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# Double Constructions of Frobenius Hom-Algebras and Connes Cocycles

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**Abstract** In this paper, we give an explicit and systematic study on the double constructions of Frobenius Hom-algebras and introduce the close relations between $\mathcal{O}$-operators and Hom-dendriform algebras. Furthermore, we study the double constructions of Connes cocycles in terms of Hom-dendriform algebras. Finally, we give a clear analogy between antisymmetric infinitesimal Hom-bialgebras and Hom-dendriform $D$-bialgebras.

**Keywords** Frobenius Hom-algebras; Hom-dendriform algebras; $\mathcal{O}$-operators; antisymmetric infinitesimal Hom-bialgebras; Hom-dendriform $D$-bialgebras

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## 1.Introduction

The Hom-algebra structures arose first in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures in which the Jacobi condition is twisted. The first examples of $q$-deformations, in which the derivations are replaced by $\sigma$-derivations, concerned the Witt and Virasoro algebras, the readers can see [1]. A general study and construction of Hom-Lie algebras and a more general framework bordering color and Lie superalgebras were considered in [2]. In the subclass of Hom-Lie algebras skewsymmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map. The notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted by a linear map was introduced in [3]. It turns out that the commutator bracket multiplication defined using the multiplication of a Hom-associative algebra leads naturally to a Hom-Lie algebra. This provided a different way of constructing Hom-Lie algebras. This paper [4] led to the development of the theory of Hom-Lie algebras. In recent years, Hom-associative algebras and Hom-Lie algebras have been investigated by some scholars in [3, 4] and [7–16].

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In [5], Bai considered the double construction of Frobenius algebra and the double construction of Connes cocycle, which is interpreted in terms of dendriform algebras. Both of them are equivalent to a kind of bialgebras, namely, antisymmetric infinitesimal bialgebras and dendriform $D$-bialgebras, respectively, and he showed that an antisymmetric solution of the associative Yang-Baxter equation corresponds to the antisymmetric part of a certain operator called $\mathcal{O}$-operator which gives a double construction of a Frobenius algebra, whereas a symmetric solution of the $D$-equation corresponds to the symmetric part of an $\mathcal{O}$-operator which gives a double construction of the Connes cocycle.

The Rota-Baxter operator has appeared in a wide range of areas both in mathematics and physics. Bai also introduced the extended $\mathcal{O}$-operator and studied the relation between the extended $\mathcal{O}$-operator and the associative Yang-Baxter equation in [6]. In [7], Makhlouf introduced Rota-Baxter Hom-operators and studied the relation between Hom-dendriform algebras and Rota-Baxter Hom-operators. Recently, As a generalization of Rota-Baxter Hom-operators, Hom-$\mathcal{O}$-operator also has a close relation with the associative Hom-Yang-Baxter equation, which we can refer to [8].

In [9], Yau introduced the definition of an infinitesimal Hom-bialgebra and studied the relationship between infinitesimal Hom-bialgebras and Hom-Lie bialgebras introduced by [10]. The main purpose of this paper is to investigate the above mentioned objects in the sense of Hom-setting. The paper is organized as follows. In Section 3, we give some related definitions of Hom-associative algebras. Then we give an explicit and systematic study on the double constructions of Frobenius Hom-algebras. In Section 4, we malnly discuss the close relations between $\mathcal{O}$-operators and Hom-dendriform algebras. In Section 5, we study the double constructions of Connes cocycles in terms of Hom-dendriform algebras. In Section 6, we give a clear analogy between antisymmetric infinitesimal Hom-bialgebras and Hom-dendriform $D$-bialgebras.

## 2. Preliminary

Throughout this paper we work over field $k$ unless otherwise specified and all algebras are finite-dimensional. In this section, we will recall from [3] and [11] the basic definitions and results on Hom-associative algebras.

**Definition 2.1** *A Hom-associative algebra is a triple $(A,\mu,\alpha)$ where $\alpha: A\longrightarrow A$ and $\mu: A\otimes A\longrightarrow A$ are linear maps, with notation $\mu(a\otimes b)=ab$ such that for any $a,b,c\in A$,*

$$\alpha(ab)=\alpha(a)\alpha(b),\quad \alpha(a)(bc)=(ab)\alpha(c).$$

A linear map $f:(A,\mu_A,\alpha_A)\longrightarrow(B,\mu_B,\alpha_B)$ is called a homomorphism of Hom-algebra if $\alpha_B f=f\alpha_A$ and $f\mu_A=\mu_B(f\otimes f)$.

**Definition 2.2** *A bilinear form $\mathcal{B}(\ ,\ )$ on a Hom-associative algebra $(A,\alpha)$ is invariant if*

$$\mathcal{B}(xy,\alpha(z))=\mathcal{B}(\alpha(x),yz),\ \text{for any } x,y,z\in A.$$

Let $(A, \circ)$ be an algebra with a bilinear operation $\circ : A \otimes A \to A$.

(a) Let $L_\circ(x)$ and $R_\circ(x)$ denote the left and right multiplication operator respectively, that is, $L_\circ(x)y = R_\circ(y)x = x \circ y$ for any $x, y \in A$. We also simply denote them by $L(x)$ and $R(x)$ respectively without confusion.

(b) Let $r = \sum_i x_i \otimes y_i \in A \otimes A$. Set

$$r_{12} = \sum_i x_i \otimes y_i \otimes 1,\ r_{13} = \sum_i x_i \otimes 1 \otimes y_i,\ r_{23} = \sum_i 1 \otimes x_i \otimes y_i,$$

where 1 is the unit, if $(A, \circ)$ is unital or a symbol playing a similar role to the unit for the non-unital cases. The operation between two $r$s is given in an obvious way. For example,

$$r_{12}r_{13} = \sum_{i,j} x_i \circ x_j \otimes y_i \otimes y_j, \quad r_{13}r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i \circ y_j,$$

$$r_{23}r_{12} = \sum_{i,j} x_i \otimes x_j \circ y_i \otimes y_j.$$

(c) Let $\sigma : A \otimes A \to A \otimes A$ be the exchanging operator defined by

$$\sigma(x \otimes y) = y \otimes x, \quad \forall x, y \in A.$$

(d) Let $A_1, A_2$ be two vector spaces and $T : A_1 \to A_2$ be a linear map. Denote the dual (linear) map by $T^* : A_2^* \to A_1^*$ defined by

$$\langle x_1, T^*(x_2^*)\rangle = \langle T(x_1), x_2^*\rangle, \quad \forall x_1 \in A_1, x_2^* \in A_2^*.$$

## 3. The double constructions of Frobenius Hom-algebras

In this section, we give some definitions of bimodules of Hom-associative algebras, matched pairs of Hom-associative algebras, Frobenius Hom-algebras and antisymmetric infinitesimal Hom-bialgebras. We give an explicit and systematic study on the double constructions of Frobenius Hom-algebras.

**Definition 3.1** *Let $(A, \alpha)$ be a Hom-associative algebra and $(V, \mu)$ a vector space. Let $l, r : A \to gl(V)$ be two linear maps, $(l, r, V, \mu)$ is called a bimodule of $(A, \alpha)$ if*

$$\mu(l(x)) = l(\alpha(x)) \circ \mu, \quad \mu(r(x)) = r(\alpha(x)) \circ \mu,$$

$$l(xy) \circ \mu = l(\alpha(x)) \circ l(y), \quad r(xy) \circ \mu = r(\alpha(x)) \circ r(y),$$

$$l(x) \circ r(y) = r(y) \circ l(x),$$

*for all $x, y \in A$ and $v \in V$.*

It is easy to check that $(l, r, V, \mu)$ is a bimodule of $(A, \alpha)$ if and only if the direct sum $A \oplus V$ of vector spaces is turned into a Hom-associative algebra by defining multiplication as follows

$$(x_1 + v_1) * (x_2 + v_2) = x_1x_2 + (l(x_1)v_2 + r(x_2)v_1),$$

for all $x_1, x_2 \in A$ and $v_1, v_2 \in V$. We denote it by $A \ltimes V$.

Let $(l, r, V, \mu)$ be a bimodule of a Hom-associative algebra $(A, \alpha)$, define by $l^*$, $r^* : A \to gl(V^*)$

$$\langle l^*(x)u^*, v\rangle = -\langle u^*, l(x)v\rangle, \quad \langle r^*(x)u^*, v\rangle = -\langle u^*, r(x)v\rangle,$$

for any $x \in A, u^* \in V^*, v \in V$.

$(l^*, r^*, V^*, \mu^*)$ is not a bimodule of $(A, \alpha)$ on $V^*$ with respect to $A^*$ in general. Follow the approach of [15], we have

**Lemma 3.2** *(1) Let $(l, r, V, \mu)$ be a bimodule of a Hom-associative algebra $(A, \alpha)$, if the following conditions hold:*

$$l(x) \circ \mu = \mu \circ l(\alpha(x)), \quad r(x) \circ \mu = \mu \circ r(\alpha(x)),$$

$$\mu \circ l(xy) = l(x) \circ l(\alpha(y)), \quad \mu \circ r(xy) = r(x) \circ r(\alpha(y)),$$

$$l(x) \circ r(y) = r(y) \circ l(x).$$

*Then $(l^*, r^*, V^*, \mu^*)$ is a bimodule of $(A, \alpha)$.*

*(2) $(l, 0, V), (0, r, V), (r^*, 0, V^*)$ and $(0, l^*, V^*)$ are bimodules of $(A, \alpha)$.*

**Theorem 3.3** *Let $(A, \cdot, \alpha)$ and $(B, \circ, \beta)$ be two Hom-associative algebras. If there are linear maps $l_A, r_A : A \to gl(B)$ and $l_B, r_B : B \to gl(A)$ such that $(l_A, r_A)$ is a bimodule of $(A, \alpha)$ and $(l_B, r_B)$ is a bimodule of $(B, \beta)$ and they satisfy the following equations*

$$\beta(l_A(x)b) = l_A(\alpha(x))\beta(b), \quad \beta(r_A(b)x) = r_A(\alpha(x))\beta(b), \tag{3.1}$$

$$\alpha(l_B(b)x) = l_B(\beta(b))\alpha(x), \quad \alpha(r_B(b)x) = r_B(\beta(b))\alpha(x), \tag{3.2}$$

$$l_A(\alpha(x))(a \circ b) = (l_A(x)a) \circ \beta(b) + l_A(r_B(a)x)\beta(b), \tag{3.3}$$

$$r_A(\alpha(x))(a \circ b) = \beta(a) \circ (r_A(x)b) + r_A(l_B(b)x)\beta(a), \tag{3.4}$$

$$l_B(\beta(a))(x \cdot y) = (l_B(a)x) \cdot \alpha(y) + l_B(r_A(x)a)\alpha(y), \tag{3.5}$$

$$r_B(\beta(a))(x \cdot y) = \alpha(x) \cdot (r_B(a)y) + r_B(l_A(y)a)\alpha(x), \tag{3.6}$$

$$l_A(l_B(a)x)\beta(b) + (r_A(x)a) \circ \beta(b) - r_A(r_B(b)x)\beta(a) - \beta(a) \circ (l_A(x)b) = 0, \tag{3.7}$$

$$l_B(l_A(x)a)\alpha(y) + (r_B(a)x) \cdot \alpha(y) - r_B(r_A(y)a)\alpha(x) - \alpha(x) \cdot (l_B(a)y) = 0, \tag{3.8}$$

*for any $x, y \in A$, $a, b \in B$. Then there exists a Hom-associative algebra structure "$*$" on the vector space $(A \oplus B, \alpha + \beta)$ given by*

$$(x + a) * (y + b) = x \cdot y + l_B(a)y + r_B(b)x + a \circ b + l_A(x)b + r_A(y)a, \tag{3.9}$$

$$(\alpha + \beta)(x + a) = \alpha(x) + \beta(a), \tag{3.10}$$

*for any $x, y \in A$ and $a, b \in B$. It is denoted by $(A \bowtie B, \alpha + \beta)$. Moreover, every Hom-associative algebra which is the direct sum of the underlying vector spaces of two Hom-subalgebras can be obtained from the above way.*

**Proof** For any $x, y, z \in A$ and $a, b, c \in B$, we get

$$(\alpha + \beta)((x + a) * (y + b)) = (\alpha + \beta)(x \cdot y + l_B(a)y + r_B(b)x + a \circ b + l_A(x)b + r_A(y)a)$$
$$= \alpha(x \cdot y + l_B(a)y + r_B(b)x) + \beta(a \circ b + l_A(x)b + r_A(y)a)$$

$$= \alpha(x\cdot y) + l_B(\beta(a))\alpha(y) + r_B(\beta(b))\alpha(x) + \beta(a\circ b) + l_A(\alpha(x))\beta(b) + r_A(\alpha(y))\beta(a)$$
$$= (\alpha+\beta)(x+a) * (\alpha+\beta)(y+b).$$

In order to check that $(A \bowtie B, \alpha+\beta)$ is a Hom-associative algebra, we have to check that

$$(\alpha+\beta)(x+a) * [(y+b) * (z+c)] = [(x+a) * (y+b)] * (\alpha+\beta)(z+c).$$

In fact, we have

$$\begin{aligned}
&(\alpha+\beta)(x+a) * [(y+b) * (z+c)]\\
&= (\alpha+\beta)(x+a)[y\cdot z + l_B(b)z + r_B(c)y + b\circ c + l_A(y)c + r_A(z)b]\\
&= (\alpha(x)+\beta(a))[y\cdot z + l_B(b)z + r_B(c)y + b\circ c + l_A(y)c + r_A(z)b]\\
&= \alpha(x)\cdot(y\cdot z) + l_B(\beta(a))(y\cdot z + l_B(b)z + r_B(c)y) + \mu_2(b\circ c + l_A(y)c +\\
&\quad r_A(z)b)\alpha(x) + \beta(a)\circ(b\circ c + l_A(y)c + r_A(z)b) + l_A(\alpha(x))(b\circ c +\\
&\quad l_A(y)c + r_A(z)b) + r_A(y\cdot z + l_B(b)z + r_B(c)y)\beta(a)\\
&= (x\cdot y + l_B(a)y + r_B(b)x)\cdot\alpha(z) + l_B(a\circ b + l_A(x)b + r_A(y)a)\alpha(z) +\\
&\quad l_A(\beta(c))(x\circ y + l_B(a)y + r_B(b)x) + (a\circ b + l_A(x)b + r_A(y)a)\circ\beta(c) +\\
&\quad l_A(x\cdot y + l_B(a)y + r_B(b)x)\beta(c) + r_A(\alpha(z))(a\circ b + l_A(x)b + r_A(y)a)\\
&= (x\cdot y + l_B(a)y + r_B(b)x + a\circ b + l_A(x)b + r_A(y)a) * (\alpha(z)+\beta(c))\\
&= [(x+a) * (y+b)] * (\alpha+\beta)(z+c).
\end{aligned}$$

And this proof is completed. □

**Definition 3.4** *Let $(A,\cdot,\alpha)$ and $(B,\circ,\beta)$ be two Hom-associative algebras. Suppose that there are linear maps $l_A, r_A : A \to gl(B)$ and $l_B, r_B : B \to gl(A)$ such that $(l_A, r_A)$ is a bimodule of $(A,\alpha)$ and $(l_B, r_B)$ is a bimodule of $(B,\beta)$. If Eqs. (3.1)–(3.8) are satisfied, then $(A, B, l_A, r_A, l_B, r_B, \alpha, \beta)$ is called a matched pair of Hom-associative algebras.*

**Definition 3.5** *A Frobenius Hom-algebra $(A,\mathcal{B},\alpha)$ is a Hom-associative algebra $(A,\alpha)$ with a nondegenerate invariant bilinear form $\mathcal{B}$. Furthermore, a Frobenius Hom-algebra $(A,\mathcal{B},\alpha)$ is symmetric if $\mathcal{B}$ is symmetric.*

**Definition 3.6** *We call $(A,\mathcal{B},\alpha)$ a double construction of Frobenius Hom-algebra if it satisfies the following conditions*

*(1) $A = A_1 \oplus A_1^*$ as the direct sum of vector spaces;*

*(2) $(A,\alpha)$ is a Hom-associative algebra and $(A_1,\alpha)$, $(A_1^*,\alpha^*)$ are Hom-associative subalgebras of $(A,\alpha)$;*

*(3) $\mathcal{B}$ is the natural symmetric bilinear form on $A_1 \oplus A_1^*$ given by*

$$\mathcal{B}(x+a^*, y+b^*) = \langle x, b^*\rangle + \langle a^*, y\rangle, \text{ for any } x, y \in A_1, a^*, b^* \in A_1^*. \tag{3.11}$$

According to [15], a bimodule $(l, r, V, \mu)$ of a Hom-associative algebra $(A,\alpha)$ is called admissible if $(l^*, r^*, V^*, \mu^*)$ is a bimodule of $(A, \alpha)$.

Let $(A, \cdot, \alpha)$ be an *admissible* Hom-associative algebra. Suppose that there is an *admissible* Hom-associative algebra structure " $\circ$ " on its dual space $A^*$. We construct a Hom-associative algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of $A$ and $A^*$ such that $(A, \cdot, \alpha)$ and $(A^*, \circ, \alpha^*)$ are Hom-associative subalgebras and the symmetric bilinear form on $A \oplus A^*$ given by Eq.(3.11) is invariant. That is, $(A \oplus A^*, \mathcal{B}, \alpha + \alpha^*)$ is a symmetric Frobenius Hom-algebra. Such a construction is called a double construction of Frobenius Hom-algebra associated to $(A, \cdot, \alpha)$ and $(A^*, \circ, \alpha^*)$ and we denote it by $(A \bowtie A^*, \mathcal{B}, \alpha + \alpha^*)$.

**Theorem 3.7** *Let $(A, \cdot, \alpha)$ be an admissible Hom-associative algebra. Suppose that there is an admissible Hom-associative algebra structure " $\circ$ " on its dual space $A^*$. Then there is a double construction of Frobenius Hom-algebra associated to $(A, \cdot, \alpha)$ and $(A^*, \circ, \alpha^*)$ if and only if $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras.*

**Proof** If $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras, for any $x, y, z \in A$ and $a^*, b^*, c^* \in A^*$, using Eq. (3.9), we have

$$\begin{aligned}
&\mathcal{B}((x + a^*)(y + b^*), \alpha(z) + \alpha^*(c^*)) \\
&= \mathcal{B}(x \cdot y + R_\cdot^*(a^*)y + L_\cdot^*(b^*)y + a^* \circ b^* + R_\circ^*(x)b^* + L_\circ^*(y)a^*, \alpha(z) + \alpha^*(c^*)) \\
&= \langle x \cdot y + R_\cdot^*(a^*)y + L_\cdot^*(b^*)y, \alpha^*(c^*)\rangle + \langle a^* \circ b^* + R_\circ^*(x)b^* + L_\circ^*(y)a^*, \alpha(z)\rangle \\
&= \langle \alpha(x), b^* \circ c^* + R_\circ^*(y)c^* + L_\circ^*(z)b^*\rangle + \langle \alpha^*(a^*), y \cdot z + R_\cdot^*(b^*)z + L_\cdot^*(c^*)y\rangle \\
&= \mathcal{B}(\alpha(x) + \alpha^*(a^*), (y + b^*)(z + c^*)).
\end{aligned}$$

So $\mathcal{B}$ is invariant. Conversely, we set

$$x * a^* = l_A(x)a^* + r_{A^*}(a^*)x, \quad a^* * x = l_{A^*}(a^*)x + r_A(x)a^*,$$

for any $x \in A$ and $a^* \in A^*$. Since

$$\begin{aligned}
&\langle l_A(x)a^*, \alpha(y)\rangle = \langle r_A(y)a^*, \alpha(x)\rangle = \langle y \cdot x, \alpha^*(a^*)\rangle, \\
&\langle l_{A^*}(b^*)x, \alpha^*(a^*)\rangle = \langle r_{A^*}(a^*)x, \alpha^*(b^*)\rangle = \langle a^* \circ b^*, \alpha(x)\rangle,
\end{aligned}$$

for any $x, y \in A$ and $a^*, b^* \in A^*$. Hence, $l_A = R_\cdot^*, r_A = L_\cdot^*, l_{A^*} = R_\circ^*, r_{A^*} = L_\circ^*$. Then $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras. □

**Proposition 3.8** *Let $(A, \cdot, \alpha)$ be an admissible Hom-associative algebra. Suppose that there is an admissible Hom-associative algebra structure "$\circ$" on its dual space $A^*$. Then $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras if and only if for any $x \in A$ and $a^*, b^* \in A^*$,*

$$\alpha^*(R_\cdot^*(x)b) = R_\cdot^*(\alpha(x))\alpha^*(b), \quad \alpha^*(L_\cdot^*(b)x) = L_\cdot^*(\alpha(x))\alpha^*(b), \tag{3.12}$$

$$\alpha(L_\cdot^*(b)x) = L_\cdot^*(\alpha^*(b))\alpha(x), \quad \alpha(L_\circ^*(b)x) = L_\circ^*(\alpha^*(b))\alpha(x), \tag{3.13}$$

$$R_\cdot^*(\alpha(x))(a^* \circ b^*) = R_\cdot^*(L_\circ^*(a^*)x)\alpha^*(b^*) + (R_\cdot^*(x)a^*) \circ \alpha^*(b^*), \tag{3.14}$$

$$R_\cdot^*(R_\circ^*(a^*)x)\alpha^*(b^*) + L_\cdot^*(x)a^* \circ \alpha^*(b^*) = L_\cdot^*(L_\circ^*(b^*)x)\alpha^*(a^*) + \alpha^*(a^*)(R_\cdot^*(x)b^*). \tag{3.15}$$

**Proof** Take $l_A = R_\cdot^*$, $r_A = L_\cdot^*$, $l_B = l_{A^*} = R_\circ^*$, $r_B = r_{A^*} = l_\circ^*$, then Eq. (3.1) is just Eq. (3.12), Eq. (3.2) is just Eq. (3.13), Eq. (3.3) is just Eq. (3.14) and Eq. (3.7) is just Eq. (3.15). By Lemma 3.2, it is easy to show that Eq. (3.3)$\Leftrightarrow$ Eq. (3.4) $\Leftrightarrow$ Eq. (3.5) $\Leftrightarrow$ Eq. (3.6) and Eq. (3.7)$\Leftrightarrow$ Eq. (3.8). □

Let $(A, \alpha)$ be a Hom-associative algebra. We can make $(A \otimes A, \alpha \otimes \alpha)$ into a bimodule of $(A, \alpha)$. For example, $(\alpha \otimes L\alpha, R\alpha \otimes \alpha)$ is a bimodule of $(A, \alpha)$ with

$$(\alpha \otimes L\alpha)(x)(a \otimes b) = (\alpha \otimes L(\alpha(x)))(a \otimes b) = \alpha(a) \otimes \alpha(x)b,$$
$$(R\alpha \otimes \alpha)(x)(a \otimes b) = (R(\alpha(x)) \otimes \alpha)(a \otimes b) = b\alpha(x) \otimes \alpha(b).$$

Similarly, $(L\alpha \otimes \alpha, \alpha \otimes R\alpha)$ is also a bimodule of $(A, \alpha)$. Define $\Delta : A \to A \otimes A$ by

$$\Delta(ab) = (L(\alpha(a)) \otimes \alpha)\Delta(b) + (\alpha \otimes R(\alpha(b)))\Delta(a), \quad \forall a, b \in A \tag{3.16}$$

which gives the notion of infinitesimal Hom-bialgebras [9].

**Theorem 3.9** *Let $(A, \cdot, \alpha)$ be an admissible Hom-associative algebra. Suppose that there is an admissible Hom-associative algebra structure "$\circ$" on its dual space $A^*$ given by a linear map $\Delta^* : A^* \otimes A^* \to A^*$. Then $(A, A^*, R^*, L^*, R_\circ^*, L_\circ^*, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras if and only if $\Delta : A \to A \otimes A$ satisfies*

$$\alpha^*(R_\cdot^*(x)b) = R_\cdot^*(\alpha(x))\alpha^*(b), \quad \alpha^*(L_\cdot^*(b)x) = L_\cdot^*(\alpha(x))\alpha^*(b), \tag{3.17}$$
$$\alpha(L_\cdot^*(b)x) = L_\cdot^*(\alpha^*(b))\alpha(x), \quad \alpha(L_\circ^*(b)x) = L_\circ^*(\alpha^*(b))\alpha(x), \tag{3.18}$$
$$\Delta(x \cdot y) = (\alpha \otimes L_\cdot(\alpha(x)))\Delta(y) + (R_\cdot(\alpha(y)) \otimes \alpha)\Delta(x), \tag{3.19}$$
$$(L_\cdot(\alpha(y)) \otimes \alpha - \alpha \otimes R_\cdot(\alpha(y)))\Delta(x) + \sigma[(L_\cdot(\alpha(x)) \otimes \alpha - \alpha \otimes R_\cdot(\alpha(x)))\Delta(y)] = 0. \tag{3.20}$$

**Proof** Clearly, Eqs. (3.12) and (3.13) correspond to Eqs. (3.17) and (3.18). Let $e_1, \dots, e_n$ be a basis of $A$ and $e_1^*, \dots, e_n^*$ its dual basis. Take $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$. Thus, we have $\Delta(e_k) = \sum_{k=1}^n f_{ij}^k e_i \otimes e_j$ and

$$R_\cdot^*(e_i)e_j^* = \sum_{k=1}^n c_{ki}^j e_k^*, \quad L_\cdot^*(e_i)e_j^* = \sum_{k=1}^n c_{ik}^j e_k^*,$$
$$R_\circ^*(e_i^*)e_j = \sum_{k=1}^n f_{ki}^j e_k, \quad L_\circ^*(e_i^*)e_j = \sum_{k=1}^n f_{ik}^j e_k.$$

Let $\alpha(e_i) = \sum_{s=1}^n p_s e_s$, $\alpha(e_l) = \sum_{j=1}^n w_j e_j$ and $\alpha(e_u) = \sum_{k=1}^n q_k e_k$. Hence the coefficient of $e_j \otimes e_k$ in Eq. (3.19) gives the following relation

$$\sum_{l=1}^n c_{mi}^l f_{jk}^l = \sum_{s,j,k,l} p_s w_j c_{ks}^j f_{ls}^m + q_k p_j c_{ks}^j f_{lu}^i,$$

which is precisely the relation given by the coefficient of $e_m^*$ in

$$R_\cdot^*(\alpha(e_i))(e_j^* \circ e_k^*) = R_\cdot^*(L_\circ^*(e_j^*)e_i)\alpha^*(e_k^*) + (R_\cdot^*(e_i)e_j^*) \circ \alpha^*(e_k^*).$$

Similarly, Eq. (3.20) corresponds to Eq. (3.15). □

**Definition 3.10** *Let $(A, \cdot, \alpha)$ be an admissible Hom-associative algebra. An antisymmetric infinitesimal Hom-bialgebra structure on $A$ is a linear map $\Delta : A \to A \otimes A$ such that*

*(i) $\Delta^* : A^* \otimes A^* \to A^*$ defines a Hom-associative algebra structure on $(A^*, \alpha^*)$;*

*(ii) $\Delta$ satisfies Eqs. (3.19) and (3.20).*

*We denote it by $(A, \Delta, \alpha)$ or $(A, A^*, \alpha)$.*

Combining Theorems 3.7 and 3.9, we have the following conclusion.

**Theorem 3.11** *Let $(A, \cdot, \alpha)$ and $(A^*, \circ, \alpha^*)$ be two admissible Hom-associative algebras. Then the following conditions are equivalent:*

*(1) There is a double construction of Frobenius Hom-algebra associated to $(A, \cdot, \alpha)$ and $(A^*, \circ, \alpha^*)$;*

*(2) $(A, A^*, R^*, L^*, R^*_\circ, L^*_\circ, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras;*

*(3) $(A, \Delta, \alpha)$ is an antisymmetric infinitesimal Hom-bialgebra.*

**Definition 3.12** *Let $(A, \Delta_A, \alpha_A)$ and $(B, \Delta_B, \alpha_B)$ be two antisymmetric infinitesimal Hom-bialgebras. A homomorphism of antisymmetric infinitesimal Hom-bialgebras $\varphi : A \to B$ is a homomorphism of Hom-associative algebras such that*

$$(\varphi \otimes \varphi)\Delta_A(x) = \Delta_B(\varphi(x)), \text{ for any } x \in A.$$

*An isomorphism of antisymmetric infinitesimal Hom-bialgebras is an invertible homomorphism of antisymmetric infinitesimal Hom-bialgebras.*

**Definition 3.13** *Let $(A_1 \ltimes A_1^*, \mathcal{B}_1, \alpha_1 + \alpha_1^*)$ and $(A_2 \ltimes A_2^*, \mathcal{B}_2, \alpha_2 + \alpha_2^*)$ be two double constructions of admissible Frobenius Hom-algebras. They are isomorphic if and only if there exists an isomorphism of admissible Hom-associative algebras $\varphi : A_1 \ltimes A_1^* \to A_2 \ltimes A_2^*$ such that*

$$\varphi(A_1) = A_2, \ \varphi(A_1^*) = A_2^*, \ (\alpha_2 + \alpha_2^*)\varphi = \varphi(\alpha_1 + \alpha_1^*),$$
$$\mathcal{B}_1(x, y) = \mathcal{B}_2(\varphi(x), \varphi(y)), \text{ for any } x, y \in A_1 \ltimes A_1^*.$$

**Proposition 3.14** *Two double constructions of admissible Frobenius Hom-algebras are isomorphic if and only if their corresponding antisymmetric infinitesimal Hom-bialgebras are isomorphic.*

**Proof** Similar to [5]. □

**Example 3.15** Let $(A, \alpha)$ be an admissible Hom-associative algebra. If the Hom-associative algebra structure on $A^*$ is trivial, then $(A, 0)$ is an antisymmetric infinitesimal Hom-bialgebra. Dually, if $(A, \alpha)$ is a trivial Hom-associative algebra, then the antisymmetric infinitesimal Hom-bialgebra structures on $A$ are in one-to-one correspondence with the Hom-associative algebra structures on $A^*$.

## 4. $\mathcal{O}$-operators and Hom-dendriform algebras

In this section, we recall the definition and properties of Hom-dendriform algebras from

[7]. Then we introduce the notion of an $\mathcal{O}$-operator and discuss the close relations between $\mathcal{O}$-operators and Hom-dendriform algebras. Furthermore we give the notion of bimodules of Hom-dendriform algebras, and discuss under which conditions a vector space can construct the bimodule of Hom-dendriform algebras and under which conditions the direct sum of two Hom-dendriform algebras can construct a Hom-dendriform algebra.

**Definition 4.1** ([7]) *A Hom-dendriform algebra $A$ is a vector space equipped with three bilinear operations $(\prec, \succ, \alpha)$ satisfying the following equations:*

$$\begin{aligned}
&\alpha(x \prec y) = \alpha(x) \prec \alpha(y),\\
&\alpha(x \succ y) = \alpha(x) \succ \alpha(y),\\
&(x \prec y) \prec \alpha(z) = \alpha(x) \prec (y * z),\\
&(x \succ y) \prec \alpha(z) = \alpha(x) \succ (y \prec z),\\
&(x * y) \succ \alpha(z) = \alpha(x) \succ (y \succ z),
\end{aligned}$$

*where $x * y = x \prec y + x \succ y$, for any $x, y, z \in A$.*

Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. For any $x \in A$, let $L_{\succ}(x), R_{\succ}(x)$ and $L_{\succ}(x), R_{\succ}(x)$ denote the left and right multiplication operators of $(A, \prec, \alpha)$ and $(A, \succ, \alpha)$, respectively, that is,

$$\begin{aligned}
&L_{\succ}(x)(y) = x \succ y, \quad R_{\succ}(x)(y) = y \succ x,\\
&L_{\prec}(x)(y) = x \prec y, \quad R_{\prec}(x)(y) = y \prec x,
\end{aligned}$$

for any $y \in A$.

Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Recall from [7] that we can define a Hom-associative algebra by

$$x * y = x \prec y + x \succ y, \quad \forall x, y \in A. \tag{4.1}$$

We call $(A, *, \alpha)$ an associated Hom-associative algebra of $(A, \succ, \prec, \alpha)$ and $(A, \succ, \prec, \alpha)$ is called a compatible Hom-dendriform algebra structure on the Hom-associative algebra $(A, *, \alpha)$. Moreover, $(L_{\succ}, R_{\prec})$ is a bimodule of the associated Hom-associative algebra $(A, *, \alpha)$.

**Proposition 4.2** ( [7]) *Let $(A, *, \alpha)$ be a Hom-associative algebra and "$\succ, \prec$" two bilinear products on $A$. Then $(A, \succ, \prec, \alpha)$ is a Hom-dendriform algebra if and only if Eq. (4.1) holds. Moreover, $(L_{\succ}, R_{\prec})$ is a bimodule of $(A, *, \alpha)$.*

**Definition 4.3** *Let $(A, \cdot, \alpha)$ be a Hom-associative algebra and $(l, r, V, \mu)$ a bimodule of $(A, \alpha)$. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to $(l, r, V, \mu)$ if $T$ satisfies*

$$\alpha T = T\mu, \quad T(u) \cdot T(v) = T(l(T(u))v + r(T(v)u)),$$

*for any $u, v \in V$.*

**Example 4.4** Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then the identity map $id$ is an $\mathcal{O}$-operator associated to the bimodule $(L, 0)$ and $(0, R)$.

**Example 4.5** Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. A linear map $R : A \to A$ is called a Rota-Baxter operator on $A$ of weight zero if $R$ satisfies

$$\alpha R = R\alpha, \quad R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)),$$

for any $x, y \in A$. Then a Rota-Baxter operator on $A$ is just an $\mathcal{O}$-operator associated to the bimodule $(L, R)$.

**Theorem 4.6** *Let $T : V \to A$ be an $\mathcal{O}$-operator of a Hom-associative algebra $(A, *, \alpha)$ associated to a bimodule $(l, r, V, \mu)$. Then there exists a Hom-dendriform algebra structure on $V$ given by*

$$u \succ v = l(T(u))v, \quad u \prec v = r(T(v))u,$$

*for any $u, v \in V$. Therefore, there exists an associated Hom-associative algebra structure on $V$ given by Eq. (4.1) and $T$ is a homomorphism of Hom-associative algebras. Moreover, $T(V) = \{T(v)|v \in V\} \subset A$ is a Hom-associative subalgebra of $A$ and there is an induced Hom-dendriform algebra structure on $T(V)$ given by*

$$T(u) \succ T(v) = T(u \succ v), \quad T(u) \prec T(v) = T(u \prec v),$$

*for any $u, v \in V$. Its corresponding associated Hom-associative algebra structure on $T(V)$ given by Eq. (4.1) is just the Hom-associative subalgebra structure of $A$ and $T$ is a homomorphism of Hom-dendriform algebras.*

**Proof** For any $u, v, w \in V$, by the definition of " $\prec, \succ$ " and $(l, r, V, \mu)$ is a bimodule, we have

$$\begin{aligned}(u \prec v) \prec \mu(w) &= r(T(\mu(w)))(u \prec v) = r(T(\mu(w)))r(T(v))u \\ &= r(T(w)T(v))\mu(u) = r(T(l(T(w))v + r(T(v)w)))\mu(u) \\ &= \mu(u) \prec (v \succ w) + \mu(u) \prec (v \prec w).\end{aligned}$$

Similar arguments can be applied to verify other axioms for a Hom-dendriform algebra. □

**Corollary 4.7** ([7]) *Let $(A, *, \alpha)$ be a Hom-associative algebra and $R$ a Rota-Baxter operator of weight zero on $A$. Then there exists a Hom-dendriform algebra structure on $A$ given by*

$$x \succ y = R(x) * y, \quad x \prec y = x * R(y),$$

*for any $x, y \in A$.*

**Corollary 4.8** *Let $(A, *, \alpha)$ be a Hom-associative algebra. There is a compatible Hom-dendriform algebra structure $(\succ, \prec)$ on $(A, *, \alpha)$ if and only if there exists an invertible $\mathcal{O}$-operator of $(A, *, \alpha)$.*

**Proof** If $T$ is an invertible $\mathcal{O}$-operator associated to the bimodule $(l, r, V, \mu)$, it is easy to check that $(A, \prec, \succ, \alpha)$ is a Hom-dendriform algebra structure given by

$$x \succ y = T(l(x)T^{-1}(y)), \quad x \prec y = T(r(y)T^{-1}(x)),$$

for any $x, y \in A$. Conversely, let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra and $(A, *, \alpha)$ the associated Hom-associative algebra. Then the identity map $id$ is an $\mathcal{O}$-operator associated to the bimodule $(L_{\succ}, R_{\prec})$ of $(A, *, \alpha)$. $\square$

**Definition 4.9** *Let* $(A, \succ, \prec, \alpha)$ *be a Hom-dendriform algebra,* $(V, \mu)$ *a vector space and* $l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec} : A \to gl(V)$ *four linear maps. Then* $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V, \mu)$ *is called a bimodule of* $(A, \succ, \prec, \alpha)$ *if the following equations hold,*

$$\begin{aligned}
&\mu(l_{\prec}(x)v) = l_{\prec}(\alpha(x))\mu(v), \quad \mu(l_{\succ}(x)v) = l_{\succ}(\alpha(x))\mu(v),\\
&\mu(r_{\prec}(x)v) = r_{\prec}(\alpha(x))\mu(v), \quad \mu(r_{\succ}(x)v) = r_{\succ}(\alpha(x))\mu(v),\\
&l_{\prec}(x \prec y)\mu = l_{\prec}(\alpha(x))l_{*}(y), \quad r_{\prec}(y * x)\mu = r_{\prec}(\alpha(x))r_{\prec}(y),\\
&l_{\prec}(y)r_{*}(x) = r_{\prec}(x)l_{\prec}(y), \quad l_{\prec}(x \succ y)\mu = l_{\succ}(\alpha(x))l_{*}(y),\\
&r_{\succ}(y \prec x)\mu = r_{\prec}(\alpha(x))r_{\succ}(y), \quad l_{\succ}(y)r_{\prec}(x) = r_{\prec}(x)l_{\succ}(y),\\
&l_{\succ}(x * y)\mu = l_{\succ}(\alpha(x))l_{\succ}(y), \quad r_{\succ}(y \succ x)\mu = r_{\succ}(\alpha(x))r_{*}(y),\\
&l_{\succ}(y)r_{\succ}(x) = r_{\succ}(x)l_{*}(y),
\end{aligned}$$

*where* $x * y = x \succ y + x \prec y$, $l_{*} = l_{\succ} + l_{\prec}$, $r_{*} = r_{\succ} + r_{\prec}$, *for any* $x, y \in A$.

**Proposition 4.10** *Let* $(A, \succ, \prec, \alpha)$ *be a Hom-dendriform algebra and* $(V, \mu)$ *a vector space. Then* $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V, \mu)$ *is a bimodule of* $(A, \succ, \prec, \alpha)$ *if and only if there exists a Hom-dendriform algebra structure on the direct sum* $A \oplus V$ *given by*

$$\begin{aligned}
&(\alpha + \mu)(x + u) = \alpha(x) + \mu(u),\\
&(x + u) \succ (y + v) = x \succ y + l_{\succ}(x)v + r_{\succ}(y)u,\\
&(x + u) \prec (y + v) = x \prec y + l_{\prec}(x)v + r_{\prec}(y)u,
\end{aligned}$$

*for all* $x, y \in A$ *and* $u, v \in V$. *We denote it by* $A \ltimes_{l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}} V$.

**Proof** $\Rightarrow$) For any $x, y, z \in A$ and $u, v, w \in V$, we have

$$\begin{aligned}
&(\alpha + \mu)[(x + u) \succ (y + v)] = (\alpha + \mu)(x \succ y + l_{\succ}(x)v + r_{\succ}(y)u)\\
&\quad = \alpha(x \succ y) + \mu(l_{\succ}(x)v + r_{\succ}(y)u)\\
&\quad = \alpha(x) \succ \alpha(y) + l_{\succ}(\alpha(x))\mu(v) + r_{\succ}(\alpha(y))\mu(u)\\
&\quad = (\alpha + \mu)(x + u) \succ (\alpha + \mu)(y + v),
\end{aligned}$$

and

$$\begin{aligned}
&[(x + u) \prec (y + v)] \prec (\alpha(w) + \mu(z))\\
&\quad = [x \prec y + l_{\prec}(x)v + r_{\prec}(y)u] \prec (\alpha(w) + \mu(z))\\
&\quad = (x \prec y) \prec \alpha(z) + l_{\prec}(x \prec y)\mu(u) + r_{\prec}(\alpha(w))(l_{\prec}(x)v + r_{\prec}(y)u)\\
&\quad = \alpha(x) \prec (y * z) + l_{\prec}(\alpha(x))l_{\prec}(y)u + l_{\prec}(\alpha(x))r_{\prec}(w)v + r_{\prec}(y * z)\mu(u)\\
&\quad = \alpha(x) \prec (y * z) + l_{\prec}(\alpha(x))(l_{\prec}(y)u + r_{\prec}(w)v) + r_{\prec}(y * z)\mu(u)\\
&\quad = (\alpha(x) + \mu(u)) \prec ((y + v) * (z + w)).
\end{aligned}$$

Similar arguments can be applied to verify other axioms for the Hom-dendriform algebra.

$\Leftarrow$. Clearly obtain by Definitions 4.1 and 4.9. □

**Example 4.11** Let $(A,\succ,\prec,\alpha)$ be a Hom-dendriform algebra. Then

$$(l_\succ,r_\succ,l_\prec,r_\prec,A,\alpha),\ (l_\succ,0,0,r_\prec,A,\alpha),\ (l_\succ+l_\prec,0,0,r_\prec+r_\succ,A,\alpha)$$

are bimodules of $(A,\succ,\prec,\alpha)$.

**Theorem 4.12** *Let $(A,\succ_A,\prec_A,\alpha)$ and $(B,\succ_B,\prec_B,\beta)$ be two Hom-dendriform algebras. Suppose there are linear maps $l_{\succ_A},r_{\succ_A},l_{\prec_A},r_{\prec_A}:A\to gl(B)$ and $l_{\succ_B},r_{\succ_B},l_{\prec_B},r_{\prec_B}:B\to gl(A)$ such that $(l_{\succ_A},r_{\succ_A},l_{\prec_A},r_{\prec_A})$ is a bimodule of $(A,\succ_A,\prec_A,\alpha)$ and $(l_{\succ_B},r_{\succ_B},l_{\prec_B},r_{\prec_B})$ is a bimodule of $(B,\succ_B,\prec_B,\beta)$ and they satisfy the following conditions*

$$\beta(l_{\prec_A}(x)b)=l_{\prec_A}(\alpha(x))\beta(b),\quad \beta(r_{\prec_A}(b)x)=r_{\prec_A}(\alpha(x))\beta(b),\tag{4.2}$$
$$\beta(l_{\succ_A}(x)b)=l_{\succ_A}(\alpha(x))\beta(b),\quad \beta(r_{\succ_A}(b)x)=r_{\succ_A}(\alpha(x))\beta(b),\tag{4.3}$$
$$\alpha(l_{\prec_B}(b)x)=l_{\prec_B}(\beta(b))\alpha(x),\quad \alpha(r_{\prec_B}(b)x)=r_{\prec_B}(\beta(b))\alpha(x),\tag{4.4}$$
$$\alpha(l_{\succ_B}(b)x)=l_{\succ_B}(\beta(b))\alpha(x),\quad \alpha(r_{\succ_B}(b)x)=r_{\succ_B}(\beta(b))\alpha(x),\tag{4.5}$$
$$l_{\prec_A}(l_{\prec_B}(a)x)\beta(b)+(r_{\prec_A}(x)a)\prec_B\beta(b)=\beta(a)\prec_B(l_A(x)b)+r_{\prec_A}(r_B(b)x)\beta(a),\tag{4.6}$$
$$l_{\prec_A}(\alpha(x))(a*_B b)=(l_{\prec_A}(x)a)\prec_B\beta(b)+l_{\prec_A}(r_{\prec_B}(a)x)\beta(b),\tag{4.7}$$
$$r_{\prec_A}(\alpha(x))(a\succ_B b)=r_{\succ_A}(l_{\prec_B}(b)x)\beta(a)+\beta(a)\succ_B(r_{\prec_A}(x)b),\tag{4.8}$$
$$l_{\prec_A}(l_{\succ_B}(a)x)\beta(b)+(r_{\succ_A}(x)a)\prec_B\beta(b)=\beta(a)\succ_B(l_{\prec_A}(x)b)+r_{\succ_A}(r_{\prec_B}(b)x)\beta(a),\tag{4.9}$$
$$l_{\succ_A}(\alpha(x))(a\prec_B b)=(l_{\succ_A}(x)a)\prec_B\beta(b)+l_{\prec_A}(r_{\succ_B}(a)x)\beta(b),\tag{4.10}$$
$$r_{\succ_A}(\alpha(x))(a*_B b)=r_{\succ_A}(l_{\succ_B}(b)x)\beta(a)+\beta(a)\succ_B(r_{\succ_A}(x)b),\tag{4.11}$$
$$\beta(a)\succ_B(l_{\succ_A}(x)b)+r_{\succ_A}(r_{\succ_B}(b)x)\beta(a)=l_{\succ_A}(l_B(a)x)\beta(b)+(r_A(x)a)\succ_B\beta(b),\tag{4.12}$$
$$l_{\succ_A}(\alpha(x))(a\succ_B b)=(l_A(x)a)\succ_B\beta(b)+l_{\succ_A}(r_B(a)x)\beta(b),\tag{4.13}$$
$$r_{\prec_B}(\beta(a))(x\prec_A y)=r_{\prec_B}(l_A(y)a)\alpha(x)+\alpha(x)\prec_A(r_B(a)y),\tag{4.14}$$
$$l_{\prec_B}(l_{\prec_A}(x)a)\alpha(y)+(r_{\prec_B}(a)x)\prec_A\beta(y)=\alpha(x)\prec_B(l_B(a)y)+r_{\prec_B}(r_A(y)a)\alpha(x),\tag{4.15}$$
$$l_{\prec_B}(\beta(a))(x*_A y)=(l_{\prec_B}(a)x)\prec_A\alpha(y)+l_{\prec_B}(r_{\prec_A}(x)a)\alpha(y),\tag{4.16}$$
$$r_{\prec_B}(\beta(a))(x\succ_A y)=r_{\succ_B}(l_{\prec_A}(y)a)\alpha(x)+\alpha(x)\succ_A(r_{\prec_B}(a)y),\tag{4.17}$$
$$l_{\prec_B}(l_{\succ_A}(x)a)\alpha(y)+(r_{\succ_B}(a)x)\prec_A\alpha(y)=\alpha(x)\succ_A(l_{\prec_B}(a)y)+r_{\succ_B}(r_{\prec_A}(y)a)\alpha(x),\tag{4.18}$$
$$l_{\succ_B}(\beta(a))(x\prec_A y)=(l_{\succ_B}(a)x)\prec_A\alpha(y)+l_{\prec_B}(r_{\succ_A}(x)a)\alpha(y),\tag{4.19}$$
$$r_{\succ_B}(\beta(a))(x*_A y)=r_{\succ_B}(l_{\succ_a}(y)a)\alpha(x)+\alpha(x)\succ_A(r_{\succ_B}(a)y),\tag{4.20}$$
$$\alpha(x)\succ_A(l_{\succ_B}(a)y)+r_{\succ_B}(r_{\succ_A}(y)a)\alpha(x)=l_{\succ_B}(l_A(x)a)\alpha(y)+(r_B(a)x)\succ_A\alpha(y),\tag{4.21}$$
$$l_{\succ_B}(\beta(a))(x\succ_A y)=(l_B(a)x)\succ_A\alpha(y)+l_{\succ_B}(r_A(x)a)\alpha(y),\tag{4.22}$$
$$r_{\prec_A}(\alpha(x))(a\prec_B b)=\beta(a)\prec_B(r_A(x)b)+r_{\prec_A}(l_B(b)x)\beta(a),\tag{4.23}$$

*for any $x,y\in A,a,b\in B$ and $l_A=l_{\succ_A}+l_{\prec_A}$, $r_A=r_{\succ_A}+r_{\prec_A}$, $l_B=l_{\succ_B}+l_{\prec_B}$, $r_B=r_{\succ_B}+r_{\prec_B}$. Then there is a Hom-dendriform algebra structure on the direct sum $A\oplus B$ given by*

$$(x+a)\succ(y+b)=(x\succ_A y+r_{\succ_B}(b)x+l_{\succ_B}(a)y)+(l_{\succ_A}(x)b+r_{\succ_A}(y)a+a\succ_B b),$$

$$(x+a)\prec(y+b)=(x\prec_A y+r_{\prec_B}(b)x+l_{\prec_B}(a)y)+(l_{\prec_A}(x)b+r_{\prec_A}(y)a+a\prec_B b),$$

*for any $x,y\in A,a,b\in B$. We denote it by $A\bowtie^{l_{\succ_A},r_{\succ_A},l_{\prec_A},r_{\prec_A}}_{l_{\succ_B},r_{\succ_B},l_{\prec_B},r_{\prec_B}} B$.*

**Proof** Similar to Theorem 3.3. □

**Definition 4.13** *Let $(A,\succ_A,\prec_A,\alpha)$ and $(B,\succ_B,\prec_B,\beta)$ be two Hom-dendriform algebras. Suppose there are linear maps $l_{\succ_A}$, $r_{\succ_A}$, $l_{\prec_A}$, $r_{\prec_A}: A\to gl(B)$ and $l_{\succ_B}$, $r_{\succ_B}$, $l_{\prec_B}$, $r_{\prec_B}: B\to gl(A)$ such that $(l_{\succ_A},r_{\succ_A},l_{\prec_A},r_{\prec_A})$ is a bimodule of $(A,\succ_A,\prec_A,\alpha)$ and $(l_{\succ_B},r_{\succ_B},l_{\prec_B},r_{\prec_B})$ is a bimodule of $(B,\succ_B,\prec_B,\beta)$. If Eqs. (4.2)–(4.23) are satisfied, then $(A, B, l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}, l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B}, \alpha, \beta)$ is called a matched pair of Hom-dendriform algebras.*

**Corollary 4.14** *Let $(A,B,l_{\succ_A},r_{\succ_A},l_{\prec_A},r_{\prec_A},l_{\succ_B},r_{\succ_B},l_{\prec_B},r_{\prec_B},\alpha,\beta)$ be a matched pair of Hom-dendriform algebras. Then $(A,B,l_{\succ_A}+l_{\prec_A},r_{\succ_A}+r_{\prec_A},l_{\succ_B}+l_{\prec_B},r_{\succ_B}+r_{\prec_B},\alpha,\beta)$ is a matched pair of the associated Hom-associative algebras $(A,*_A,\alpha)$ and $(B,*_B,\beta)$.*

**Proof** Let $(A,*_A,\alpha)$ and $(B,*_B,\beta)$ be two associated Hom-associative algebras. For any $x,y\in A$ and $a,b\in B$. Define

$$(x+a)*(y+b)=x*_A y+l_B(a)y+r_B(b)x+a*_B b+l_A(x)b+r_A(y)a,$$
$$(\alpha+\beta)(x+a)=\alpha(x)+\beta(a)$$

where $l_A=l_{\succ_A}+l_{\prec_A},r_A=r_{\succ_A}+r_{\prec_A},l_B=l_{\succ_B}+l_{\prec_B},r_B=r_{\succ_B}+r_{\prec_B}$. By Theorem 3.3, we can obtain $(A\bowtie B,*,\alpha+\beta)$ is a Hom-associative algebras. By Definition 3.4, we know that $(A,B,l_{\succ_A}+l_{\prec_A},r_{\succ_A}+r_{\prec_A},l_{\succ_B}+l_{\prec_B},r_{\succ_B}+r_{\prec_B},\alpha,\beta)$ is a matched pair of the associated Hom-associative algebras $(A,*_A,\alpha)$ and $(B,*_B,\beta)$. □

## 5. The double constructions of Connes cocycles

In this section, we introduce the definition of a nondegenerate Connes cocycle on a Hom-associative algebra $(A,\alpha)$ and get that there exists a compatible Hom-dendriform algebra structure on $(A,\alpha)$. Moreover, we mainly discuss the double constructions of Connes cocycles in terms of Hom-dendriform algebras.

**Definition 5.1** *An antisymmetric bilinear form $\omega(\ ,\ )$ on a Hom-associative algebra $(A,\alpha)$ is a cyclic 2-cocycle in the sense of Connes if*

$$\omega(xy,\alpha(z))+\omega(yz,\alpha(x))+\omega(zx,\alpha(y))=0, \tag{5.1}$$

*for any $x,y,z\in A$. We also call $\omega$ a Connes cocycle.*

**Theorem 5.2** *Let $(A,*,\alpha)$ be a Hom-associative algebra and $\omega$ a nondegenerate Connes cocycle. Then there exists a compatible Hom-dendriform algebra structure "$\succ,\prec$" on $(A,*,\alpha)$ given by*

$$\omega(x\succ y,\alpha(z))=\omega(\alpha(y),z*x),\quad \omega(x\prec y,\alpha(z))=\omega(\alpha(x),y*z), \tag{5.2}$$

*for any $x,y,z\in A$.*

**Proof** For any $x, y \in A$, define a linear map $T: A \to A^*$ such that $T\alpha = \alpha^* T$ by $\langle T(x), y\rangle = \omega(x, y)$. Then $T$ is invertible and $T^{-1}$ is an $\mathcal{O}$-operator of the Hom-associative algebra $(A, *, \alpha)$ associated to the bimodule $(R_*^*, L_*^*)$. By Corollary 4.8. there is a compatible Hom-dendriform algebra structure " $\succ, \prec$ " on $(A, *, \alpha)$, which is given by

$$x \succ y = T^{-1} R_*^*(x) T(y), \quad x \prec y = T^{-1} L_*^*(y) T(x),$$

for any $x, y \in A$. □

**Definition 5.3** *We call $(A, \omega, \alpha)$ a double construction of Connes cocycle if it satisfies the following conditions*

*(1) $A = A_1 \oplus A_1^*$ as the direct sum of vector spaces;*

*(2) $(A, \alpha)$ is an admissible Hom-associative algebra and $(A_1, \alpha)$, $(A_1^*, \alpha^*)$ are an admissible Hom-associative subalgebras of $(A, \alpha)$;*

*(3) $\omega$ is the natural antisymmetric bilinear form on $A_1 \oplus A_1^*$ given by*

$$\omega(x + a^*, y + b^*) = -\langle x, b^*\rangle + \langle a^*, y\rangle, \text{ for any } x, y \in A_1, a^*, b^* \in A_1^*, \tag{5.3}$$

*and $\omega$ is a Connes cocycle of $(A, \alpha)$.*

Let $(A, *_A, \alpha)$ be an admissible Hom-associative algebra and suppose that there is an admissible Hom-associative algebra structure " $*_{A^*}$ " on its dual space $A^*$. We construct a Hom-associative algebra structure on $A \oplus A^*$ of the underlying vector spaces of $A$ and $A^*$ such that both $A$ and $A^*$ are subalgebras and the antisymmetric bilinear form on $A \oplus A^*$ given by Eq.(5.3) is a Connes cocycle on $A \oplus A^*$. Such a construction is called a double construction of Connes cocycle associated to $(A, *_A, \alpha)$ and $(A^*, *_{A^*}, \alpha^*)$, we denote it by $(T(A) = A \bowtie A^*, \omega, \alpha + \alpha^*)$.

**Corollary 5.4** *Let $(T(A) = A \bowtie A^*, \omega, \alpha + \alpha^*)$ be a double construction of Connes cocycle. Then there exists a compatible Hom-dendriform algebra structure " $\succ, \prec$ " on $T(A)$ given by Eq. (4.2).*

**Definition 5.5** *Let $(T(A_1) = A_1 \bowtie A_1^*, \omega_1, \alpha_1 + \alpha_1^*)$ and $(T(A_2) = A_2 \bowtie A_2^*, \omega_2, \alpha_2 + \alpha_2^*)$ be two double constructions of Connes cocycles. They are isomorphic if there exists an isomorphism of Hom-associative algebras $\varphi: T(A_1) \to T(A_2)$ satisfying the conditions*

$$\varphi(A_1) = A_2, \quad \varphi(A_1^*) = A_2^*, \quad \omega_1(x, y) = \omega_2(\varphi(x), \varphi(y)), \tag{5.4}$$

*for any $x, y \in A_1$.*

**Proposition 5.6** *Two double constructions of Connes cocycles $(T(A_1) = A_1 \bowtie A_1^*, \omega_1, \alpha_1 + \alpha_1^*)$ and $(T(A_2) = A_2 \bowtie A_2^*, \omega_2, \alpha_2 + \alpha_2^*)$ are isomorphic if and only if there exists a Hom-dendriform algebra isomorphism $\varphi: T(A_1) \to T(A_2)$ satisfying Eq. (5.3), where the Hom-dendriform algebra structures on $T(A_1)$ and $T(A_2)$ are given by Eq. (5.2), respectively.*

**Proof** Straightforward. □

**Theorem 5.7** *Let $(A, \succ_A, \prec_A, \alpha)$ be an admissible Hom-dendriform algebra and $(A, *_A, \alpha)$*

*the associated admissible Hom-associative algebra. Suppose that there is an admissible Hom-dendriform algebra structure* " $\succ_{A^*}, \prec_{A^*}$ " *on its dual space* $A^*$ *and* $(A^*, *_{A^*})$ *is the associated admissible Hom-associative algebra. Then there exists a double construction of Connes cocycle associated to* $(A, *_A, \alpha)$ *and* $(A, *_{A^*}, \alpha^*)$ *if and only if* $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ *is a matched pair of Hom-associative algebras.*

**Proof** $\Leftarrow$. If $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras, for any $x, y, z \in A$ and $a^*, b^*, c^* \in A^*$, using Eq. (5.2), we have

$$
\begin{aligned}
&\omega((x+a^*)(y+b^*), \alpha(z)+\alpha^*(c^*))\\
&= \omega(x \cdot y + R^*_{\prec_A}(a^*)y + L^*_{\succ_A}(b^*)y +\\
&\quad a^* \circ b^* + R^*_{\prec_{A^*}}(x)b^* + L^*_{\succ_{A^*}}(y)a^*, \alpha(z)+\alpha^*(c^*))\\
&= -\langle x \cdot y + R^*_{\prec_A}(a^*)y + L^*_{\succ_A}(b^*)y, \alpha^*(c^*)\rangle +\\
&\quad \langle a^* \circ b^* + R^*_{\prec_{A^*}}(x)b^* + L^*_{\succ_{A^*}}(y)a^*, \alpha(z)\rangle\\
&= -\langle \alpha(x), b^* \circ c^* + R^*_{\prec_{A^*}}(y)c^* + L^*_{\succ_{A^*}}(z)b^*\rangle +\\
&\quad \langle \alpha^*(a^*), y \cdot z + R^*_{\prec_A}(b^*)z + L^*_{\succ_A}(c^*)y\rangle.
\end{aligned}
$$

So we can prove that $\omega$ satisfies Eq. (5.1). Then there exists a double construction of Connes cocycle associated to $(A, *_A, \alpha)$ and $(A, *_{A^*}, \alpha^*)$.

$\Rightarrow$. If there exists a double construction of Connes cocycle associated to $(A, *_A, \alpha)$ and $(A, *_{A^*}, \alpha^*)$, we take

$$
x * a^* = l_A(x)a^* + r_{A^*}(a^*)x, \quad a^* * x = l_{A^*}(a^*)x + r_A(x)a^*,
$$

for any $x \in A$ and $a^* \in A^*$. Since

$$
\begin{aligned}
&\langle l_A(x)a^*, \alpha(y)\rangle = \langle r_A(y)a^*, \alpha(x)\rangle = \langle y \cdot x, \alpha^*(a^*)\rangle,\\
&\langle l_{A^*}(b^*)x, \alpha^*(a^*)\rangle = \langle r_{A^*}(a^*)x, \alpha^*(b^*)\rangle = \langle a^* \circ b^*, \alpha(x)\rangle,
\end{aligned}
$$

for any $x, y \in A$ and $a^*, b^* \in A^*$. Hence, $l_A = R^*_{\prec_A}$, $r_A = L^*_{\succ_A}$, $l_{A^*} = R^*_{\prec_{A^*}}$, $r_{A^*} = L^*_{\succ_{A^*}}$. Then $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras. □

**Theorem 5.8** *Let* $(A, \succ_A, \prec_A, \alpha)$ *be an admissible Hom-dendriform algebra and* $(A, *_A, \alpha)$ *the associated admissible Hom-associative algebra. Suppose that there is an admissible Hom-dendriform algebra structure* " $\succ_{A^*}, \prec_{A^*}$ " *on its dual space* $A^*$ *and* $(A^*, *_{A^*})$ *is the associated admissible Hom-associative algebra. Then* $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ *is a matched pair of the Hom-associative algebras if and only if*

$$
\begin{aligned}
&(A, A^*, R^*_{\prec_A} + R^*_{\prec_{A^*}}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A} + L^*_{\succ_A},\\
&\quad R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}}, \alpha, \alpha^*)
\end{aligned}
$$

*is a matched pair of Hom-dendriform algebras.*

**Proof** $\Rightarrow$. If $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ is a matched pair of the Hom-associative algebras, then $(A \bowtie^{R^*_{\prec_A}, L^*_{\succ_A}}_{R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}} A^*, \omega, \alpha + \alpha^*)$ is a double construction of Connes cocycle by Theorem 5.7. Hence there exists a compatible Hom-dendriform algebra structure given by Eq. (5.2), for any $x \in A$ and $a^* \in A^*$, other products are given by

$$\begin{aligned}
x \succ a^* &= (R^*_{\prec_A} + R^*_{\succ_A})(x)a^* - L^*_{\prec_{A^*}}(a^*)x,\\
x \prec a^* &= (L^*_{\prec_{A^*}} + L^*_{\succ_{A^*}})(a^*)x - R^*_{\succ_A}(x)a^*,\\
a^* \succ x &= (R^*_{\prec_{A^*}} + R^*_{\succ_{A^*}})(a^*)x - L^*_{\prec_A}(x)a^*,\\
a^* \prec x &= (L^*_{\prec_A} + L^*_{\succ_A})(x)a^* - R^*_{\succ_{A^*}}(a^*)x.
\end{aligned}$$

By a simple and direct computation, we get that

$$\begin{aligned}
&(A, A^*, R^*_{\prec_A} + R^*_{\prec_{A^*}}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A} + L^*_{\succ_A},\\
&R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}}, \alpha, \alpha^*)
\end{aligned}$$

is a matched pair of Hom-dendriform algebras.

$\Leftarrow$. This follows from Corollary 4.11. $\square$

**Theorem 5.9** *Let $(A, \succ_A, \prec_A, \alpha)$ be an admissible Hom-dendriform algebra whose products are given by two linear maps $\beta^*_\prec, \beta^*_\succ : A \otimes A \to A$. Suppose that there is an admissible Hom-dendriform algebra structure " $\succ_{A^*}, \prec_{A^*}$ " on its dual space $A^*$ given by two linear maps $\Delta^*_\prec, \Delta^*_\succ : A^* \otimes A^* \to A^*$. Then $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ is a matched pair of the Hom-associative algebras if and only if the following equations hold for any $x, y \in A$ and $a^*, b^* \in A^*$:*

$$\Delta_\prec(x *_A y) = (\alpha \otimes L_{\prec_A}(\alpha(x)))\Delta_\prec(y) + (R_A(\alpha(y)) \otimes \alpha)\Delta_\prec(x), \tag{5.5}$$

$$\Delta_\succ(x *_A y) = (\alpha \otimes L_A(\alpha(x)))\Delta_\succ(y) + (R_{\prec_A}(\alpha(y)) \otimes \alpha)\Delta_\succ(x), \tag{5.6}$$

$$\beta_\prec(a^* *_{A^*} b^*) = (\alpha^* \otimes L_{\prec_{A^*}}(\alpha^*(a^*)))\beta_\prec(b^*) + (R_{A^*}(\alpha^*(b^*)) \otimes \alpha^*)\beta_\prec(a^*), \tag{5.7}$$

$$\beta_\succ(a^* *_{A^*} b^*) = (\alpha^* \otimes L_{A^*}(\alpha^*(a^*)))\beta_\succ(b^*) + (R_{\prec_{A^*}}(\alpha^*(b^*)) \otimes \alpha^*)\beta_\succ(a^*), \tag{5.8}$$

$$\begin{aligned}
&(L_A(\alpha(x)) \otimes \alpha - \alpha \otimes R_{\prec_A}(\alpha(x)))\Delta_\prec(y) +\\
&\quad \sigma[(L_{\succ_y}(\alpha(y)) \otimes \alpha - \alpha \otimes R_A(\alpha(y)))\Delta_\succ(x)] = 0,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
&(L_{A^*}(\alpha^*(a^*)) \otimes \alpha^* - \alpha^* \otimes R_{\prec_{A^*}}(\alpha^*(a^*)))\beta_\prec(b^*) +\\
&\quad \sigma[(L_{\succ_{A^*}}(\alpha^*(b^*)) \otimes \alpha^* - \alpha^* \otimes R_{A^*}(\alpha^*(b^*)))\beta_\succ(a^*)] = 0,
\end{aligned} \tag{5.10}$$

$$\alpha^*(R^*_{\prec_A}(x)b) = R^*_{\prec_A}(\alpha(x))\alpha^*(b),\ \alpha^*(L^*_{\succ_A}(b)x) = L^*_{\succ_A}(\alpha(x))\alpha^*(b), \tag{5.11}$$

$$\alpha(R^*_{\prec_{A^*}}(b)x) = R^*_{\prec_{A^*}}(\alpha^*(b))\alpha(x),\ \alpha(L^*_{\succ_{A^*}}(b)x) = L^*_{\succ_{A^*}}(\alpha^*(b))\alpha(x), \tag{5.12}$$

*where $L_A = L_{\succ_A} + L_{\prec_A}$, $R_A = R_{\succ_A} + R_{\prec_A}$, $L_{A^*} = L_{\succ^*_A} + L_{\prec^*_A}$, $R_{A^*} = R_{\succ^*_A} + R_{\prec^*_A}$.*

**Proof** Clearly Eqs. (3.1) and (3.2) correspond to Eqs. (5.11) and (5.12). Let $e_1, \ldots, e_n$ be a basis of $A$ and $e^*_1, \ldots, e^*_n$ its dual basis. Set

$$e_i \succ_A e_j = \sum_{k=1}^n a^k_{ij} e_k, \quad e_i \prec_A e_j = \sum_{k=1}^n b^k_{ij} e_k,$$

$$e^*_i \succ_{A^*} e^*_j = \sum_{k=1}^n c^k_{ij} e^*_k, \quad e_i \prec_{A^*} e_j = \sum_{k=1}^n d^k_{ij} e^*_k.$$

Let $\alpha(e_i) = \sum_{s=1}^{n} p_s e_s$ and $\alpha(e_k^*) = \sum_{t=1}^{n} w_t e_t^*$. Hence the coefficient of $e_l^*$ in

$$R_{\prec_A}^*(\alpha(e_i))(e_j^* *_{A^*} e_k^*) = R_{\prec_A}^*(L_{A^*}^*(e_j^*)e_i)\alpha^*(e_k^*) + R_{\prec_A}^*(e_i)e_j^* *_{A^*} \alpha^*(e_k^*)$$

gives the following relation

$$\sum_{m,s} p_s b_{ls}^m (c_{jk}^m + d_{jk}^m) = \sum_{m,t} [w_t c_{jm}^i b_{lm}^t + w_t b_{mi}^j (c_{jt}^m + d_{jt}^m)],$$

which is precisely the relation given by the coefficient of $e_l^* \otimes e_i^*$ in

$$\beta_\prec(e_j^* *_{A^*} e_k^*) = (\alpha^* \otimes L_{\prec_{A^*}}(\alpha^*(e_j^*)))\beta_\prec(e_k^*) + (R_{A^*}(\alpha^*(e_k^*)) \otimes \alpha^*)\beta_\prec(e_j^*).$$

So Eq. (3.3) in the case $l_A = R_{\prec_A}^*$, $r_A = L_{\prec_A}^*$, $l_B = l_{A^*} = R_{\prec_{A^*}}^*$, $r_B = r_{A^*} = L_{\succ_{A^*}}^*$ is Eq. (5.7). Similarly, we have the following correspondences:

Eq. (3.4) ⇔ Eq. (5.8), Eq. (3.5) ⇔ Eq. (5.5), Eq. (3.6) ⇔ Eq. (5.6)

Eq. (3.7) ⇔ Eq. (5.10), Eq.(3.8) ⇔ Eq. (5.9).

Therefore, the conclusion holds. □

**Definition 5.10** *Let $(A, \alpha)$ be a vector space. An admissible Hom-dendriform $D$-bialgebra structure on $A$ is a set of linear maps $(\Delta_\prec, \Delta_\succ, \beta_\prec, \beta_\succ)$ such that $\Delta_\prec, \Delta_\succ : A \to A \otimes A$, $\beta_\prec, \beta_\succ : A^* \to A^* \otimes A^*$ and*

*(1) $(\Delta_\prec^*, \Delta_\succ^*) : A^* \otimes A^* \to A^*$ defines an admissible Hom-dendriform algebra structure $(\succ_{A^*}, \prec_{A^*})$ on $A^*$;*

*(2) $(\beta_\prec^*, \beta_\succ^*) : A \otimes A \to A$ defines a Hom-dendriform algebra structure $(\succ_A, \prec_A)$ on $A$;*

*(3) Eqs. (5.5)–(5.12) are satisfied.*

*We denote it by $(A, A^*)$.*

**Theorem 5.11** *Let $(A, \prec_A, \succ_A, \alpha)$ and $(A^*, \prec_{A^*}, \succ_{A^*}, \alpha^*)$ be two admissible Hom-dendriform algebras. Let $(A, *_A, \alpha)$ and $(A^*, *_{A^*}, \alpha^*)$ be the associated admissible Hom-associative algebras, respectively. Then the following conditions are equivalent:*

*(1) There is a double construction of Connes cocycle associated to $(A, *_A, \alpha)$ and $(A^*, *_{A^*}, \alpha^*)$;*

*(2) $(A, A^*, R_{\prec_A}^*, L_{\succ_A}^*, R_{\prec_{A^*}}^*, L_{\succ_{A^*}}^*, \alpha, \alpha^*)$ is a matched pair of the Hom-associative algebras;*

*(3) $(A, A^*, R_{\prec_A}^* + R_{\prec_{A^*}}^*, -L_{\prec_A}^*, -R_{\succ_A}^*, L_{\succ_A}^* + L_{\succ_A}^*, R_{\succ_{A^*}}^* + R_{\prec_{A^*}}^*, -L_{\prec_{A^*}}^*, -R_{\succ_{A^*}}^*, L_{\succ_{A^*}}^* + L_{\prec_{A^*}}^*, \alpha, \alpha^*)$ is a matched pair of Hom-dendriform algebras;*

*(4) $(A, A^*)$ is an admissible Hom-dendriform $D$-bialgebra.*

**Proof** This follows from Theorems 5.7–5.9. □

## 6. Comparison between antisymmetric infinitesimal Hom-bialgebras and Hom-dendriform $D$-bialgebras

In this section, we mainly consider the case that a Hom-dendriform $D$-bialgebra is an antisymmetric infinitesimal Hom-bialgebra.

**Theorem 6.1** *Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec, \alpha, \alpha^*)$ be an admissible Hom-dendriform $D$-bialgebra.*

*Then $(A, A^*)$ is an antisymmetric infinitesimal Hom-bialgebra if and only if the following equations hold:*

$$\alpha^*(R^*_{\prec_A}(x)b) = R^*_{\prec_A}(\alpha(x))\alpha^*(b), \quad \alpha^*(L^*_{\succ_A}(b)x) = L^*_{\succ_A}(\alpha(x))\alpha^*(b), \tag{6.1}$$

$$\alpha^*(R^*_{\succ_A}(x)b) = R^*_{\succ_A}(\alpha(x))\alpha^*(b), \quad \alpha^*(L^*_{\prec_A}(b)x) = L^*_{\prec_A}(\alpha(x))\alpha^*(b), \tag{6.2}$$

$$\alpha(R^*_{\prec_{A^*}}(b)x) = R^*_{\prec_{A^*}}(\alpha^*(b))\alpha(x), \quad \alpha(L^*_{\succ_{A^*}}(b)x) = L^*_{\succ_{A^*}}(\alpha^*(b))\alpha(x), \tag{6.3}$$

$$\alpha(R^*_{\succ_{A^*}}(b)x) = R^*_{\succ_{A^*}}(\alpha^*(b))\alpha(x), \quad \alpha(L^*_{\prec_{A^*}}(b)x) = L^*_{\prec_{A^*}}(\alpha^*(b))\alpha(x), \tag{6.4}$$

$$\langle L^*_{\prec_{A^*}}(b^*)y, L^*_{\prec_A}(x)a^*\rangle = \langle R^*_{\succ_{A^*}}(a^*)x, R^*_{\succ_A}(y)b^*\rangle, \tag{6.5}$$

$$\begin{aligned}&\langle L^*_{\prec_{A^*}}(b^*)y, R^*_{\succ_A}(x)a^*\rangle + \langle L^*_{\prec_{A^*}}(a^*)x, R^*_{\succ_A}(y)b^*\rangle \\ &\quad = \langle R^*_{\succ_{A^*}}(b^*)x, L^*_{\prec_A}(y)a^*\rangle + \langle R^*_{\succ_{A^*}}(a^*)y, L^*_{\prec_A}(x)b^*\rangle,\end{aligned} \tag{6.6}$$

*for any $x, y \in A^*$ and $a^*, b^* \in A^*$.*

**Proof** Similar to [5]. □

**Example 6.2** Let $(A, *_A, \alpha)$ be an admissible Hom-associative algebra and $\omega$ a Connes cocycle on $(A, *_A, \alpha)$. Then there exists an admissible antisymmetric infinitesimal Hom-bialgebra whose Hom-associative algebra structure on $A^*$ is given by a nondegenerate solution $r$ of associative Hom-Yang-Baxter equation introduced by [9] as follows:

$$\Delta(x) = (\alpha \otimes L(\alpha(x)) - R(\alpha(x)) \otimes \alpha)r,$$

for all $x \in A$, where $r : A^* \to A$ is given by $\omega(x, y) = \langle r^{-1}(x), y\rangle$. On the other hand, there exists a compatible Hom-dendriform algebra structure "$\succ_A, \prec_A$" on $(A, *_A, \alpha)$ given by Eq. (6.2), that is,

$$\omega(x \succ_A y, \alpha(z)) = \omega(\alpha(y), z *_A x), \quad \omega(x \prec_A y, \alpha(z)) = \omega(\alpha(x), y *_A z),$$

for all $x, y, z \in A$. Moreover, there exists a compatible Hom-dendriform algebra structure "$\succ_{A^*}$, $\prec_{A^*}$" on the Hom-associative algebra $A^*$ given by

$$a^* \succ_{A^*} b^* = r^{-1}(r(a^*) \succ_A r(b^*)), \quad a^* \prec_{A^*} b^* = r^{-1}(r(a^*) \prec_A r(b^*)),$$

for all $a^*, b^* \in A$. Furthermore, it is easy to show that

$$\begin{aligned}&L^*_{\succ_A}(x)a^* = r^{-1}(r(a^*) *_A x), \quad R^*_{\succ_A}(x)a^* = -r^{-1}(x \prec_A r(a^*)), \\ &L^*_{\prec_A}(x)a^* = -r^{-1}(r(a^*) \succ_A x), \quad R^*_{\prec_A}(x)a^* = r^{-1}(x * r(a^*)), \\ &L^*_{\succ_{A^*}}(a^*)x = x *_A r(a^*), \quad R^*_{\succ_{A^*}}(a^*) = -r(a^*) \prec_A x, \\ &L^*_{\prec_{A^*}}(a^*)x = -x \succ_A r(a^*), \quad R^*_{\prec_{A^*}}(a^*) = r(a^*) *_A x,\end{aligned}$$

for all $x \in A$ and $a^* \in A^*$. Therefore, $(A, A^*)$ is a Hom-dendriform $D$-bialgebra if and only if $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}, \alpha, \alpha^*)$ is a matched pair of Hom-associative algebras, if and only if $x *_A [y *_A z] = 0$ for any $x, y, z \in A$. In this case, by Eq. (6.2), it is equivalent to

$$\alpha(x) \succ_A (y \succ_A z) = \alpha(x) \prec_A (y \prec_A z) = \alpha(x) \succ_A (y \prec_A z) = 0,$$

for all $x, y, z \in A$. Therefore, under these conditions, Eqs. (6.5) and (6.6) hold.

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# References

[1] Naihong HU. *q-Witt algebras, q-Lie algebras, q-holomorph structure and representation.* Algebra Colloq., 1999, **6**(1): 51–70.

[2] J. HARTWIG, D. LARSSON, S. SILVESTROV. *Deformations of Lie algebras using $\sigma$-derivations.* J. Algebra, 2006, **295**(2): 314–361.

[3] A. MAKHLOUF, S. SILVESTROV. *Hom-algebra structures.* J. Gen. Lie Theory Appl., 2008, **2**(2): 51–64.

[4] Yunhe SHENG. *Representations of hom-Lie algebras.* Algebr. Represent. Theory, 2012, **15**(6): 1081–1098.

[5] Chengming BAI. *Double constructions of Frobenius algebras, Connes cocycles and their duality.* J. Noncommut. Geom., 2010, **4**(4): 475–530.

[6] Chengming BAI, Li GUO, Xiang NI. *$\mathcal{O}$-operators on associative algebras and associative Yang-Baxter equations.* Pacific J. Math., 2012, **256**(2): 257–289.

[7] A. MAKHLOUF. *Hom-Dendriform Algebras and Rota-Baxter Hom-Algebras.* World Sci. Publ., Hackensack, NJ, 2012.

[8] Yuanyuan CHEN, Liangyun ZHANG. *Hom-$\mathcal{O}$-operators and Hom-Yang-Baxter equations.* Adv. Math. Phys., 2015, Art. ID 823756, 11 pp.

[9] D. YAU. *Infinitesimal Hom-bialgebras and Hom-Lie bialgebras.* 2010, arXiv: 1001.5000.

[10] D. YAU. *The Hom-Yang-Baxter equation and Hom-Lie algebras.* J. Math. Phys., 2011, **52**(5): 053502, 19 pp.

[11] S. BENAYADI, A. MAKHLOUF. *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms.* J. Geom. Phys., 2014, **76**: 38–60.

[12] Liqiang CAI, Yunhe SHENG. *Purely Hom-Lie bialgebras.* Sci. China Math., 2018, **61**(9): 1553–1566.

[13] A. MAKHLOUF, S. SILVESTROV. *Hom-algebras and Hom-coalgebras.* J. Algebra Appl., 2010, **9**(4): 553–589.

[14] A. MAKHLOUF, D. YAU. *Rota-Baxter Hom-Lie-admissible algebras.* Comm. Algebra, 2014, **42**(3): 1231–1257.

[15] Yunhe SHENG, Chengming BAI. *A new approach to hom-Lie bialgebras.* J. Algebra, 2014, **399**(2): 232–250.

[16] D. YAU. *Hom-algebra and homology.* J. Lie Theory, 2009, **19**(2): 409–421.