moreover, if $\mathrm{Ext}^{\geqslant 1}_{\mathcal{AY}}(N,L) = 0$, *then* $\mathrm{Ext}^k_{\mathcal{AY}}(N,L'') \cong \mathrm{Ext}^{k+1}_{\mathcal{AY}}(N,L')$ *for any* $k \geqslant 1$.

Lemma 2.2 ([17, Lemma 4.3,4.4]) *Assume that*

$$\begin{array}{ccc} M & \xrightarrow{f_1} & N \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

Diagram 1 The diagram such that $gf_1 = fg_1$

is a commutative diagram in $\mathcal{A}$ and $D \in \mathcal{A}$. Then the followings hold

(1) If this diagram is a pullback of f and g, and $\mathrm{Hom}_{\mathcal{A}}(D, g)$ *is epic, then* $\mathrm{Hom}_{\mathcal{A}}(D, g_1)$ *is also epic;*

(2) If this diagram is a pushout of f_1 and g_1, and $\mathrm{Hom}_{\mathcal{A}}(g_1, D)$ *is epic, then* $\mathrm{Hom}_{\mathcal{A}}(g, D)$ *is also epic.*

Proposition 2.3 *Let $N \in \mathcal{A}$ and $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair. Then for any non-negative integer n, the following conditions are equivalent*

(1) $\mathcal{Y}$-$id(N) \leqslant n$;

(2) $\mathrm{Ext}^{n+k}_{\mathcal{AY}}(-, N) = 0$ *for all $k \geqslant 1$;*

(3) $\mathrm{Ext}^{n+1}_{\mathcal{AY}}(-, N) = 0$.

Proof (1)$\Rightarrow$(2). Just to prove that there exists a proper $\mathcal{Y}$-coresolution $\mathbf{Y}$, such that $Y^i = 0$ for $i > n$. From (1) we get an exact sequence $0 \to N \to W^0 \to W^1 \to \cdots \to W^n \to 0$ in $\mathcal{A}$ with each $W^i \in \mathcal{Y}$. Since $(\mathcal{X}, \mathcal{Y})$ is complete, we get a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{Y})$-exact sequence $0 \to N \to Y^0 \to Y^1 \to \cdots \to Y^{n-1} \to C^n \to 0(*)$ in $\mathcal{A}$ with each $Y^i \in \mathcal{Y}$. Consider the following commutative diagram (Diagram 2)

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & C^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_N} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & W^0 & \longrightarrow & W^1 & \longrightarrow & \cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^n & \longrightarrow & 0 \end{array}$$

Diagram 2 The induced diagram of proper $\mathcal{Y}$-coresolution $\mathbf{Y}$

Consequently, the mapping cone $0 \to N \to Y^0 \oplus N \to Y^1 \oplus W^0 \to \cdots \to C^n \oplus W^{n-1} \to W^n \to 0$ is exact. Since $N \to Y^0 \oplus N$ is split, the sequence $0 \to Y^0 \to Y^1 \oplus W^0 \to \cdots \to C^n \oplus W^{n-1} \to W^n \to 0$ is exact. Note that $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\mathcal{Y}$ is closed under direct summand, then $C^n \in \mathcal{Y}$, which means $(*)$ is a proper $\mathcal{Y}$-coresolution.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1). Let $\mathbf{Y}$ be a proper $\mathcal{Y}$-coresolution of N and $N^i = \mathrm{Ker}(Y^i \to Y^{i+1})$ for $i \geqslant 1$. Consider the $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{Y})$-exact sequences

$$0 \to N^i \to Y^i \to N^{i+1} \to 0 \qquad (*_i)$$

for $i \geqslant 1$. The case $n = 0$, $\mathrm{Ext}^1_{\mathcal{AY}}(-, N) = 0$, then $(*_i)$ is also exact under $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{N})$. Note that $N_0 \cong N$, so $(*_i)$ is split, $N \in \mathcal{Y}$ because $\mathcal{Y}$ is closed under direct summands. Now suppose $n \geqslant 1$ and $\mathrm{Ext}^{n+1}_{\mathcal{AY}}(-, N) = 0$. Following Lemma 2.1 (2), we can conclude that $\mathrm{Ext}^1_{\mathcal{AY}}(-, N^n) \cong \mathrm{Ext}^{n+1}_{\mathcal{AY}}(-, N) = 0$, then $N^n \in \mathcal{Y}$ by the case $n = 0$. Therefore, $\mathcal{Y}$-$id(N) \leqslant n$. □

The following lemmas are from [11].

Lemma 2.4 *Let $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ be a complete hereditary cotorsion triple and $M \in \mathcal{A}$. Then*

(1) M admits a proper $\mathcal{X}$-resolution $\mathbf{X}$ such that $\mathbf{X}^+$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{Y})$*-exact;*

(2) M admits a proper $\mathcal{Y}$-coresolution $\mathbf{Y}$ such that ${}^+\mathbf{Y}$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$*-exact.*

Lemma 2.5 *Let $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ be a complete hereditary cotorsion triple in $\mathcal{A}$. Then for any objects*

$M,N\in\mathcal{A}$ and any $k\in\mathbb{Z}$, there is isomorphism

$$\mathrm{Ext}^k_{\mathcal{X}\mathcal{A}}(M,N)\cong\mathrm{Ext}^k_{\mathcal{A}\mathcal{Y}}(M,N).$$

The whole article assumes that $(\mathcal{X},\mathcal{Z},\mathcal{Y})$ is a complete hereditary cotorsion triple, and n is a non-negative integer. The term $\mathcal{Y}$ is always part of $(\mathcal{X},\mathcal{Z},\mathcal{Y})$, which can ensure any object N of $\mathcal{A}$ admits a proper $\mathcal{Y}$-coresolution, and induce relative cohomology functor $\mathrm{Ext}^*_{\mathcal{A}\mathcal{Y}}(-,N)$.

3. Self-orthogonal-$\mathcal{Y}$ subcategories

We start with the following definition.

Definition 3.1 *Let $\mathcal{N}$ be a subcategory of $\mathcal{A}$. $\mathcal{N}$ is called a self-orthogonal-$\mathcal{Y}$ subcategory, if $\mathrm{Ext}^{k\geqslant 1}_{\mathcal{A}\mathcal{Y}}(N,N')=0$ for any objects $N,N'\in\mathcal{N}$.*

We denote ${}_n\widehat{\mathcal{N}}_{\mathcal{Y}}=\{M\in\mathcal{A}|$ there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to N^n\to\cdots\to N^1\to N^0\to M\to 0$, with each $N^i\in\mathcal{N}\}$. $\widehat{\mathcal{N}}_{\mathcal{Y}}=\{M\in\mathcal{A}|M\in{}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$ for some $n\}$. ${}^{\mathcal{Y}\perp}\mathcal{N}=\{M\in\mathcal{A}|\ \mathrm{Ext}^{k\geqslant 1}_{\mathcal{A}\mathcal{Y}}(M,N)=0$ for any $N\in\mathcal{N}\}$. And $\mathscr{Y}_{\mathcal{N}}=\{M\in\mathcal{A}|$ there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to M\to N^0\xrightarrow{f_0}N^1\xrightarrow{f_1}\cdots$, with each $N^i\in\mathcal{N}$ and $\mathrm{Ker}f_i\in{}^{\mathcal{Y}\perp}\mathcal{N}\}$.

Dually we can get symbols ${}_n\widetilde{\mathcal{N}}_{\mathcal{Y}}$, $\widetilde{\mathcal{N}}_{\mathcal{Y}}$, $\mathcal{N}^{\perp\mathcal{Y}}$ and ${}_{\mathcal{N}}\mathscr{Y}$. It is clear $\mathscr{Y}_{\mathcal{N}}\subseteq{}^{\mathcal{Y}\perp}\mathcal{N}$ and ${}_{\mathcal{N}}\mathscr{Y}\subseteq\mathcal{N}^{\perp\mathcal{Y}}$. We shall discuss properties of self-orthogonal-$\mathcal{Y}$ subcategories.

Lemma 3.2 *Let $\mathcal{N}$ be a self-orthogonal-$\mathcal{Y}$ subcategory of $\mathcal{A}$. Then $\mathrm{Ext}^{k\geqslant 1}_{\mathcal{A}\mathcal{Y}}(M',M)=0$ for any object $M\in\widehat{\mathcal{N}}_{\mathcal{Y}}$ and $M'\in{}^{\mathcal{Y}\perp}\mathcal{N}$.*

Proof For any object $M\in\widehat{\mathcal{N}}_{\mathcal{Y}}$, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to N^n\to\cdots\to N^1\to N^0\to M\to 0$, with each $N^i\in\mathcal{N}$. Since Lemma 2.1 (2), $\mathrm{Ext}^k_{\mathcal{A}\mathcal{Y}}(M',M)\cong\mathrm{Ext}^{k+n}_{\mathcal{A}\mathcal{Y}}(M',N^n)=0$ for any $M'\in{}^{\mathcal{Y}\perp}\mathcal{N}$ and $k\geqslant 1$.

A subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extension, if for any short $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to A\to B\to C\to 0$ with $A,C\in\mathcal{B}$, it induces that $B\in\mathcal{B}$.

Lemma 3.3 *Let $\mathcal{N}$ be a self-orthogonal-$\mathcal{Y}$ subcategory of $\mathcal{A}$. Then both ${}_{\mathcal{N}}\mathscr{Y}$ and $\mathscr{Y}_{\mathcal{N}}$ are closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extension and direct summands.*

Proof For any $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to A\to B\to C\to 0$ with $A,C\in\mathscr{Y}_{\mathcal{N}}$, it is also $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{N})$-exact by Lemma 2.1. Following [18, Lemma 1.10], we have $B\in\mathscr{Y}_{\mathcal{N}}$. Therefore, $\mathscr{Y}_{\mathcal{N}}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions.

Let $U=U_1\oplus U_2$ in $\mathscr{Y}_{\mathcal{N}}$. There is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to U\to N^0\to U'\to 0$ with $N^0\in\mathcal{N}$ and $U'\in\mathscr{Y}_{\mathcal{N}}$. Consider the following pushout diagram (Diagram 3)

Since the up row is split and the middle column is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact, we obtain the middle row is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact by Lemma 2.2. Note that $0\to U_2\to H\to U'\to 0$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact, then the exact sequence $0\to U_2\oplus U_1\to H\oplus U_1\to U'\to 0$ is also $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. Because $U,U'\in\mathscr{Y}_{\mathcal{N}}$, $U=U_1\oplus U_2$ and $\mathscr{Y}_{\mathcal{N}}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extension, we can see $H\oplus U_1\in\mathscr{Y}_{\mathcal{N}}$. So U_1 is a direct summand of some object in $\mathscr{Y}_{\mathcal{N}}$, and $U\in\mathscr{Y}_{\mathcal{N}}$ deduced by recursiveness. Thus, $\mathscr{Y}_{\mathcal{N}}$ is closed under direct summands. □

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & U & \longrightarrow & U_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & N^0 & \longrightarrow & H & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U' & = & U' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram 3 The pushout diagram of $U \to N^0$ and $U \to U_2$

Dually, we can deduce ${}_{\mathcal{N}}\mathscr{Y}$ is also closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extension and direct summands.

Let $\mathcal{W},\mathcal{H}$ be subcategories of $\mathcal{A}$. We say that $\mathcal{W}$ is $\mathcal{Y}$-cogenerator of $\mathcal{H}$, if $\mathcal{W} \subseteq \mathcal{H}$ and for any object $H \in \mathcal{H}$, there is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to H \to W \to H' \to 0$ with $W \in \mathcal{W}$ and $H' \in \mathcal{H}$ (see [19]).

Lemma 3.4 *Suppose that $\mathcal{N}$ and $\mathcal{H}$ are subcategories of $\mathcal{A}$, $\mathcal{H}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions, and $\mathcal{N}$ is $\mathcal{Y}$-cogenerator of $\mathcal{H}$. If $0 \to Z \to M^1 \to M^2 \to \cdots \to M^n \to Z' \to 0$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence with each $M^i \in \mathcal{H}$, then there are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequences $0 \to Z' \to V^n \to U^n \to 0$ with $U^n \in \mathcal{H}$, and $0 \to Z \to N^1 \to \cdots \to N^{n-1} \to N^n \to V^n \to 0$ with each $N^i \in \mathcal{N}$.*

Proof We prove it by induction on n.

The case $n = 1$, there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to Z \to M^1 \to Z' \to 0$ with $M^1 \in \mathcal{H}$. Because $\mathcal{N}$ is $\mathcal{Y}$-cogenerator of $\mathcal{H}$, we have another $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to M^1 \to N^1 \to U^1 \to 0$ with $N^1 \in \mathcal{N}$ and $U^1 \in \mathcal{H}$. Consider pushout diagram (Diagram 4)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & M^1 & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & N^1 & \longrightarrow & V^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U^1 & = & U^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram 4 The pushout diagram of $M^1 \to N^1$ and $M^1 \to Z'$

Following Lemma 2.2, the right column is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. Note that the up row is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact, then the middle row is also $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. The conclusion is tenable.

Suppose that the conclusion is tenable for $n-1$. We shall prove that the conclusion is tenable for n. Let $Z'' = \mathrm{Ker}(M^n \to Z')$. Then by induction hypothesis we have $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequences $0 \to Z'' \to \bar{V}^{n-1} \to \bar{U}^{n-1} \to 0$ and $0 \to Z \to N^1 \to \cdots \to N^{n-1} \to \bar{V}^{n-1} \to 0$, with $\bar{U}^{n-1} \in \mathcal{H}$ and each $N^i \in \mathcal{N}$. Consider the following pushout diagram (Diagram 5)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Z'' & \longrightarrow & M^n & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \bar{V}^{n-1} & \longrightarrow & X & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bar{U}^{n-1} & = & \bar{U}^{n-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Diagram 5 The pushout diagram of $Z'' \to \bar{V}^{n-1}$ and $Z'' \to M^n$

in which the middle row and column are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact by Lemma 2.2. Consider exact sequence $0 \to M^n \to X \to \bar{U}^{n-1} \to 0$ with $M^n, \bar{U}^{n-1} \in \mathcal{H}$, we get $X \in \mathcal{H}$ because $\mathcal{H}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions. Since $\mathcal{N}$ is $\mathcal{Y}$-cogenerator of $\mathcal{H}$, there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to X \to N \to U^n \to 0$ with $N \in \mathcal{N}$ and $U^n \in \mathcal{H}$. Now we can construct the following pushout diagram (Diagram 6)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{V}^{n-1} & \longrightarrow & X & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{V}^{n-1} & \longrightarrow & N & \longrightarrow & V^n & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U^n & = & U^n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram 6 The pushout diagram of $X \to N$ and $X \to Z'$

where the right column and middle row are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact by Lemma 2.2. Consequently, the conclusion is tenable for n. $\square$

In particular, we get the following result.

Corollary 3.5 *Suppose that $\mathcal{N}$ and $\mathcal{H}$ are subcategories of $\mathcal{A}$, $\mathcal{H}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions, and $\mathcal{N}$ is $\mathcal{Y}$-cogenerator of $\mathcal{H}$. If $Z' \in {}_n\widehat{\mathcal{H}}_{\mathcal{Y}}$, then there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to Z' \to V \to U \to 0$ with $U \in \mathcal{H}$ and $V \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$.*

Proposition 3.6 *Let the subcategory $\mathcal{N}$ of $\mathcal{A}$ be both self-orthogonal-$\mathcal{Y}$ and closed under direct summands. Then the followings are equivalent for any object $M \in {}_{\mathcal{N}}\mathcal{Y}$*

(1) $M \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$;

(2) $\mathrm{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M, M') = 0$ for any object $M' \in \mathcal{N}^{\perp_{\mathcal{Y}}}$;

(3) $\mathrm{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M, M') = 0$ for any object $M' \in {}_{\mathcal{N}}\mathcal{Y}$.

Particularly, $\widehat{\mathcal{N}}_{\mathcal{Y}}$ is closed under direct summands.

Proof (1)⇒(2). Suppose that $M \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$, there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0 \to N^n \to \cdots \to N^1 \to N^0 \to M \to 0$$

with each $N^i \in \mathcal{N}$. According to Lemma 2.1, $\mathrm{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M, M') \cong \mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(N^n, M') = 0$ for any object $M' \in \mathcal{N}^{\perp_{\mathcal{Y}}}$.

(2)⇒(3) is clear.

(3)⇒(1). Let $M \in {}_{\mathcal{N}}\mathcal{Y}$. Then there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$\cdots \to N^2 \xrightarrow{f_2} N^1 \xrightarrow{f_1} N^0 \xrightarrow{f_0} M \to 0$$

with each $N^i \in \mathcal{N}$ and $\mathrm{Im} f_i \in \mathcal{N}^{\perp_{\mathcal{Y}}}$. Following Lemma 2.1, we obtain the isomorphism

$$\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(\mathrm{Ker} f_{n-1}, \mathrm{Ker} f_n) \cong \mathrm{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M, \mathrm{Ker} f_n)$$

by applying the functor $\mathrm{Hom}_{\mathcal{A}}(-, \mathrm{Ker} f_n)$ to this sequence. Note that $\mathrm{Ker} f_n \in {}_{\mathcal{N}}\mathcal{Y}$, then $\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(\mathrm{Ker} f_{n-1}, \mathrm{Ker} f_n) = 0$. So we have that $0 \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker} f_{n-1}, \mathrm{Ker} f_n) \to \mathrm{Hom}_{\mathcal{A}}(N^n, \mathrm{Ker} f_n) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker} f_n, \mathrm{Ker} f_n) \to 0$ is exact. Consequently, the exact sequence $0 \to \mathrm{Ker} f_n \to N^n \to \mathrm{Ker} f_{n-1} \to 0$ is split. Since $N^n \in \mathcal{N}$ and $\mathcal{N}$ is closed under direct summands, then $\mathrm{Ker} f_n \in \mathcal{N}$, which means $M \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$.

The final statement comes directly from Lemma 3.3 by $\widehat{\mathcal{N}}_{\mathcal{Y}} \in {}_{\mathcal{N}}\mathscr{Y}$. □

4. n-$\mathcal{Y}$-cotilting subcategories

In this section, we introduce the concept and examples of n-$\mathcal{Y}$-cotilting subcategories with respect to a complete hereditary cotorsion triple $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$. Finally the characterization of it is given.

Definition 4.1 *Assume that the subcategory $\mathcal{N}$ of $\mathcal{A}$ is closed under direct summands. $\mathcal{N}$ is said to be n-$\mathcal{Y}$-cotilting (with respect to $(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$) provided*

(1) $\mathcal{Y}$-id$\,\mathcal{N} \leqslant n$ i.e, $\mathcal{Y}$-id$\,N \leqslant n$ for all $N \in \mathcal{N}$;

(2) $\mathcal{N}$ is a $\mathcal{Y}$-cogenerator of ${}^{\mathcal{Y}\perp}\mathcal{N}$;

(3) $\mathcal{Y} \subseteq {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$.

For convenience, we denote $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$=$\{M \in \mathcal{A}|$ there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to M \to N^1 \to N^2 \to \cdots \to N^n$, with each $N^i \in \mathcal{N}\}$, and $\mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N}) = \mathrm{Copres}^1_{\mathcal{Y}}(\mathcal{N})$.

Obviously $\mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N})$ is closed under direct summands.

Proposition 4.2 *Assume that $\mathcal{N}$ is a subcategory of $\mathcal{A}$. Then the following conditions are equivalent*

(1) $\mathcal{N}$ is a $\mathcal{Y}$-cogenerator of ${}^{\mathcal{Y}\perp}\mathcal{N}$;

(2) $\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$ and ${}^{\mathcal{Y}\perp}\mathcal{N}=\mathcal{Y}_{\mathcal{N}}$;

(3) $\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$, each object of ${}^{\mathcal{Y}\perp}\mathcal{N}$ admits a left $\mathcal{N}$-approximation and ${}^{\mathcal{Y}\perp}\mathcal{N}\subseteq \mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N})$.

Proof (1)⇒(2). It is easy to see that $\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$ and $\mathcal{Y}_{\mathcal{N}}\subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$. On the other hand, for any $M\in {}^{\mathcal{Y}\perp}\mathcal{N}$, since $\mathcal{N}$ is a $\mathcal{Y}$-cogenerator of ${}^{\mathcal{Y}\perp}\mathcal{N}$, we obtain a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to M\to N\to M'\to 0$ with $N\in\mathcal{N}$ and $M'\in {}^{\mathcal{Y}\perp}\mathcal{N}$. Repeating the process for N', we finally get $M\in\mathcal{Y}_{\mathcal{N}}$. Therefore, ${}^{\mathcal{Y}\perp}\mathcal{N}=\mathcal{Y}_{\mathcal{N}}$.

(2)⇒(3). Let $M\in {}^{\mathcal{Y}\perp}\mathcal{N}=\mathcal{Y}_{\mathcal{N}}$. By definition we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to M\xrightarrow{f} N\to M'\to 0$ with $N\in\mathcal{N}$ and $M'\in {}^{\mathcal{Y}\perp}\mathcal{N}$. Then this sequence is also exact under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{N})$ by Lemma 2.1. One can see $M\xrightarrow{f} N$ is a left $\mathcal{N}$-approximation of M and $M'\in \mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N})$. So the conclusion (3) holds.

(3)⇒(1). For any $M\in {}^{\mathcal{Y}\perp}\mathcal{N}$, by (3) we know $M'\in\mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N})$, there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0\to M\xrightarrow{\alpha} N^1\to M'\to 0$ with $N^1\in\mathcal{N}$. On the other hand, M admits a left $\mathcal{N}$-approximation $\beta: M\to N$ with $N\in\mathcal{N}$, which derives a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{N})$-exact sequence

$$0\to M\xrightarrow{\beta} N\to M''\to 0 \qquad (*)$$

we show that $(*)$ is desired. Firstly we construct the following commutative diagram (Diagram 7)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\beta} & N & \longrightarrow & M'' & \longrightarrow & 0\\
 & & \| & & \gamma\downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\alpha} & N^1 & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

Diagram 7 The commutative diagram induced by left $\mathcal{N}$-approximation of M

for any $Y\in\mathcal{Y}$ and any morphism $f: M\to Y$, there is a morphism $g: N^1\to Y$ such that $g\alpha=f$ since $0\to M\xrightarrow{\alpha} N^1\to M'\to 0$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. Let $h=g\gamma$. Then $h\in\mathrm{Hom}_{\mathcal{A}}(N,Y)$ and $h\beta=g\gamma\beta=g\alpha=f$. So $(*)$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. We now only need to prove $M''\in {}^{\mathcal{Y}\perp}\mathcal{N}$. Indeed, for any $N'\in\mathcal{N}$, there is a long exact sequence

$$\cdots\to\mathrm{Hom}_{\mathcal{A}}(N,N')\xrightarrow{\beta^*}\mathrm{Hom}_{\mathcal{A}}(M,N')\to\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(M'',N')\to\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(N,N')\to\cdots$$

by applying $\mathrm{Hom}_{\mathcal{A}}(-,N')$ to $(*)$. Note that $\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(N,N')=0$ because β is left $\mathcal{N}$-approximation. So $\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(M'',N')=0$. Since $\mathrm{Ext}^{k+1}_{\mathcal{A}\mathcal{Y}}(M'',N')\cong\mathrm{Ext}^1_{\mathcal{A}\mathcal{Y}}(M,N')=0$, we obtain $\mathrm{Ext}^k_{\mathcal{A}\mathcal{Y}}(M'',N')=0$ for any $k\geqslant 1$, which means $M''\in {}^{\mathcal{Y}\perp}\mathcal{N}$. □

Here are some examples of n-$\mathcal{Y}$-cotilting subcategories.

Example 4.3 *(1) $\mathcal{Y}$ is n-$\mathcal{Y}$-cotilting subcategory.*

(2) Assume that $T \in R$-Mod where R-Mod is the category of left R-modules. Then the two statements are in agreement

(a) T is an n-cotilting module;

(b) Prod$T$ *is an n-$\mathcal{I}$-cotilting subcategory with respect to complete hereditary cotorsion triple ($\mathcal{P}$,R-Mod, $\mathcal{I}$), where* Prod$T$ *consists of all left R-modues isomorphic to direct summands of arbitrary products of copies of T.*

A commutative noetherian ring R is said to be virtually Gorenstein, if R has finite Krull dimension and $\mathcal{GP}^{\perp} = {}^{\perp}\mathcal{GI}$. Following [11, Theorm 4.5], a virtually Gorenstein ring R is n-Gorenstein if and only if $\mathcal{GI}$ is n-$\mathcal{GP}$-tilting subcategory.

As an application of n-$\mathcal{Y}$-cotilting subcategory, we shall discuss the relation of n-Gorenstein ring and n-$\mathcal{GI}$-cotilting subcategories.

Theorem 4.4 *Assume that R is a virtually Gorenstein ring. If $\mathcal{GP}$ is an n-$\mathcal{GI}$-cotilting subcategory, then R is an n-Gorenstein. Moreover, $\mathcal{GI}$ is an n-$\mathcal{GP}$-tilting subcategory.*

Proof Following [14, Theorem 9.1.11], we only need to show that the projective dimension of all injective R-modules is at most n. For any injective R-module M, M is also Gorenstein injective. Since $\mathcal{GP}$ is n-$\mathcal{GI}$-cotilting subcategory, then the Gorenstein projective dimension of M is at most n. By [20, Theorem 2.2], we obtain that the projective dimension of M is equal to its Gorenstein projective dimension. Therefore, R is n-Gorenstein.

According to [11, Theorem 4.5], $\mathcal{GI}$ is n-$\mathcal{GP}$-tilting subcategory. □

Proposition 4.5 *Assume that $\mathcal{N}$ is n-$\mathcal{Y}$-cotilting subcategory of $\mathcal{A}$. Then*

$$\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N}) = {}^{\mathcal{Y}\perp}\mathcal{N}.$$

Proof Since Definition 4.1 and Proposition 4.2, we obtain ${}^{\mathcal{Y}\perp}\mathcal{N} = \mathcal{Y}_{\mathcal{N}} \subseteq \mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$. It is only to prove $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N}) \subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$. For any $C \in \mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$ and $N \in \mathcal{N}$, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to C \to N^1 \to N^2 \to \cdots \to N^{n-1} \to N^n \to I \to 0$, where $I = \mathrm{Coker}(N^{n-1} \to N^n)$ and each $N^i \in \mathcal{N}$. Since Lemma 2.1 and $\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$, we obtain $\mathrm{Ext}^k_{\mathcal{AY}}(C,N) \cong \mathrm{Ext}^{k+n}_{\mathcal{AY}}(I,N)$ for $k \geqslant 1$. Note that $\mathcal{Y}$-id$\mathcal{N} \leqslant n$, then $\mathrm{Ext}^{k+n}_{\mathcal{AY}}(I,N) = 0$ by Proposition 2.3. So $\mathrm{Ext}^k_{\mathcal{AY}}(C,N) = 0$, which means $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N}) \subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$. Therefore, $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})= {}^{\mathcal{Y}\perp}\mathcal{N}$. □

For any subcategory $\mathcal{V}$ of $\mathcal{A}$, it is obvious $\mathcal{X} \in {}^{\mathcal{Y}\perp}\mathcal{V}$ by Proposition 2.3. Then $\mathcal{V} \subseteq \mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{V})$. $\mathcal{V}$ is said to be closed under n-$\mathcal{Y}$-kernels provided that $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{V}) \subseteq \mathcal{V}$, which means $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{V}) = \mathcal{V}$.

Lemma 4.6 *Let $\mathcal{N}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{Y}$-id$\mathcal{N} \leqslant n$ if and only if ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels.*

Proof ($\Rightarrow$). For any $C \in \mathrm{Copres}^n_{\mathcal{Y}}({}^{\mathcal{Y}\perp}\mathcal{N})$, there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0 \to C \to M^1 \to M^2 \to \cdots \to M^{n-1} \to M^n \to I \to 0,$$

where $I = \mathrm{Coker}(M^{n-1} \to M^n)$ and each $M^i \in {}^{\mathcal{Y}\perp}\mathcal{N}$. Note that ${}^{\mathcal{Y}\perp}\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$, then

$\mathrm{Ext}^k_{\mathcal{AY}}(C,N)\cong \mathrm{Ext}^{k+n}_{\mathcal{AY}}(I,N)=0$ for any $N\in\mathcal{N}$ and $k\geqslant 1$ by Lemma 2.1 and Proposition 2.3. We have $C\in {}^{\mathcal{Y}\perp}\mathcal{N}$, which means $\mathrm{Copres}^n_{\mathcal{Y}}({}^{\mathcal{Y}\perp}\mathcal{N})\subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$. Thus ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels.

($\Leftarrow$). It suffices to prove $\mathrm{Ext}^{n+1}_{\mathcal{AY}}(M,N)=0$ for any $N\in\mathcal{N}$ and $M\in\mathcal{A}$ by Proposition 2.3. Note that $(\mathcal{X},\mathcal{Z},\mathcal{Y})$ is a complete hereditary cotorsion triple, by lemma 2.4, we obtain a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0\to K\to X^n\to X^{n-1}\to\cdots\to X^2\to X^1\to M\to 0$$

where $X^i\in\mathcal{X}\subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$. Since ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels, we have $K\in {}^{\mathcal{Y}\perp}\mathcal{N}$. Therefore, $\mathrm{Ext}^{n+1}_{\mathcal{AY}}(M,N)\cong \mathrm{Ext}^1_{\mathcal{AY}}(K,N)=0$ by Lemma 2.1. □

Proposition 4.7 *Assume that $\mathcal{N}$ is a self-orthogonal-$\mathcal{Y}$ subcategory of $\mathcal{A}$. Then* $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})=\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})$.

Proof By assumption, we obtain $\mathcal{N}\in\mathcal{Y}_{\mathcal{N}}$, then $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})\subseteq\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})$. We only need to prove $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})\subseteq\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$. For any $C\in\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})$, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0\to C\to M^1\to M^2\to\cdots\to M^{n-1}\to M^n\to I\to 0,$$

where $I=\mathrm{Coker}(M^{n-1}\to M^n)$ and each $M^i\in\mathcal{Y}_{\mathcal{N}}$. Since $\mathcal{N}$ is a $\mathcal{Y}$-cogenerator of $\mathcal{Y}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions by Lemma 3.3. Following Lemma 3.4, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0\to C\to N^1\to N^2\to\cdots\to N^{n-1}\to N^n\to V^n\to 0,$$

where each $N^i\in\mathcal{N}$. It is clear $C\in\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$. Then $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})\subseteq\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})$. In conclusion, $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})=\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})$.

Proposition 4.8 *Assume that $\mathcal{N}$ is a subcategory of $\mathcal{A}$ with $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})={}^{\mathcal{Y}\perp}\mathcal{N}$, and each object in ${}^{\mathcal{Y}\perp}\mathcal{N}$ admits a left $\mathcal{N}$-approximation. Then the followings hold*

(1) ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels and $\mathcal{Y}$-id$\mathcal{N}\leqslant n$;

(2) If $\mathcal{N}$ is closed under direct summands, then $\mathcal{Y}\subseteq {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$.

Proof (1) By Lemma 4.6, we only need to show ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels. Since $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})={}^{\mathcal{Y}\perp}\mathcal{N}$, we obtain $\mathcal{N}$ is self-orthogonal-$\mathcal{Y}$ and ${}^{\mathcal{Y}\perp}\mathcal{N}\subseteq\mathrm{Cogen}_{\mathcal{Y}}(\mathcal{N})$. Note that each object in ${}^{\mathcal{Y}\perp}\mathcal{N}$ admits left $\mathcal{N}$-approximation, then ${}^{\mathcal{Y}\perp}\mathcal{N}=\mathcal{Y}_{\mathcal{N}}$ by Proposition 4.2. According to Proposition 4.7, we get

$$\mathrm{Copres}^n_{\mathcal{Y}}({}^{\mathcal{Y}\perp}\mathcal{N})=\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{Y}_{\mathcal{N}})=\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N})={}^{\mathcal{Y}\perp}\mathcal{N}$$

which means ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under n-$\mathcal{Y}$-kernels.

(2) Let $\mathcal{N}$ be closed under direct summands. For any $Y\in\mathcal{Y}$, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence

$$0\to Z\to X^n\to X^{n-1}\to\cdots\to X^2\to X^1\to Y\to 0$$

with $X^i \in \mathcal{X} \subseteq {}^{\mathcal{Y}\perp}\mathcal{N}$ by Lemma 2.4. Following (1), we get $Z \in {}^{\mathcal{Y}\perp}\mathcal{N}$. It is clear ${}^{\mathcal{Y}\perp}\mathcal{N}$ is closed under $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-extensions by Lemma 2.1. Note that $\mathcal{N}$ is a $\mathcal{Y}$-cogenerator of ${}^{\mathcal{Y}\perp}\mathcal{N}$ since Proposition 4.2. According to Corollary 3.5, we have a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact sequence $0 \to Y \to V \to U \to 0$ with $V \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$ and $U \in {}^{\mathcal{Y}\perp}\mathcal{N}$. Then $V \cong Y \oplus U$ by Lemma 3.2. Also because ${}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$ is closed under direct summands by Lemma 3.6, we obtain $Y \in {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$, which means $Y \subseteq {}_n\widehat{\mathcal{N}}_{\mathcal{Y}}$. □

We can now state one of our main results which follows immediately by Propositions 4.5 and 4.8. It is similar to [7, Theorem 3.11].

Theorem 4.9 *Assume that the subcategory $\mathcal{N}$ of $\mathcal{A}$ is closed under direct summands, and every object in ${}^{\mathcal{Y}\perp}\mathcal{N}$ admits a left $\mathcal{N}$-approximation. Then $\mathcal{N}$ is n-$\mathcal{Y}$-cotilting (with respect to $(\mathcal{X},\mathcal{Z},\mathcal{Y})$), if and only if* $\mathrm{Copres}^n_{\mathcal{Y}}(\mathcal{N}) = {}^{\mathcal{Y}\perp}\mathcal{N}$.

5. n-Costar subcategories and n-$\mathcal{I}$-cotilting subcategories

In this section, R is an associative ring with nonzero identity. R-Mod is the subcategory of all left R-modules. We use the term "subcategory" to stand for a full additive subcategory of R-Mod closed under isomorphisms. $\mathcal{P}$ denotes the subcategory of projective left R-modules and $\mathcal{I}$ denotes the subcategory of injective left R-modules. If $\alpha : X \to Y$ and $\beta : Y \to Z$ are homomorphisms, we denote by $\alpha\beta$ the composition of α and β. Let $\mathcal{M}$ be a subcategory of R-Mod, we denote

${}^{\perp_{1\leqslant i\leqslant n}}\mathcal{M} = \{N \in R\text{-Mod} | \mathrm{Ext}^i_R(N,M) = 0 \text{ for any } M \in \mathcal{M} \text{ and any } 1 \leqslant i \leqslant n\}$,

${}^{\perp_{i\geqslant 1}}\mathcal{M} = \{N \in R\text{-Mod} | \mathrm{Ext}^i_R(N,M) = 0 \text{ for any } M \in \mathcal{M} \text{ and any } i \geqslant 1\}$.

Obviously, $(\mathcal{P}, R\text{-Mod}, \mathcal{I})$ is a complete hereditary cotorsion triple and ${}^{\mathcal{I}\perp}\mathcal{M} = {}^{\perp_{i\geqslant 1}}\mathcal{M}$.

And denote by $\mathrm{Copres}^n(\mathcal{M})$ the subcategory of $N \in R$-Mod such that there exists an exact sequence $0 \to N \to M_1 \to M_2 \to \cdots \to M_n$ with each $M_i \in \mathcal{M}$. It is obvious that $\mathrm{Cogen}(\mathcal{M}) = \mathrm{Copres}^1(\mathcal{M})$, $\mathrm{Copres}^1(\mathcal{M})$ is closed under direct summands, and $\mathrm{Copres}^{n+1}(\mathcal{M}) \subseteq \mathrm{Copres}^n(\mathcal{M})$ for any non-negative integer n. Dually we can define $\mathcal{M}^{\perp_{1\leqslant i\leqslant n}}$, $\mathcal{M}^{\perp_{i\geqslant 1}}$ and $\mathrm{Pres}^n(\mathcal{M})$. If the short exact sequence $0 \to U \to V \to W \to 0$ is still exact under the functor $\mathrm{Hom}_R(-,\bar{M})$ for any $\bar{M} \in \mathcal{M}$, then we say this exact sequence stays exact under the functor $\mathrm{Hom}_R(-,\mathcal{M})$. We say $\mathcal{M}$ is closed under n-kernels if $\mathrm{Copres}^n(\mathcal{M}) \subseteq \mathcal{M}$, i.e., $\mathrm{Copres}^n(\mathcal{M}) = \mathcal{M}$. A set $\mathcal{M}' \subseteq \mathcal{M}$ is a class of representatives (of isomorphism types) of $\mathcal{M}$ in case each $M \in \mathcal{M}$ is isomorphic to some element of $\mathcal{M}'$ (see [21]). Clearly, if $\mathcal{M}'$ is a class of representatives of $\mathcal{M}$, then $\mathrm{Cogen}(\mathcal{M}) = \mathrm{Cogen}(\mathcal{M}')$, and $\mathrm{Gen}(\mathcal{M}) = \mathrm{Gen}(\mathcal{M}')$.

Let us start with the concept of n-quasi-injective subcategories.

Definition 5.1 *Let $n \geqslant 1$ and $\mathcal{M}$ be a subcategory of R-Mod, which is closed under direct summands. $\mathcal{M}$ is said to be an n-quasi-injective subcategory if for any exact sequence $0 \to U \to M \to W \to 0$ with $M \in \mathcal{M}$ and $W \in \mathrm{Copres}^{n-1}(\mathcal{M})$, the induced sequence $0 \to \mathrm{Hom}_R(W,\bar{M}) \to \mathrm{Hom}_R(M,\bar{M}) \to \mathrm{Hom}_R(U,\bar{M}) \to 0$ is also exact for any $\bar{M} \in \mathcal{M}$.*

It is clear that ${}_RT$ is an n-quasi-injective module is equivalent to $\mathrm{Prod}_R T$ is an n-quasi-injective subcategory, where $\mathrm{Prod}_R T$ denotes the subcategory of all left R-modules N that are isomorphic to direct summand of T^{λ} for some cardinal λ. If $\mathcal{M}$ is an n-quasi-injective subcategory, then $\mathcal{M}$ is an m-quasi-injective subcategory for all $m \geqslant n$.

We now introduce a useful lemma.

Lemma 5.2 *Suppose that $\mathcal{M}$ is a subcategory of R-Mod which is closed under direct summands, $0 \to U \to M_1 \to I_1 \to 0$ and $0 \to U \to M_2 \to I_2 \to 0$ are exact sequences with $M_1, M_2 \in \mathcal{M}$. If both sequences stay exact under the functor* $\mathrm{Hom}_R(-, \mathcal{M})$*, then $M_1 \oplus I_2 \cong M_2 \oplus I_1$.*

Proof By assumption, we have that both $0 \to U \to M_1 \to I_1 \to 0$ and $0 \to U \to M_2 \to I_2 \to 0$ stay exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$, where $M_1, M_2 \in \mathcal{M}$. Then by the decomposition lemma we can construct the following commutative diagram with row exact (Diagram 8):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & M_1 & \longrightarrow & I_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & M_2 & \longrightarrow & I_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & M_1 & \longrightarrow & I_1 & \longrightarrow & 0
\end{array}$$

Diagram 8 The commutative diagram induced by the decomposition lemma

Following the dually conclusion of [12, Lemma2.2], we obtain $M_1 \oplus I_2 \cong M_2 \oplus I_1$. □

Below we give an equivalent characterization of n-quasi-injective subcategories.

Proposition 5.3 *Suppose that $n \geqslant 1$ and $\mathcal{M}$ is a subcategory of R-Mod, which is closed under direct summands. Then the followings are equivalent.*

(1) $\mathcal{M}$ is an n-quasi-injective subcategory;

(2) For any exact sequence $\delta : 0 \to U \to M \to W \to 0$ with $M \in \mathcal{M}$ and $U \in \mathrm{Copres}^n(\mathcal{M})$, we have that $W \in \mathrm{Copres}^{n-1}(\mathcal{M})$ if only if δ is still exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$.

Proof (1)⇒ (2). For any exact sequence $\delta : 0 \to U \to M \to W \to 0$ with $M \in \mathcal{M}$ and $U \in \mathrm{Copres}^n(\mathcal{M})$, if $W \in \mathrm{Copres}^{n-1}(\mathcal{M})$, then it is clear that the induced sequence $0 \to \mathrm{Hom}_R(W, \bar{M}) \to \mathrm{Hom}_R(M, \bar{M}) \to \mathrm{Hom}_R(U, \bar{M}) \to 0$ is still exact for any $\bar{M} \in \mathcal{M}$ by (1), which means δ is still exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$. If δ is still exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$, by $U \in \mathrm{Copres}^n(\mathcal{M})$ we can get an exact sequence $0 \to U \to M' \to W' \to 0$ with $M' \in \mathcal{M}$ and $U' \in \mathrm{Copres}^{n-1}(\mathcal{M})$, which is also exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$ according to Definition 5.1. So we obtain that $W' \oplus M \cong W \oplus M'$ by applying Lemma 5.2. Therefore, $W \in \mathrm{Copres}^{n-1}(\mathcal{M})$.

(2)⇒(1) is obvious. □

We introduce the concept and characterizations of n-costar subcategories as follows.

Definition 5.4 *Suppose that $n \geqslant 1$ and $\mathcal{M}$ is a subcategory of R-Mod which is closed under direct summands and direct products. $\mathcal{M}$ is called an n-costar subcategory, if $\mathcal{M}$ is an* $(n+1)$-

quasi-injective subcategory and $\mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M})$.

Lemma 5.5 *Suppose that* $\mathcal{M}$ *is an* n*-costar subcategory and* $0 \to U \xrightarrow{i} V \xrightarrow{\pi} W \to 0$ *is a short exact sequence in* R*-Mod. If* $V, W \in \mathrm{Copres}^n(\mathcal{M})$*, then* $U \in \mathrm{Copres}^n(\mathcal{M})$.

Proof If $V, W \in \mathrm{Copres}^n(\mathcal{M})$, then there are exact sequences

$$0 \to W \xrightarrow{\beta} M_W \to W' \to 0 \text{ and } 0 \to V \xrightarrow{\alpha} M_V \to V_1 \to 0$$

with $W', V_1 \in \mathrm{Copres}^n(\mathcal{M})$ and $M_W, M_V \in \mathcal{M}$. Since $\mathcal{M}$ is an n-costar subcategory, we can get the following exact commutative diagram (Diagram 9):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{\pi} & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i\alpha} & & \downarrow{\scriptstyle (\alpha,\pi\beta)} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & M_V & \xrightarrow{(1,0)} & M_V \oplus M_W & \xrightarrow{\binom{0}{1}} & M_W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \longrightarrow & V' & \longrightarrow & W' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Diagram 9 The diagram corresponding to U

Because the exact sequence $0 \to V \to M_V \to V_1 \to 0$ is exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$ by assumption, for any $\bar{M} \in \mathcal{M}$ and any homomorphism $f : V \to \bar{M}$, there exists a homomorphism $g : M_V \to \bar{M}$ such that $\alpha g = f$. It is easy to see that $(\alpha, \pi\beta)\binom{g}{0} = \alpha g = f$ and $\binom{g}{0} : M_V \oplus M_W \to \bar{M}$, which means the exact sequence $0 \to V \to M_V \oplus M_W \to V' \to 0$ is also exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$. Note that $V \in \mathrm{Copres}^n(\mathcal{M})$ and $\mathcal{M}$ is $(n+1)$-quasi-injective subcategory, we can get $V' \in \mathrm{Copres}^n(\mathcal{M})$ by Proposition 5.3. Repeat the above process to $0 \to U' \to V' \to W' \to 0$ and continue. It is not difficult to draw the conclusion $U \in \mathrm{Copres}^n(\mathcal{M})$. □

Now let us talk about the closure of $\mathrm{Copres}^n(\mathcal{M})$ under kernels of monomorphism, cokernels of epimorphism and extensions, by assumption that $\mathcal{M}$ is an n-costar subcategory.

Proposition 5.6 *Suppose that* $\mathcal{M}$ *is an* n*-costar subcategory and* $0 \to U \xrightarrow{i} V \xrightarrow{\pi} W \to 0$ *is a short exact sequence which stays exact under the functor* $\mathrm{Hom}_R(-, \mathcal{M})$*. Then the followings hold:*

(1) If $V, W \in \mathrm{Copres}^n(\mathcal{M})$*, then* $U \in \mathrm{Copres}^n(\mathcal{M})$*;*

(2) If $U, W \in \mathrm{Copres}^n(\mathcal{M})$*, then* $V \in \mathrm{Copres}^n(\mathcal{M})$*;*

(3) If $U, V \in \mathrm{Copres}^n(\mathcal{M})$*, then* $W \in \mathrm{Copres}^n(\mathcal{M})$.

Proof (1) It is clear by Lemma 5.5.

(2) If $U, W \in \mathrm{Copres}^n(\mathcal{M})$, then there are exact sequences $0 \to U \xrightarrow{\alpha} M_U \to U' \to 0$ and $0 \to W \xrightarrow{\gamma} M_W \to W' \to 0$ with $U', W' \in \mathrm{Copres}^n(\mathcal{M})$ and $M_U, M_W \in \mathcal{M}$, which are still exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$, since $\mathcal{M}$ is an n-costar subcategory. Note that $0 \to U \xrightarrow{i} V \xrightarrow{\pi} W \to 0$ is also exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$ and $M_W \in \mathcal{M}$, so we can get a homomorphism $\xi : V \to M_U$ such that $i\xi = \alpha$. Consider the following commutative diagram (Diagram 10):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{\pi} & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (\xi, \pi\gamma)} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M_U & \xrightarrow{(1,0)} & M_U \oplus M_W & \xrightarrow{\binom{0}{1}} & M_W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \longrightarrow & V' & \longrightarrow & W' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Diagram 10 The diagram corresponding to V

where $V' = \mathrm{Coker}(V \to M_U \oplus M_W)$. For any $\bar{M} \in \mathcal{M}$, applying the functor $\mathrm{Hom}_R(-, \bar{M})$ to the diagram, we can obtain the following exact commutative diagram (Diagram 11):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(W', \bar{M}) & \longrightarrow & \mathrm{Hom}_R(V', \bar{M}) & \longrightarrow & \mathrm{Hom}_R(U', \bar{M}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M_W, \bar{M}) & \longrightarrow & \mathrm{Hom}_R(M_U \oplus M_W, \bar{M}) & \longrightarrow & \mathrm{Hom}_R(M_U, \bar{M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(W, \bar{M}) & \longrightarrow & \mathrm{Hom}_R(V, \bar{M}) & \longrightarrow & \mathrm{Hom}_R(U, \bar{M}) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

Diagram 11 The induced diagram by Diagram 7

By the snake lemma, we get the sequence $0 \to \mathrm{Hom}_R(W', \bar{M}) \to \mathrm{Hom}_R(V', \bar{M}) \to \mathrm{Hom}_R(U', \bar{M}) \to 0$ is exact, which means $0 \to U' \to V' \to W' \to 0$ is exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$. Repeat the above process to $0 \to U' \to V' \to W' \to 0$ and continue. It is not difficult to draw the conclusion $V \in \mathrm{Copres}^n(\mathcal{M})$.

(3) If $U, V \in \mathrm{Copres}^n(\mathcal{M})$, then there is an exact sequence $0 \to V \xrightarrow{\beta} M_V \to V' \to 0$ with $V' \in \mathrm{Copres}^n(\mathcal{M})$ and $M_V \in \mathcal{M}$. Consider the following pushout diagram (Diagram 12):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{\pi} & W & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{\mu} & M_V & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V' & = & V' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram 12 The pushout diagram of $V \to M_V$ and $V \to W$

For any $\bar{M} \in \mathcal{M}$ and any homomorphism $g : U \to \bar{M}$, there is a homomorphism $h : V \to \bar{M}$ such that $ih = g$ by the fact that upper row stays exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$. Since the middle column is also exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$, there exists a homomorphism $f : M_V \to \bar{M}$ such that $\beta f = h$. Then $\mu f = (i\beta)f = i(\beta f) = ih = g$, which means the middle row is still exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$. Following Proposition 5.3, we can get $Y \in \mathrm{Copres}^n(\mathcal{M})$. Also because $V' \in \mathrm{Copres}^n(\mathcal{M})$, it is easy to see that $U \in \mathrm{Copres}^n(\mathcal{M})$ by Lemma 5.5. □

According to the proof of Proposition 5.6, we can get the following corollary.

Corollary 5.7 *Suppose that $\mathcal{M}$ is an n-costar subcategory and $\delta : 0 \to U \to V \to W \to 0$ is a short exact sequence with $U, V, W \in \mathrm{Copres}^n(\mathcal{M})$, then δ stays exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$.*

Some characterizations of n-costar subcategories are given below.

Theorem 5.8 *Suppose that $n \geqslant 1$ and $\mathcal{M}$ is a subcategory of R-Mod, which is closed under direct summands and direct products. $\mathcal{M}$ has a class of representatives. Then the followings are equivalent.*

(1) $\mathcal{M}$ is an n-costar subcategory;

(2) For any short exact sequence $\delta : 0 \to U \to M \to W \to 0$ with $M \in \mathcal{M}$ and $U \in \mathrm{Copres}^n(\mathcal{M})$, then $W \in \mathrm{Copres}^n(\mathcal{M})$ if and only if the induced sequence $0 \to \mathrm{Hom}_R(W, \bar{M}) \to \mathrm{Hom}_R(M, \bar{M}) \to \mathrm{Hom}_R(U, \bar{M}) \to 0$ for any $\bar{M} \in \mathcal{M}$.

(3) For any short exact sequence $\delta : 0 \to U \to V \to W \to 0$ with $U, V \in \mathrm{Copres}^n(\mathcal{M})$, then $W \in \mathrm{Copres}^n(\mathcal{M})$ if and only if the induced sequence $0 \to \mathrm{Hom}_R(W, \bar{M}) \to \mathrm{Hom}_R(V, \bar{M}) \to \mathrm{Hom}_R(U, \bar{M}) \to 0$ is also exact for any $\bar{M} \in \mathcal{M}$.

Proof (1)$\Rightarrow$ (3). Let $\mathcal{M}$ be an n-costar subcategory. Then $\mathcal{M}$ is $(n+1)$-quasi-injective and $\mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M})$. For any short exact sequence $\delta : 0 \to U \to V \to W \to 0$ with $U, V \in \mathrm{Copres}^n(\mathcal{M})$, by Proposition 5.6 and Corollary 5.7, it is not difficult to prove that the conclusion holds.

(3)$\Rightarrow$ (2). It is clear by $\mathcal{M} \subseteq \mathrm{Copres}^n(\mathcal{M})$.

(2)$\Rightarrow$ (1). By assumption (2), we can get $\mathcal{M}$ is $(n+1)$-quasi-injective subcategory. Note that $\mathrm{Copres}^{n+1}(\mathcal{M}) \subseteq \mathrm{Copres}^n(\mathcal{M})$, so we only need to prove $\mathrm{Copres}^n(\mathcal{M}) \subseteq \mathrm{Copres}^{n+1}(\mathcal{M})$. For any $U \in \mathrm{Copres}^n(\mathcal{M})$, we have $U \in \mathrm{Cogen}(\mathcal{M})$. Let $\mathcal{M}'$ be a class of representatives of $\mathcal{M}$, then $U \in \mathrm{Cogen}(\mathcal{M}')$. There exists a set $(M_i)_{i\in I}$ in $\mathcal{M}'$ and an monomorphism $U \to \prod_{i\in I} M_i$. Let $H = \prod_{M'\in\mathcal{M}'} M'$, and $M_U = H^{\mathrm{Hom}_R(U,H)}$. Then there is an exact sequence $o \to U \to M_U \to U' \to 0$ which is still exact under the functor $\mathrm{Hom}_R(-,\mathcal{M})$. By assumption we get $U' \in \mathrm{Copres}^n(\mathcal{M})$, which means $U \in \mathrm{Copres}^{n+1}(\mathcal{M})$. So $\mathrm{Copres}^n(\mathcal{M}) \subseteq \mathrm{Copres}^{n+1}(\mathcal{M})$. Consequently, $\mathcal{M}$ is an n-costar subcategory. □

Theorem 5.9 *Suppose that $n \geqslant 1$ and $\mathcal{M}$ is a subcategory of R-Mod, which is closed under direct summands and direct products. Then the followings are equivalent.*

(1) $\mathcal{M}$ is an n-costar subcategory and $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions;

(2) $\mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$.

Proof (1)$\Rightarrow$ (2). Since $\mathcal{M}$ is an n-costar subcategory, we have $\mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M})$ and we only need to prove that $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$. For any $N \in \mathrm{Copres}^n(\mathcal{M})$, any $\bar{M} \in \mathcal{M}$ and any extension of N by $\bar{M}$: $0 \to \bar{M} \to E \to N \to 0$, because $\bar{M} \in \mathcal{M} \subseteq \mathrm{Copres}^n(\mathcal{M})$ and $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions by (1), we get $E \in \mathrm{Copres}^n(\mathcal{M})$. According to Corollary 5.7, we know the induced sequence $0 \to \mathrm{Hom}_R(N,\bar{M}) \to \mathrm{Hom}_R(E,\bar{M}) \to \mathrm{Hom}_R(\bar{M},\bar{M}) \to 0$ is exact, which means $0 \to \bar{M} \to E \to N \to 0$ is split. Therefore, $\mathrm{Ext}^1_R(N,\bar{M}) = 0$ for any $\bar{M} \in \mathcal{M}$, then $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$.

(2)$\Rightarrow$ (1). It is easy to see that $\mathcal{M}$ is an n-costar subcategory by (2), so let us prove that $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions. For any extension $0 \to U \to V \to W \to 0$ with $U, W \in \mathrm{Copres}^n(\mathcal{M})$, by assumption $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$, we can get this sequence is exact under $\mathrm{Hom}_R(-,\mathcal{M})$. It is not difficult to prove $V \in \mathrm{Copres}^n(\mathcal{M})$ according to Proposition 5.6, which means $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions. □

The following lemma shall be used to the proof Theorem 5.13.

Proposition 5.10 *Suppose that $\mathcal{M}$ is an n-costar subcategory and $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions. Then $\mathrm{Copres}^k(\mathrm{Copres}^n(\mathcal{M})) = \mathrm{Copres}^k(\mathcal{M})$ for any $k \geqslant 1$. Especially, $\mathrm{Copres}^n(\mathcal{M})$ is closed under n-kernels.*

Proof It is easy to prove the conclusion by induction on k. □

As well known, every left R-module has an injective envelope, so it also has a left $\mathcal{I}$-approximation. Then we can get the following lemma by Theorem 4.9.

Lemma 5.11 *Suppose that n is a non-negative integer and $\mathcal{M}$ is a subcategory of R-Mod closed under direct summands. Then $\mathcal{M}$ is n-$\mathcal{I}$-cotilting subcategory with respect to cotorsion triple $(\mathcal{P}, R\text{-Mod}, \mathcal{I})$ if and only if $\mathrm{Copres}^n(\mathcal{M}) = ^{\perp_{i\geqslant 1}} \mathcal{M}$.*

By Proposition 5.6 and Theorem 5.9, we can obtain the following lemma.

Lemma 5.12 *Suppose that $\mathcal{M}$ is an n-costar subcategory such that $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions, and $0 \to U \to V \to W \to 0$ is a short exact sequence with $U, V \in \mathrm{Copres}^n(\mathcal{M})$. Then $W \in \mathrm{Copres}^n(\mathcal{M})$ if and only if $W \in^{\perp} \mathcal{M}$.*

We can now state our main result as follows.

Theorem 5.13 *Let n be a non-negative integer and $\mathcal{M}$ be a subcategory of R-Mod closed under direct summands and direct products. $\mathcal{M}$ has a class of representatives. Then the following conditions are equivalent.*

(1) $\mathcal{M}$ is an n-$\mathcal{I}$-cotilting subcategory;

(2) $\mathrm{Copres}^n(\mathcal{M}) =^{\perp_{1\leqslant i\leqslant n}} \mathcal{M}$;

(3) $\mathcal{M}$ is an n-costar subcategory with $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M})$;

(4) $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M}) = \mathrm{Copres}^{n+1}(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$.

Proof (1)⇒(2). It is clear by Lemma 5.11.

(2)⇒(3). Since $\mathcal{P} \subseteq^{\perp_{1\leqslant i\leqslant n}} \mathcal{M} = \mathrm{Copres}^n(\mathcal{M})$ by (2). We only need to prove that $\mathcal{M}$ is an n-costar subcategory. For any exact sequence $0 \to U \to M_U \to W \to 0$ with $M_U \in \mathcal{M}$ and $U \in \mathrm{Copres}^n(\mathcal{M})$. It is easy to see that $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions by (2). Then this sequence is exact under the functor $\mathrm{Hom}_R(-, \mathcal{M})$ if and only if $W \in^{\perp} \mathcal{M}$, if and only if $W \in^{\perp_{1\leqslant i\leqslant n}} \mathcal{M} = \mathrm{Copres}^n(\mathcal{M})$ since $M_U, U \in \mathrm{Copres}^n(\mathcal{M}) =^{\perp_{1\leqslant i\leqslant n}} \mathcal{M}$. According to Theorem 5.8, one can see that $\mathcal{M}$ is n-costar subcategory.

(3)⇒(4). We only need to prove that $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp} \mathcal{M}$. For any $W \in \mathrm{Copres}^n(\mathcal{M})$, there exists exact sequence $0 \to W' \to P_W \to W \to 0$ with $P_W \in \mathcal{P}$. Note that $P_W, W \in \mathrm{Copres}^n(\mathcal{M})$, we have $W' \in \mathrm{Copres}^n(\mathcal{M})$ by Lemma 5.5. Consider the following long exact sequence

$$\cdots \to \mathrm{Hom}_R(P_W, \bar{M}) \to \mathrm{Hom}_R(W', \bar{M}) \to \mathrm{Ext}^1_R(W, \bar{M}) \to \mathrm{Ext}^1(P_W, \bar{M}) \to \cdots$$

Following $P_W \in \mathcal{P}$ and Corollary 5.7, we obtain that $W \in^{\perp} \mathcal{M}$.

(4)⇒(1). According to Theorem 5.9, we know $\mathcal{M}$ is an n-costar subcategory and $\mathrm{Copres}^n(\mathcal{M})$ is closed under extensions. By Lemma 5.11, we only need to prove that $\mathrm{Copres}^n(\mathcal{M}) =^{\perp_{i\geqslant 1}} \mathcal{M}$.

Firstly, we shall prove $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp_{i\geqslant 1}} \mathcal{M}$. For any $U \in \mathrm{Copres}^n(\mathcal{M})$, there is a projective resolution $\cdots \to P_k \xrightarrow{f_k} \cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} U \to 0$. Let $U_i = ker f_i$. Then $U_i \in \mathrm{Copres}^n(\mathcal{M})$ by Lemma 5.5 and $\mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M})$, which implies that $U_i \in^{\perp} \mathcal{M}$ for every $i \geqslant 1$. Then we get $\mathrm{Ext}^j(U, \bar{M}) \cong \mathrm{Ext}^1(U_{j-1}, \bar{M}) = 0$ for any $j \geqslant 1$ and $\bar{M} \in \mathcal{M}$, which means $\mathrm{Copres}^n(\mathcal{M}) \subseteq^{\perp_{i\geqslant 1}} \mathcal{M}$.

And then we prove $^{\perp_{i\geqslant 1}}\mathcal{M} \subseteq \mathrm{Copres}^n(\mathcal{M})$. For any $V \in^{\perp_{i\geqslant 1}} \mathcal{M}$, there is a projective resolution $\cdots \to Q_k \xrightarrow{g_k} \cdots \to Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} U \to 0$, it is easy to see that $V_i \in^{\perp_{i\geqslant 1}} \mathcal{M}$. By Lemma 5.10, we know $V_n \in \mathrm{Copres}^n(\mathcal{M})$. Note that $V_{n-1} \in^{\perp_{i\geqslant 1}} \mathcal{M}$ and $Q_n \in \mathcal{P} \subseteq \mathrm{Copres}^n(\mathcal{M})$, one can see that $V_{n-1} \in \mathrm{Copres}^n(\mathcal{M})$ by Lemma 5.11. Repeat the process and continue, it is easy to get $V \in \mathrm{Copres}^n(\mathcal{M})$. Thus $^{\perp_{i\geqslant 1}}\mathcal{M} \subseteq \mathrm{Copres}^n(\mathcal{M})$. □

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