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Gorenstein Hereditary and FP-Injectivity over Formal Triangular Matrix Rings

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Abstract In this paper we study Gorenstein (semiheredity) heredity, finite presentedness and FP-injectivity of modules over a formal triangular matrix ring. We provide necessary and sufficient conditions for such rings to be Gorenstein (semihereditary) hereditary and investigate when a triangular matrix ring is an n-FC ring.

Keywords Gorenstein (semihereditary) hereditary; formal triangular matrix ring; FP-injective module; n-FC ring

MR(2010) Subject Classification 16E65; 16E60; 16E30

1. Introduction

Auslander and Bridger [1] introduced the concept of Gorenstein dimension for finitely generated modules over a two-sided noetherian ring. This idea was extended by Enochs and Jenda [2] to the concepts of Gorenstein projective, Gorenstein injective and Gorenstein flat modules over an arbitrary ring, and developed Gorenstein homological algebra. Bennis and Mahdou [3] studied the global Gorenstein dimension of a ring R. In the paper [4] published recently, Mahdou and Tamekkante investigated the rings of (weak) global Gorenstein dimension at most one, which we called Gorenstein (semi) hereditary rings. More recently, Gao and Wang [5] showed that a ring R is Gorenstein semihereditary if and only if every finitely generated submodule of a projective module is Gorenstein projective, and pointed out that every Gorenstein hereditary ring is coherent.

FP-injective modules are similar to injective modules. Pinzon [6] proved that if R is a coherent ring, then every R-module has an FP-injective cover. Ding and Chen [7] introduced the concept of n-FC rings. Gorenstein modules have nice properties when the ring in question is an n-FC ring. Mao and Ding [8] proposed the concept of Gorenstein FP-injective modules, and proved that a left coherent ring R is left noetherian if and only if every FP-injective left R-module is Gorenstein FP-injective.

Formal triangular matrix rings play an important role in ring theory and the representation theory of algebras. In the paper [9], Enochs and other authors introduced Gorenstein regular

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rings and characterized when a left module over a formal triangular matrix rings is Gorenstein projective or Gorenstein injective.

The organization and the main results of the paper are as follows. In Section 2, we collect preliminary notions and results on formal triangular matrix rings that will be useful throughout the paper and we fix some notations. In Section 3, we provide necessary and sufficient conditions for such rings to be Gorenstein (semihereditary) hereditary. In Section 4, we study the FP-injective modules over a triangular matrix ring and investigate when a triangular matrix ring is an n-FC ring.

2. Preliminaries

All rings are assumed to be associative and with a nonzero identity element, and modules are assumed to be unitary. Unless otherwise stated, modules are assumed to be left modules. For any ring R, we use R-Mod to denote the category of left R-modules, and use ${}_RM$ (resp., M_R) to denote a left (resp., right) R-module.

Recall that a ring is left Gorenstein regular [9] if the classes of left modules with finite projective dimension and finite injective dimension coincide and the injective and projective finitistic left dimensions are finite.

Recall that a left R-module M is called Gorenstein projective in [10] if there exists an exact sequence $\cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$ of projective R-modules with $M \cong \mathrm{Im} d^{-1}$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module. The Gorenstein injective modules are defined dually.

Throughout the paper, we fix a formal triangular matrix ring

$$T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix},$$

where R and S are two (arbitrary but fixed) rings, and ${}_RM_S$ is an R-S-bimodule, which is a ring under componentwise addition and multiplication given by the rule:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ 0 & ss' \end{pmatrix},$$

$r, r' \in R$, $s, s' \in S$ and $m, m' \in M$. We shall adopt the well-known description of T-Mod from [11] which is afforded by the equivalence of category T-Mod with a category Ω, described below.

Let Ω denote the category whose objects are triple $(X, Y)_f$, or simply (X, Y) if f is clear, where $X \in R$-Mod, $Y \in S$-Mod and $f : M \otimes_S Y \to X$ is a map in R-Mod. If $(X, Y)_f$ and $(X', Y')_g$ are objects in Ω, the morphisms from $(X, Y)_f$ to $(X', Y')_g$ in Ω are pairs (φ_1, φ_2), where $\varphi_1 : X \to X'$ is a map in R-Mod, $\varphi_2 : Y \to Y'$ is a map in S-Mod satisfying the condition $\varphi_1 f = g(1_M \otimes \varphi_2)$, where 1_M denotes the identity map on M. The left T-module corresponding to the triple $(X,Y)_f$ is the additive group $X\oplus Y$ with the left T-action given by

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \big(rx+f(m\otimes y), sy\big).$$

Conversely, if a module ${}_TV$ is given then by using the idempotents $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and ring identifications $Te_1 \equiv R$ and $Te_2 \equiv S$, the triple $(X,Y)_f$ corresponding to ${}_TV$ is constructed, where $X = e_1V, Y = e_2V$ and $f : M \otimes e_2V \to e_1V$ is given by $f(m \otimes e_2v) = \left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right)e_2v = e_1\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right)v$. Thus, the regular module ${}_TT$ corresponds to $(R\oplus M, S)_f$, where f is the map $M\otimes_S S \to R\oplus M$ given by $f(m\otimes s) = (0, ms)$. If V is a T-module corresponding to $(X,Y)_f$, we let $\tilde{f} : Y \to \mathrm{Hom}_R(M,X)$ be given by $\tilde{f}(y)(m) = f(y\otimes m)$ for $y\in Y$, $m\in M$. Note that $\tilde{f}$ is a B-homomorphism. In a similar way, we can get right T-modules and $T_T = (R, M\oplus S)_g$, where g is the map $R\otimes_R M \to M\oplus S$ given by $g(r\otimes m) = (rm, 0)$.

Recall that the T_2-extension of a ring R is given by

$$T_2 = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$$

and every module over $T_2(R)$ is a homomorphism $\varphi : Y \to X$ of R-modules.

Lemma 2.1 ([12]) *The T-module $(X,Y)_f$ is flat (projective) if and only if the modules Y and Coker(f) are flat (projective) and $f : M\otimes_S Y \to X$ is monic.*

In particular, $(X,0)$ is flat (projective) if and only if X is a flat (projective) R-module, and $(M\otimes_S Y, Y)$ is flat (projective) if and only if Y is a flat (projective) S-module.

The following statements are useful: (i) M_S has finite flat dimension;

(ii) R is left Gorenstein regular, and ${}_RM$ has finite projective dimension.

We just refer to them by mentioning their assigned numbers. Whenever we think about Gorenstein projective T-modules, the above statements (i) and (ii) are satisfied.

Lemma 2.2 ([9]) *Suppose that both above the statements (i) and (ii) are satisfied. Then a T-module $(X,Y)_\varphi$ is Gorenstein projective if and only if the following two conditions hold:*

(i) Y and Cokerφ *are Gorenstein projective S- and R-modules, respectively;*

(ii) φ is a monomorphism.

In particular, $(X,0)$ is Gorenstein projective if and only if X is Gorenstein projective. $(M\otimes_S Y, Y)$ is Gorenstein projective if and only if Y is Gorenstein projective.

Lemma 2.3 ([13]) *(i)* $\mathrm{Ext}^i_T((X,0),(X',Y')) \cong \mathrm{Ext}^i_R(X,X')$ *for any $i\ge 0$;*

(ii) $\mathrm{Ext}^i_T((X,Y),(0,Y')) \cong \mathrm{Ext}^i_S(Y,Y')$ *for any $i\ge 0$;*

(iii) $\mathrm{Ext}^1_T((0,Y),(X,0)) \cong \mathrm{Hom}_R(M\otimes_S Y, X)$ *for any $i\ge 0$.*

Lemma 2.4 ([13]) *We have*

(i) For a left S-module Y, if $\mathrm{Ext}^i_S(Y,\mathrm{Hom}_R(M,I)) = 0$ *for any injective left R-module I, where $i\ge 1$, then* $\mathrm{Ext}^{n+1}_T((0,Y),(X,0)) \cong \mathrm{Ext}^n_R(M\otimes_S Y, X)$*, where $n\ge 0$;*

(ii) For a right R-module X, if $\mathrm{Ext}^i_{R^{op}}(X,\mathrm{Hom}_{S^{op}}(M,I)) = 0$ *for any injective right S-*

module I, where $i \geq 1$, then $\mathrm{Ext}^{n+1}_{T^{op}}((X,0),(0,Y)) \cong \mathrm{Ext}^n_{S^{op}}(X \otimes_R M, Y)$, *where* $n \geq 0$.

3. Gorenstein hereditary property

Following [4], a ring R is called left Gorenstein hereditary if every submodule of a projective module is a Gorenstein projective module. A ring R is called left *Gorenstein semihereditary* if it is left coherent and every submodule of a flat module is a Gorenstein flat module. By [4], if R is a ring with finite Gorenstein global dimension, then R is left Gorenstein hereditary if and only if every left ideal of R is a Gorenstein projective left R-module. By [5], a ring R is left Gorenstein semihereditary if and only if every finitely generated submodule of a projective module is Gorenstein projective.

Definition 3.1 *A module is said to be a Gorenstein (semihereditary) hereditary module if all its (finitely generated) submodules are Gorenstein projective modules.*

Lemma 3.2 ([12]) *Let $(X,Y)_f$ be a left T-module. Then $(X,Y)_f$ is finitely generated if and only if $X/\mathrm{Im}f$ and Y are finitely generated.*

Proposition 3.3 *Let M be a flat S-module. If the ring $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is left Gorenstein hereditary, then R and S are left Gorenstein hereditary. The converse is true provided that $(X,Y)_f$ is projective and the module ${}_R(X/\mathrm{Im}f_1)$ is projective for every submodule $Y_1 \leq Y$.*

Proof Let the ring T be left Gorenstein hereditary. We take an arbitrary submodule X of a projective R-module P. By our assumption, $(X,0)$ is a Gorenstein projective submodule of the projective T-module $(P,0)$. We obtain that X is Gorenstein projective. Therefore, R is left Gorenstein hereditary.

For any projective S-module Q, we take an arbitrary submodule Y in Q. Since M_S is flat, we know that $(M \otimes_S Y, Y)$ is a submodule of the projective T-module $(M \otimes_S Q, Q)$. Thus we get that $(M \otimes_S Y, Y)$ is Gorenstein projective. This shows that Y is a Gorenstein projective module. Thus, S is left Gorenstein hereditary.

Conversely, we take an arbitrary submodule $(X_1,Y_1)_{f_1}$ of a projective T-module $(X,Y)_f$. We know that Y and $\mathrm{Coker}(f)$ are projective modules, and $f: M \otimes_S Y \to X$ is an R-monomorphism. Since now M is flat and f is monic, we get that f_1 is monic. Since $X/\mathrm{Im}f_1$ and Y are projective modules, we have that $X_1/\mathrm{Im}f_1$ and Y_1 are Gorenstein projective modules. Thus, $(X_1,Y_1)_{f_1}$ is a Gorenstein projective module. Therefore, the ring T is left Gorenstein hereditary.□

Corollary 3.4 *Let R be left Gorenstein regular. If the T_2-extension of a ring R is Gorenstein hereditary, then R is Gorenstein hereditary.*

Proposition 3.5 *Let R and S be left Gorenstein semihereditary rings, and M be a flat S-module. If $(X,Y)_f$ is a projective T-module and the module $X/\mathrm{Im}f_1$ is a projective module for every finitely generated submodule Y_1 of Y. Then ring $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is left Gorenstein semihereditary.*

Proof Using Lemma 3.2 together with the similar arguments of Proposition 3.3, we obtain the result immediately. □

Proposition 3.6 *If $(X,Y)_f$ is a left Gorenstein hereditary module in T-Mod, then X and Y are Gorenstein hereditary, and $X/\mathrm{Im}f_1$ is Gorenstein hereditary for every submodule Y_1 of Y. The converse is true provided that M_S is flat and f is monic.*

Proof Let $(X,Y)_f$ be a Gorenstein hereditary module. We take an arbitrary submodule $(X_1,Y_1)_{f_1}$ of $(X,Y)_f$. By the assumption, $(X_1,Y_1)_{f_1}$ is Gorenstein projective. This shows that f_1 is monic, $X_1/\mathrm{Im}f_1$ and Y_1 are Gorenstein projective by Lemma 2.2, proving that $X/\mathrm{Im}f_1$ and Y are Gorenstein hereditary.

We take an arbitrary submodule X_1 in X. Then $(X_1,0)$ is a Gorenstein projective submodule of $(X,Y)_f$. Hence, X_1 is a Gorenstein projective module. Consequently, X is Gorenstein hereditary.

Conversely, let $(X_1,Y_1)_{f_1}$ be a submodule of $(X,Y)_f$. Since now M is flat and f is monic, we get that f_1 is monic. Since $X/\mathrm{Im}f_1$ is a Gorenstein hereditary module, we have that $X_1/\mathrm{Im}f_1$ is a Gorenstein projective module. Since Y is a Gorenstein hereditary module, we obtain that Y_1 is a Gorenstein projective module. Thus, $(X_1,Y_1)_{f_1}$ is a Gorenstein projective module. Consequently, $(X,Y)_f$ is a Gorenstein hereditary module. □

Proposition 3.7 *If $(X,Y)_f$ is a left Gorenstein semihereditary module in T-Mod, then X and Y are Gorenstein semihereditary, and $X/\mathrm{Im}f_1$ is Gorenstein semihereditary for every submodule Y_1 of Y. The converse is true provided that M_S is flat and f is monic.*

Proof By Lemma 3.2 and the similar arguments of Proposition 3.6, we get the result. □

Theorem 3.8 *Let $T=\begin{pmatrix} R & M \\ 0 & S\end{pmatrix}$ be a triangular matrix ring with finite Gorenstein global dimension. Then T is left Gorenstein hereditary if and only if the following conditions hold:*

(i) The rings R and S are left Gorenstein hereditary;

(ii) ${}_RM_S$ is flat as a right S-module;

(iii) M/ML is a Gorenstein hereditary R-module for any left ideal L of the ring S.

Proof We note that the ring T is left Gorenstein hereditary if and only if the left T-modules $(R,0)$ and (M,S) are Gorenstein hereditary.

Suppose T is left Gorenstein hereditary. By Proposition 3.6, the rings R and S are Gorenstein hereditary, and M is a Gorenstein hereditary R-module. In addition, for any left ideal L of the ring S, we have that the R-module M/ML is Gorenstein hereditary and the canonical homomorphism $M\otimes_S L\to ML$ is an isomorphism. The last property is equivalent to the property that M is a flat S-module.

Conversely, we assume that conditions (i)–(iii) hold. It follows from Proposition 3.6 that the left T-modules $(R,0)$ and (M,S) are Gorenstein hereditary. □

Corollary 3.9 *Let $T_2=\begin{pmatrix} R & R \\ 0 & R\end{pmatrix}$ be a ring with finite Gorenstein global dimension and R*

be left Gorenstein regular. Then the T_2-extension of a ring R is Gorenstein (semihereditary) hereditary if and only if R is Gorenstein (semihereditary) hereditary and R/L is a Gorenstein (semihereditary) hereditary module for any ideal L of the ring R.

4. FP-injectivity

Following [14], a left R-module M is called *FP-injective* (or absolutely pure, or weakly injective) if $\mathrm{Ext}_R^1(N, M) = 0$ for all finitely presented left R-modules N. The *FP-injective dimension* of M, denoted by $FP\text{-}id(M)$, is defined to be the least nonnegative integer n such that $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for all finitely presented left R-modules N. If no such n exists, set $FP\text{-}id(M) = \infty$. A ring is said to be an n-FC ring [7] if R is a left and right coherent ring with $FP\text{-}id({}_RR) \leq n$ and $FP\text{-}id(R_R) \leq n$ for some integer $n \geq 0$. A ring R is called an FC ring if R is 0-FC. Clearly every n-Gorenstein ring is n-FC, but the converse is not true in general.

Lemma 4.1 *Let $(X, Y)_f$ be a T-module.*

(i) If f is a monomorphism, then $(X, Y)_f$ is a finitely presented T-module if and only if $X/\mathrm{Im}f$ and Y are finitely presented;

(ii) If f is an epimorphism, then $(X, Y)_f$ is a finitely presented T-module if and only if $\mathrm{Ker}f$ is finitely generated and Y is finitely presented.

Proof (i) The question when f is monic is solved by [15, Corollary 3.8]. We have that $(M \otimes_S Y, Y)$ is a finitely presented T-module if and only if Y is finitely presented.

(ii) Suppose f is epic. Note that the following exact sequence

$$0 \to (\mathrm{Ker}f, 0) \to (M \otimes_S Y, Y) \to (X, Y) \to 0.$$

If $(\mathrm{Ker}f, 0)$ is finitely generated and $(M \otimes_S Y, Y)$ is finitely presented, then $(X, Y)_f$ is finitely presented by [16, 25.1(i)]. Conversely assume $(X, Y)_f$ is finitely presented, we proved that $\mathrm{Ker}f$ is finitely generated by [17, 1.2.3]. □

Proposition 4.2 *If $(X, Y)_f$ is a finitely presented Gorenstein projective T-module if and only if f is monic, $X/\mathrm{Im}f$ and Y are Gorenstein projective finitely presented.*

Proof The result is obtained by Lemmas 2.2 and 4.1 (i). □

Lemma 4.3 *If the ${}_RM_S$ is a finitely generated bimodule and ${}_SY$ is a finitely generated S-module, then $M \otimes_S Y$ is a finitely generated R-module.*

Proof Assume that m_i $(i = 1, \ldots, m)$, y_j $(j = 1, \ldots, n)$ are generators of ${}_RM_S$ and ${}_SY$, respectively, then

$$\begin{aligned} m \otimes y &= \sum_{i=1,j=1}^{m,n} m_i s_i \otimes s_j y_j = \sum_{i=1,j=1}^{m,n} m_i s_i s_j \otimes y_j \\ &= \sum_{i=1,j=1}^{m,n} \sum_{\lambda} r_\lambda m_\lambda \otimes y_j = \sum_{i=1,j=1}^{m,n} \sum_{\lambda} r_\lambda (m_\lambda \otimes y_j), \end{aligned}$$

$$s_i, s_j \in S,\ r_\lambda \in R,\ \lambda = 1, \ldots, m.$$

Thus, $M \otimes_S Y$ is a finitely generated R-module. □

Remark 4.4 *(i) By Lemma 4.1 (i), $(X,0)$ is finitely presented if and only if X is finitely presented;*

(ii) By Lemmas 4.1 (ii) and 4.3, if ${}_RM_S$ is finitely generated, then $(0,Y)$ is finitely presented if and only if Y is finitely presented.

Proposition 4.5 *Let ${}_RM_S$ be a finitely generated bimodule. If ${}_RX$ and ${}_SY$ are finitely presented, then $(X,Y)_f$ is a finitely presented T-module.*

Proof Firstly, consider the following natural short exact sequence of T-modules

$$0 \to (X,0) \to (X,Y) \to (0,Y) \to 0.$$

Then from [16, 25.1(ii)], it is enough to show that $(X,0)$ and $(0,Y)$ are finitely presented T-modules. So, the proof is completed by Remark 4.4 immediately. □

Lemma 4.6 *The following statements are true.*

(i) We have a natural isomorphism $\mathrm{Hom}_T((M\otimes_S Y', Y'), (X,Y)) \cong \mathrm{Hom}_S(Y', Y)$ for any S-module Y'. If M_S is flat, then we have $\mathrm{Ext}^i_T((M\otimes_S Y', Y'), (X,Y)) \cong \mathrm{Ext}^i_S(Y', Y)$, for any $i \geq 0$;

(ii) We have a natural isomorphism $\mathrm{Hom}_T((X,Y), (X', \mathrm{Hom}_R(M,X'))) \cong \mathrm{Hom}_R(X,X')$ for any R-module X'. If ${}_RM$ is projective, then we have $\mathrm{Ext}^i_T((X,Y), (X', \mathrm{Hom}_R(M,X'))) \cong \mathrm{Ext}^i_R(X,X')$, for any $i \geq 0$.

Proof It is easy to show the Lemma by the definition of T-modules and routine calculation. □

Remark 4.7 For any R-module X and any S-module Y, we have $\mathrm{Hom}_T((M\otimes_S Y, Y), (X,0)) = 0$ and $\mathrm{Hom}_T((0,Y), (X, \mathrm{Hom}_R(M,X))) = 0$.

Lemma 4.8 ([18]) *(i) Let R be a left coherent ring. Then $(X,Y)_f$ is a finitely presented left T-module if and only if ${}_R(X/\mathrm{Im} f)$ and ${}_SY$ are finitely presented, ${}_R(\mathrm{Ker} f)$ is finitely generated;*

(ii) Let S be a right coherent ring. Then $(X,Y)_f$ is a finitely presented right T-module if and only if $(Y/\mathrm{Im} f)_S$ and X_R are finitely presented, $(\mathrm{Ker} f)_S$ is finitely generated.

Proposition 4.9 *Let R be a left coherent ring, and ${}_RM_S$ be flat as a right S-module. If $(X,Y)_f$ is an FP-injective T-module, then X and Y are FP-injective.*

Proof Suppose $(X,Y)_f$ is FP-injective. Then we have $\mathrm{Ext}^1_T((P,Q),(X,Y)) = 0$ for all finitely presented T-modules (P,Q).

Since $\mathrm{Ext}^1_R(P,X) \cong \mathrm{Ext}^1_T((P,0),(X,Y)) = 0$ for all finitely presented R-modules P, it follows that X is FP-injective. By Lemma 4.6, we have $\mathrm{Ext}^1_S(Q,Y) \cong \mathrm{Ext}^1_T((M\otimes_S Q, Q),(X,Y)) = 0$ for all finitely presented S-modules Q. It shows that Y is FP-injective. □

Theorem 4.10 *Let $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ be an n-FC ring. Then*

(i) The rings R and S are left and right coherent rings. M/ML is locally coherent for any finitely generated left ideal L of S. $M/L'M$ is locally coherent for any finitely generated right ideal L' of R;

(ii) FP-$id(_RR) \leq n$, FP-$id(_RM) \leq n$, FP-$id(S_S) \leq n$, and FP-$id(M_S) \leq n$;

(iii) If $_RM_S$ is flat as a right S-module, then FP-$id(_SS) \leq n$;

(iv) If $_RM_S$ is flat as a left R-module, then FP-$id(R_R) \leq n$.

Proof (i) Suppose T is a left and right coherent ring. Applying [15, Theorem 4.2], we deduce that R and S are left and right coherent rings, and M/ML is locally coherent for any finitely generated left ideal L of S, and $M/L'M$ is locally coherent for any finitely generated right ideal L' of R. The converse is true provided that $_RM_S$ is flat.

(ii) By our assumption, we have FP-$id(_TT) \leq n$ and FP-$id(T_T) \leq n$. We only show that FP-$id(_RR) \leq n$ and FP-$id(_RM) \leq n$, and the other can be proved similarly. Note that we have a decomposition of left T-module $(_TT) = (R, 0) \oplus (M, S)$. Therefore, $\mathrm{Ext}_T^{n+i}((X, Y), (R, 0)) = 0$ and $\mathrm{Ext}_T^{n+i}((X, Y), (M, S)) = 0$, for any $i \geq 1$.

Since $\mathrm{Ext}_T^i((X, 0), (R, 0)) \cong \mathrm{Ext}_R^i(X, R)$, where $i \geq 0$. Then, for all finitely presented R-modules X, we have $\mathrm{Ext}_R^{n+i}(X, R) \cong \mathrm{Ext}_T^{n+i}((X, 0), (R, 0)) = 0$, where $i \geq 1$. It shows that FP-$id(_RR) \leq n$. Similarly, $\mathrm{Ext}_R^{n+i}(X, M) \cong \mathrm{Ext}_T^{n+i}((X, 0), (M, S)) = 0$ for all finitely presented R-modules X, where $i \geq 1$. It shows that FP-$id(_RM) \leq n$.

(iii) Since M_S is flat, we have $\mathrm{Ext}_T^{n+i}((M \otimes_S Y, Y), (M \otimes_S S, S)) \cong \mathrm{Ext}_S^{n+i}(Y, S)) = 0$ for all finitely presented S-modules Y by Lemma 4.6, where $i \geqq 1$. It follows that FP-$id(_SS) \leq n$.

(iv) The proof is similar to (iii). □

Theorem 4.11 $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ *is an n-FC ring if the following conditions hold:*

(i) R and S are n-FC rings;

(ii) M/ML is locally coherent for any finitely generated left ideal L of S, and $M/L'M$ is locally coherent for any finitely generated right ideal L' of R;

(iii) $_RM_S$ is flat as an R-S-bimodule;

(iv) For any $i \geq 0$, $\mathrm{Ext}_R^{n+i}(A, R) = 0$, $\mathrm{Ext}_R^{n+i}(B, M) = 0$, $\mathrm{Ext}_{S^{op}}^{n+i}(C, M) = 0$, and $\mathrm{Ext}_{S^{op}}^{n+i}(D, S) = 0$, whenever $_RA$, $_RB$, C_S, D_S are finitely generated.

Proof By the conditions (i)–(iii), we get that T is a left and right coherent ring. For FP-$id(_TT) \leq n$ and FP-$id(T_T) \leq n$, we only prove that FP-$id(_TT) \leq n$, and the other can be obtained similarly.

By Lemma 4.8, $(X, Y)_f$ is a finitely presented left T-module if and only if $X/\mathrm{Im}f$ and Y are finitely presented, and $\mathrm{Ker}f$ is finitely generated. Then we have the following two exact sequences

$$0 \longrightarrow (\mathrm{Ker}f, 0) \longrightarrow (M \otimes_S Y, Y) \longrightarrow (\mathrm{Im}f, Y) \longrightarrow 0, \tag{4.1}$$

$$0 \longrightarrow (\mathrm{Im}f, Y) \longrightarrow (X, Y) \longrightarrow (X/\mathrm{Im}f, 0) \longrightarrow 0. \tag{4.2}$$

Applying the functor $\mathrm{Hom}_T(-,(R,0))$ to the above exact sequences (4.1) and (4.2), we have two long exact sequences

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Ker}f,0),(R,0)) \longrightarrow \mathrm{Ext}_T^{n+1}((M\otimes_S Y,Y),(R,0)) \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Im}f,Y),(R,0))\\ \longrightarrow \mathrm{Ext}_T^{n}((\mathrm{Ker}f,0),(R,0)) \longrightarrow \mathrm{Ext}_T^{n}((M\otimes_S Y,Y),(R,0)) \longrightarrow \mathrm{Ext}_T^{n}((\mathrm{Im}f,Y),(R,0)) \longrightarrow \cdots\\ \longrightarrow \mathrm{Hom}_T((\mathrm{Ker}f,0),(R,0)) \longrightarrow \mathrm{Hom}_T((M\otimes_S Y,Y),(R,0)) \longrightarrow \mathrm{Hom}_T((\mathrm{Im}f,Y),(R,0)) \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Im}f,Y),(R,0)) \longrightarrow \mathrm{Ext}_T^{n+1}((X,Y),(R,0)) \longrightarrow \mathrm{Ext}_T^{n+1}((X/\mathrm{Im}f,0),(R,0)) \longrightarrow \cdots\\ \longrightarrow \mathrm{Hom}_T((\mathrm{Im}f,Y),(R,0)) \longrightarrow \mathrm{Hom}_T((X,Y),(R,0)) \longrightarrow \mathrm{Hom}_T((X/\mathrm{Im}f,0),(R,0)) \longrightarrow 0.\end{aligned}$$

Since $FP\text{-}id({}_RR) \le n$, we have

$$\mathrm{Ext}_R^{n+i}(X/\mathrm{Im}f,R) = \mathrm{Ext}_T^{n+i}((X/\mathrm{Im}f,0),(R,0)) = 0, \tag{4.3}$$

where $i \ge 1$. Since M_S is flat, we can construct a projective resolution of $(M\otimes_S Y,Y)$, and show that $\mathrm{Ext}_T^{n+i}((M\otimes_S Y,Y),(R,0)) = 0$ for any $i \ge 1$. By the condition (iv), we have $\mathrm{Ext}_R^{n+i}(\mathrm{Ker}f,R) = 0$, where $i \ge 0$. Therefore,

$$\mathrm{Ext}_T^{n+i}((\mathrm{Im}f,Y),(R,0)) = 0, \tag{4.4}$$

where $i \ge 1$. Now by (4.3) and (4.4), we get that $\mathrm{Ext}_T^{n+i}((X,Y),(R,0)) = 0$ for any $i \ge 1$.

Similarly, applying the functor $\mathrm{Hom}_T(-,(M,S))$ to the above two exact sequences (4.1) and (4.2), we have the following two long exact sequences

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Ker}f,0),(M,S)) \longrightarrow \mathrm{Ext}_T^{n+1}((M\otimes_S Y,Y),(M,S)) \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Im}f,Y),(M,S))\\ \longrightarrow \mathrm{Ext}_T^{n}((\mathrm{Ker}f,0),(M,S)) \longrightarrow \mathrm{Ext}_T^{n}((M\otimes_S Y,Y),(M,S)) \longrightarrow \mathrm{Ext}_T^{n}((\mathrm{Im}f,Y),(M,S)) \longrightarrow \cdots\\ \longrightarrow \mathrm{Hom}_T((\mathrm{Ker}f,0),(M,S)) \longrightarrow \mathrm{Hom}_T((M\otimes_S Y,Y),(M,S)) \longrightarrow \mathrm{Hom}_T((\mathrm{Im}f,Y),(M,S)) \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}_T^{n+1}((\mathrm{Im}f,Y),(M,S)) \longrightarrow \mathrm{Ext}_T^{n+1}((X,Y),(M,S)) \longrightarrow \mathrm{Ext}_T^{n+1}((X/\mathrm{Im}f,0),(M,S)) \longrightarrow \cdots\\ \longrightarrow \mathrm{Hom}_T((\mathrm{Im}f,Y),(M,S)) \longrightarrow \mathrm{Hom}_T((X,Y),(M,S)) \longrightarrow \mathrm{Hom}_T((X/\mathrm{Im}f,0),(M,S)) \longrightarrow 0.\end{aligned}$$

By the condition (iii) and Lemma 4.6, we have

$$\mathrm{Ext}_T^{n+i}((M\otimes_S Y,Y),(M\otimes_S S,S)) \cong \mathrm{Ext}_S^{n+i}(Y,S) = 0$$

for all finitely presented S-modules Y, where $i \ge 1$. Note that the condition (iv) on M yields that $\mathrm{Ext}_R^{n+i}(\mathrm{Ker}f,M) = 0$ for any $i \ge 0$, whenever ${}_R(\mathrm{Ker}f)$ is finitely generated. Thus,

$$\mathrm{Ext}_T^{n+i}((\mathrm{Im}f,Y),(M,S)) = 0, \tag{4.5}$$

where $i \ge 1$. Since $FP\text{-}id({}_RM) \le n$, we get that

$$\mathrm{Ext}_T^{n+i}((X/\mathrm{Im}f,0),(M,S)) \cong \mathrm{Ext}_R^{n+i}((X/\mathrm{Im}f,M) = 0, \tag{4.6}$$

where $i \ge 1$. Consequently, by (4.5) and (4.6), we have $\mathrm{Ext}_T^{n+i}((X,Y),(M,S)) = 0$ for any $i \ge 1$. This completes the proof. □

Lemma 4.12 ([18]) *(i) Let R be a left coherent ring, and ${}_RM_S$ be finitely presented as a left*

R-module, then a left T-module $(X,Y)_f$ is finitely presented if and only if X and Y are finitely presented;

(ii) Let S be a right coherent ring, and ${}_RM_S$ be finitely presented as a right S-module, then a right T-module $(X,Y)_f$ is finitely presented if and only if X and Y are finitely presented.

Lemma 4.13 *The following statements are true.*

(i) Let R be a left coherent ring, then $(0,Y)$ is FP-injective if and only if Y is FP-injective;

(ii) Let R be a left coherent ring, and ${}_RM$ be finitely generated and projective as a left R-module. Then $(X, \mathrm{Hom}_R(M,X))$ is FP-injective if and only if X is FP-injective;

(iii) Let R be a left coherent ring, and ${}_RM$ be finitely presented as a left R-module. If $\mathrm{Hom}_R(M\otimes_S Q, X)=0$ whenever ${}_SQ$ is finitely presented, then X is FP-injective if and only if $(X,0)$ is FP-injective.

Proof (i) It is obvious since $\mathrm{Ext}^1_S(Q,Y)\cong \mathrm{Ext}^1_T((P,Q),(0,Y))$ whenever (P,Q) is a finitely presented module.

(ii) Suppose that $(X,\mathrm{Hom}_R(M,X))$ is an FP-injective T-module. Let (P,Q) be a finitely presented module. Since $\mathrm{Ext}^1_T((P,0),(X,\mathrm{Hom}_R(M,X)))\cong \mathrm{Ext}^1_R(P,X))=0$ for all finitely presented R-modules P, we have that X is FP-injective. Conversely, if X is FP-injective, then applying the functor $\mathrm{Hom}_T(-,(X,\mathrm{Hom}_R(M,X)))$ to the short exact sequence

$$0\longrightarrow (P,0)\longrightarrow (P,Q)\longrightarrow (0,Q)\longrightarrow 0, \tag{4.7}$$

we have the following long exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}_T((0,Q),(X,\mathrm{Hom}_R(M,X))\longrightarrow \mathrm{Hom}_T((P,Q),(X,\mathrm{Hom}_R(M,X))\\ &\longrightarrow \mathrm{Hom}_T((P,0),(X,\mathrm{Hom}_R(M,X))\longrightarrow \mathrm{Ext}^1_T((0,Q),(X,\mathrm{Hom}_R(M,X))\\ &\longrightarrow \mathrm{Ext}^1_T((P,Q),(X,\mathrm{Hom}_R(M,X))\longrightarrow \mathrm{Ext}^1_T((P,0),(X,\mathrm{Hom}_R(M,X))\longrightarrow\cdots.\end{aligned}$$

As ${}_RM$ is projective, we have $\mathrm{Ext}^1_T((0,Q),(X,\mathrm{Hom}_R(M,X)))=0$ by Lemma 4.6. It induces that $(X,\mathrm{Hom}_R(M,X))$ is an FP-injective T-module.

(iii) Suppose that X is FP-injective. Similarly, applying functor $\mathrm{Hom}_T(-,(X,0))$ to the short exact sequence (4.7), we have the following long exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}_T((0,Q),(X,0))\longrightarrow \mathrm{Hom}_T((P,Q),(X,0))\longrightarrow \mathrm{Hom}_T((P,0),(X,0))\\ &\longrightarrow \mathrm{Ext}^1_T((0,Q),(X,0))\longrightarrow \mathrm{Ext}^1_T((P,Q),(X,0))\longrightarrow \mathrm{Ext}^1_T((P,0),(X,0))\longrightarrow\cdots.\end{aligned}$$

As $\mathrm{Ext}^1_T((0,Q),(X,0)\cong \mathrm{Hom}_R(M\otimes_S Q,X)$ and $\mathrm{Ext}^1_T((P,0),(X,0))\cong \mathrm{Ext}^1_R(P,X)=0$, and $\mathrm{Hom}_T((0,Q),(X,0))=0$, we obtain the following long exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}_T((P,Q),(X,0))\longrightarrow \mathrm{Hom}_T((P,0),(X,0))\\ &\longrightarrow \mathrm{Hom}_T(M\otimes_S Q,X)\longrightarrow \mathrm{Ext}^1_T((P,Q),(X,0))\longrightarrow 0.\end{aligned}$$

By the assumption that $\mathrm{Hom}_R(M\otimes_S Q,X)=0$, we get $\mathrm{Ext}^1_T((P,Q),(X,0))=0$. Conversely, it is obvious.

We have finished the proof. □

Theorem 4.14 *Assume that R is a left coherent ring and ${}_RM$ is finitely generated and projective as a left R-module. Then $(X,Y)_f$ is an FP-injective T-module if and only if X is FP-injective, $\tilde{f}$ is epic, and* $\mathrm{Ker}\tilde{f}$ is FP-injective.

Proof Assume $(X,Y)_f$ is FP-injective, then we have $\mathrm{Ext}_T^1((P,Q),(X,Y))=0$ for all finitely presented modules (P,Q).

By Lemma 4.12, we know that $(P,0)$ is finitely presented for all finitely presented R-modules P. Thus $\mathrm{Ext}_R^1(P,X)\cong \mathrm{Ext}_T^1((P,0),(X,Y))=0$. It follows that X is FP-injective. Moreover, by Lemma 4.13 (ii), we get that $(X,\mathrm{Hom}_R(M,X))$ is an FP-injective T-module. Considering the short exact sequence

$$0\to (M,0)\to (M,S)\to (0,S)\to 0,$$

and noting that (M,S) is finitely presented, we get the following long exact sequence

$$0\to \mathrm{Hom}_T((0,S),(X,Y))\to \mathrm{Hom}_T((M,S),(X,Y))\to \mathrm{Hom}_T((M,0),(X,Y))\to 0.$$

Let the homomorphism $\Phi:\mathrm{Hom}_T((M,S),(X,Y))\to \mathrm{Hom}_S(S,Y)$ be given by $\Phi(\alpha,\beta)=\beta$, where $(\alpha,\beta)\in \mathrm{Hom}_T((M,S),(X,Y))$, $\alpha=f(1_M\otimes\beta)$. From the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_T((M,S),(X,Y)) & \xrightarrow{(1_M,0)^*} & \mathrm{Hom}_T((M,0),(X,Y)) \\
\Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\cong} \\
Y\cong \mathrm{Hom}_S(S,Y) & \xrightarrow{\tilde{f}} & \mathrm{Hom}_R(M,X),
\end{array}$$

we get that $\tilde{f}$ is an epimorphism, and the short exact sequence

$$0\to (0,\mathrm{Ker}\tilde{f})\to (X,Y)\to (X,\mathrm{Hom}_R(M,X))\to 0. \tag{4.8}$$

For a finitely presented S-module Q, we apply the functor $\mathrm{Hom}_T((0,Q),-)$ to the short exact sequence (4.8). Since $(X,Y)_f$ is FP-injective, we have the following long exact sequence

$$\begin{aligned}
0&\to \mathrm{Hom}_T((0,Q),(0,\mathrm{Ker}\tilde{f}))\to \mathrm{Hom}_T((0,Q),(X,Y))\\
&\to \mathrm{Hom}_T((0,Q),(X,\mathrm{Hom}_R(M,X)))\to \mathrm{Ext}_T^1((0,Q),(0,\mathrm{Ker}\tilde{f}))\to 0.
\end{aligned}$$

Since $\mathrm{Hom}_T((0,Q),(X,\mathrm{Hom}_R(M,X)))=0$, we get that $\mathrm{Ker}\tilde{f}$ is FP-injective.

Conversely, we know that $(0,\mathrm{Ker}\tilde{f})$ and $(X,\mathrm{Hom}_R(M,X))$ are FP-injective by Lemma 4.13 (i) and (ii), respectively. Therefore, $(X,Y)_f$ is FP-injective. □

Proposition 4.15 *Assume that R is a left coherent ring, ${}_RM_S$ is finitely presented as a left R-module and ${}_RM_S$ is flat as a right S-module. Let $\mathrm{Hom}_R(M\otimes_S Q,X)=0$ for every finitely presented S-module Q. Then $(X,Y)_f$ is an FP-injective T-module if and only if X and Y are FP-injective.*

Proof Assume that $(X,Y)_f$ is FP-injective. As $(P,0)$ is finitely presented for all finitely presented R-modules P by Lemma 4.12, we get that

$$\mathrm{Ext}_R^1(P,X)\cong \mathrm{Ext}_T^1((P,0),(X,Y))=0.$$

It follows that X is FP-injective. By the assumption and Lemma 4.6 (i), we obtain that

$$\mathrm{Ext}_T^1((M\otimes_S Q, Q),(X,Y))\cong \mathrm{Ext}_S^1(Q,Y)=0$$

for all finitely presented S-modules Q. And so we deduce that Y is FP-injective. Conversely, it is obvious for sufficiency by Lemma 4.13 (i) and (iii). □

Propsiton 4.16 *Let T be a left and right coherent ring. Then FP-$id(_TT)\leq n$ and FP-$id(T_T)\leq n$ if the following conditions hold:*

(i) $\mathrm{Ext}_S^i(Y,\mathrm{Hom}_R(M,I))=0$ for a left S-module Y and any injective left R-module I, where $i\geq 1$;

(ii) $\mathrm{Ext}_{R^{op}}^i(X,\mathrm{Hom}_{S^{op}}(M,I))=0$ for a right R-module X and any injective right S-module I, where $i\geq 1$;

(iii) FP-$id(_RR)\leq n-1$, FP-$id(R_R)\leq n$, FP-$id(_SS)\leq n$, FP-$id(S_S)\leq n-1$, FP-$id(_RM)\leq n-1$, and FP-$id(M_S)\leq n-1$.

Proof We only prove that FP-$id(_TT)\leq n$. Assume that $(X,Y)_f$ is a finitely presented T-module. Now by Lemma 4.12, we know that $(X,Y)_f$ is a finitely presented T-module if and only if X and Y are finitely presented. In particular, $(M\otimes_S Y,Y)$ is finitely presented if and only if $M\otimes_S Y$ and Y are finitely presented if and only if Y is finitely presented.

Since FP-$id(_RR)\leq n-1$, we have

$$\mathrm{Ext}_T^{n+i}((X,0),(R,0))\cong \mathrm{Ext}_R^{n+i}(X,R)=0 \text{ for any } i\geq 1,$$

and

$$\mathrm{Ext}_T^{n+i+1}((0,Y),(R,0))\cong \mathrm{Ext}_R^{n+i}(M\otimes Y,R)=0 \text{ for any } i\geq 0.$$

In a word, we obtain $\mathrm{Ext}_T^{n+i}((X,Y),(R,0))=0$, where $i\geq 1$.

Since FP-$id(_RM)\leq n-1$, we have

$$\mathrm{Ext}_T^{n+i}((X,0),(M,S))\cong \mathrm{Ext}_R^{n+i}(X,M)=0 \text{ for any } i\geq 1,$$

and

$$\mathrm{Ext}_T^{n+i+1}((0,Y),(M,0))\cong \mathrm{Ext}_R^{n+i}(M\otimes Y,M)=0 \text{ for any } i\geq 0.$$

Since FP-$id(_SS)\leq n$, we get

$$\mathrm{Ext}_T^{n+i}((0,Y),(0,S))\cong \mathrm{Ext}_S^{n+i}(Y,S)=0 \text{ for any } i\geq 1.$$

It follows that $\mathrm{Ext}_T^{n+i}((X,Y),(M,S))=0$, where $i\geq 1$. Consequently, we have FP-$id(_TT)\leq n$. □

Corollary 4.17 *Let $T_2=\left(\begin{smallmatrix} R & R\\ 0 & R\end{smallmatrix}\right)$ be a left and right coherent ring. If FP-$id(_RR_R)\leq n-1$, then we have FP-$id(_{T_2}T_2)\leq n$ and FP-$id(T_{2T_2})\leq n$.*

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