Journal of Mathematical Research with Applications
Nov., 2020, Vol. 40, No. 6, pp. 609–616
DOI:10.3770/j.issn:2095-2651.2020.06.005
Http://jmre.dlut.edu.cn

The Uniqueness of Meromorphic Functions Sharing Small Functions Dealing with Multiple Values

Yang TAN[1], **Yinying KONG**[2,*]

1. *School of Applied Mathematics, Beijing Normal University, Zhuhai, Guangdong* 519085*, P. R. China;*
2. *Research Institute of Innovation Competitiveness of Guangdong, HongKong and Macao Bay Area, Guangdong University of Finance and Economics, Guangdong* 510320*, P. R. China*

Abstract In this paper, we investigate two meromorphic functions which share small functions dealing with multiple values. By constructing auxiliary functions, especially by the deep analysis of counting functions we obtain some results which extend the existing results of some scholars.

Keywords meromorphic function; small function; uniqueness; multiple value

MR(2010) Subject Classification 30D30; 30D35

1. Introduction

In this paper, we use $\mathbb{C}$ to denote the complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane. We adopt the standard notations of the Nevanlinna theory [1, 2]. Let $f(z)$ be a nonconstant meromorphic function in the complex $\mathbb{C}$. A meromorphic function $a(z)$ in $\mathbb{C}$ is called a small function with respect to $f(z)$ if $T(r, a(z)) = o(T(r, f(z)))$ as $r \to \infty$, possibly outside a set E of r of finite linear measure. We use $S(f)$ to denote the set of meromorphic functions in $\mathbb{C}$ which are small functions with respect to $f(z)$. Obviously, $S(f)$ is a field and contains $\mathbb{C}$.

Let $f(z)$ be a nonconstant meromorphic function in the complex $\mathbb{C}$, $a(z) \in S(f) \cup \{\infty\}$ and k be a positive integer or ∞, we use $\overline{N}_{k)}(r, \frac{1}{f-a})$ and $\overline{N}_{(k+1}(r, \frac{1}{f-a})$ to denote the counting function of zeros of $f(z) - a(z)$ with multiplicities $\leq k$ and $\geq k+1$ (ignoring multiplicities); $\overline{E}_{k)}(a, f)$ denotes the set of distinct zeros of $f(z) - a(z)$ with multiplicities $\leq k$. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex $\mathbb{C}$ and $a(z) \in \{S(f) \cap S(g)\} \cup \{\infty\}$, we use $\overline{N}_0(r, a, f, g)$ to denote the counting function of common zeros of $f(z) - a(z)$ and $g(z) - a(z)$ ignoring multiplicities. Let

$$\overline{N}_{12}(r, a, f, g) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - 2\overline{N}_0(r, a, f, g). \tag{1.1}$$

Received April 9, 2020; Accepted May 29, 2020
Supported by the National Natural Science Foundation of China (Grant Nos. 11871108; 11571362), Teacher Research Capacity Promotion Program of Beijing Normal University Zhuhai, Guangdong Natural Science Foundation (Grant No. 2018A030313954), Guangdong University (New Generation Information Technology) Key Field Project (Grant No. 2020ZDZX3019) and Project of Guangdong Province Innovative Team (Grant No. 2020WCXTD011).
* Corresponding author
E-mail address: shutongtan@sina.com (Yang TAN); kongcoco@hotmail.com (Yinying KONG)

By (1.1) we know that $\overline{N}_{12}(r,a,f,g)$ denotes the counting function of different zeros of $f(z)-a(z)$ and $g(z)-a(z)$ ignoring multiplicities. If $\overline{N}(r,\frac{1}{f-a})-\overline{N}_0(r,a,f,g)=0$ and $\overline{N}(r,\frac{1}{g-a})-\overline{N}_0(r,a,f,g)=0$, then we say that $f(z)$ and $g(z)$ share $a(z)$ IM; if $\overline{N}(r,\frac{1}{f-a})-\overline{N}_0(r,a,f,g)=S(r,f)$ and $\overline{N}(r,\frac{1}{g-a})-\overline{N}_0(r,a,f,g)=S(r,g)$, then we say that $f(z)$ and $g(z)$ share $a(z)$ "IM".

In 1929, Nevanlinna proved the following well-known theorems:

Theorem 1.1 ([3]) *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex $\mathbb{C}$. If they share five distinct values $a_j\in\overline{\mathbb{C}}$ $(j=1,2,\ldots,5)$ IM in the whole complex plane $\mathbb{C}$, then $f(z)\equiv g(z)$.*

Question 1 Does Theorem 1.1 hold if a_j $(j=1,2,\ldots,5)$ instead of five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$?

Dealing with this question, many mathematicians came into the study of this subject [4–7]. In 2000, Yuhua Li and Jianyong Qiao obtained the following theorem:

Theorem 1.2 ([8]) *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex $\mathbb{C}$. If they share five distinct elements $a_j(z)\in(S(f)\cup S(g))\cup\{\infty\}$ $(j=1,2,\ldots,5)$ IM in the whole complex plane $\mathbb{C}$, then $f(z)\equiv g(z)$.*

In this paper, we will investigate the uniqueness of meromorphic functions sharing small functions dealing with multiple values. We obtain some results which extend Theorem 1.2.

2. Lemmas

Now we will give some Lemmas of this paper as follows.

Lemma 2.1 ([9]) *Let $f(z)$ be a nonconstant meromorphic function and $a_j(z)$ $(j=1,2,\ldots,5)$ be five distinct elements in $S(f)\cup\{\infty\}$. Then we have*

$$(3-\varepsilon)T(r,f)\le\sum_{j=1}^{5}\overline{N}(r,\frac{1}{f-a_j})+S(r,f),$$

where ε is a sufficiently small positive number.

Lemma 2.2 *Let $f(z)$ be a nonconstant meromorphic function and $a_j(z)$ $(j=1,2,\ldots,5)$ be five distinct elements in $S(f)\cup\{\infty\}$, k_j $(j=1,2,\ldots,5)$ be five positive integers. Then we obtain*

$$\sum_{j=1}^{5}\overline{N}_{(k_j+1}(r,\frac{1}{f-a_j})\le\sum_{j=1}^{5}\frac{1}{k_j+1}T(r,f)+S(r,f).$$

Proof We notice that

$$(k_j+1)\overline{N}_{(k_j+1}(r,\frac{1}{f-a_j})\le N(r,\frac{1}{f-a_j})\le T(r,f)+S(r,f),\quad j=1,2,\ldots,5.$$

That is

$$\overline{N}_{(k_j+1}(r,\frac{1}{f-a_j})\le\frac{1}{k_j+1}T(r,f)+S(r,f).$$

So we have

$$\sum_{j=1}^{5}\overline{N}_{(k_j+1}(r,\frac{1}{f-a_j})\leq\sum_{j=1}^{5}\frac{1}{k_j+1}T(r,f)+S(r,f).$$

Then we prove this Lemma. □

Lemma 2.3 *Let $f(z)$ be a nonconstant meromorphic function and $a(z)\in S(f)$ and $a(z)\not\equiv 0$. Then we have*

$$m(r,\frac{a'f-af'}{f-a})=S(r,f),\quad m(r,\frac{a'f-af'}{f(f-a)})=S(r,f).$$

Proof We notice that

$$\frac{a'f-af'}{f-a}=a'-\frac{a(f'-a')}{f-a},\quad \frac{a'f-af'}{f(f-a)}=\frac{f'}{f}-\frac{f'-a'}{f-a}.$$

By the lemma of the logarithmic derivative we can prove Lemma 2.3. □

3. Main results

In this section, we will give the main results of this paper.

Theorem 3.1 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a_j $(j=1,2,\ldots,5)$ be five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$. If $f(z)\not\equiv g(z)$, then we have*

$$(\frac{1}{3}-\varepsilon)[T(r,f)+T(r,g)]\leq\sum_{j=1}^{5}\overline{N}_{12}(r,a_j,f,g)+S(r,f)+S(r,g).$$

Proof Firstly, we will prove the following

$$\overline{N}_0(r,a_5,f,g)\leq\sum_{j=1}^{4}\overline{N}_{12}(r,a_j,f,g)+S(r,f)+S(r,g). \tag{3.1}$$

If $\overline{N}_0(r,a_5,f,g)=S(r,f)+S(r,g)$, obviously, (3.1) holds. So we suppose that

$$\overline{N}_0(r,a_5,f,g)\neq S(r,f)+S(r,g). \tag{3.2}$$

Set

$$L(\omega)=\frac{\omega-a_1}{\omega-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}. \tag{3.3}$$

Let $F(z)=L(f(z))$, $G(z)=L(g(z))$, $b_j=L(a_j)$ $(j=1,2,3)$, $a=L(a_4)$, $b=L(a_5)$. Then from (3.3) we know that $b_1=0$, $b_2=\infty$, $b_3=1$

$$T(r,F)=T(r,f)+S(r,f),\quad T(r,G)=T(r,g)+S(r,g). \tag{3.4}$$

Since a_j $(j=1,2,\ldots,5)$ are five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$, we have that b_j $(j=1,2,3)$, a and b are five distinct elements in $\{S(F)\cap S(G)\}\cup\{\infty\}$, $a,b\not\equiv 0,1,\infty$ and $a\not\equiv b$. Noticing that $f(z)\not\equiv g(z)$, we have

$$F(z)\not\equiv G(z). \tag{3.5}$$

By (3.2) and (3.4) we obtain

$$\overline{N}_0(r,b,F,G)\neq S(r,F)+S(r,G). \tag{3.6}$$

Set

$$H=\frac{F'(a'G-aG')(F-G)}{F(F-1)G(G-a)}-\frac{G'(a'F-aF')(F-G)}{G(G-1)F(F-a)}. \tag{3.7}$$

Then we have

$$H=\frac{(F-G)Q}{F(F-1)(F-a)G(G-1)(G-a)}, \tag{3.8}$$

where

$$Q=F'(F-a)(a'G-aG')(G-1)-(a'F-aF')(F-1)G'(G-a). \tag{3.9}$$

If $H\equiv 0$, by (3.5) and (3.7) we have

$$\frac{F'(a'G-aG')}{(F-1)(G-a)}\equiv\frac{G'(a'F-aF')}{(G-1)(F-a)}. \tag{3.10}$$

If a is a constant, we notice that $a\neq 1$, then from (3.10) we obtain $F(z)\equiv G(z)$. This contradicts (3.5). So a is not a constant. Then by (3.10) we have

$$\frac{F'(a'G-aG')}{G'(a'F-aF')}-1\equiv\frac{(F-1)(G-a)}{(G-1)(F-a)}-1.$$

This is

$$\frac{a'[(F'-G')G-(F-G)G']}{G'(a'F-aF')}\equiv\frac{(1-a)(F-G)}{(G-1)(F-a)}.$$

Then we obtain

$$\frac{F'-G'}{F-G}\equiv\frac{(1-a)G'(a'F-aF')}{a'G(F-a)(G-1)}+\frac{G'}{G}. \tag{3.11}$$

From (3.6) we know that there is z_0 which is a common zero of $F-b$ and $G-b$ but neither zero nor pole of $a,a',b,b-1,b-a$. Obviously, from (3.10) we know that z_0 is a pole of the left of (3.10) but not a pole of the right of (3.10). This is a contradiction. So it must be

$$H\not\equiv 0. \tag{3.12}$$

If z_1 is a common zero of $F-b$ and $G-b$ but neither zero nor pole of $a,b,b-1,b-a$. Obviously, z_1 is a zero of $F-G$ and z_1 is not a pole of

$$\frac{Q}{F(F-1)(F-a)G(G-1)(G-a)}.$$

From (3.8) we know that z_1 is a zero of H. By (3.11) we have

$$\begin{aligned}\overline{N}_0(r,b,F,G)&\leq N(r,\frac{1}{H})+S(r,F)+S(r,G)\\&\leq m(r,H)+N(r,H)+S(r,F)+S(r,G).\end{aligned} \tag{3.13}$$

By (3.7) we obtain

$$H=\frac{F'}{F-1}\cdot\frac{a'G-aG'}{G(G-a)}-(\frac{F'}{F-1}-\frac{F'}{F})\cdot\frac{a'G-aG'}{G-a}-$$

$$(\frac{G'}{G-1} - \frac{G'}{G}) \cdot \frac{a'F - aF'}{F-a} + \frac{G'}{G-1} \cdot \frac{a'F - aF'}{F(F-a)}. \tag{3.14}$$

From (3.14) and Lemma 2.3 we have

$$m(r, H) = S(r, F) + S(r, G). \tag{3.15}$$

By (3.13) and (3.15) we obtain

$$\overline{N}_0(r, b, F, G) \leq N(r, H) + S(r, F) + S(r, G). \tag{3.16}$$

From (3.7) we know that the poles of H only possibly come from the zeros of $F, G, F-1, G-1, F-a$ and $G-a$, the poles of F, G, a. Let S_0 be the set of all zeros of $a, a-1$ and all poles of a. Set

$$A_j = \{z|F(z) - b_j(z) = 0\} \backslash S_0, \quad B_j = \{z|G(z) - b_j(z) = 0\} \backslash S_0,$$

where $b_1 = 0, b_2 = \infty, b_3 = 1, b_4 = a$. Then we know that the poles of H possibly only come from the set

$$\bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q \bigcup S_0.$$

Let

$$S_1 = \bigcup_{1 \leq p \leq 4} \{A_p \bigcap B_p\},$$

$$S_2 = \Big\{ \bigcup_{1 \leq p \leq 4} A_p \Big\} \backslash \Big\{ \bigcup_{1 \leq q \leq 4} B_q \Big\},$$

$$S_3 = \Big\{ \bigcup_{1 \leq q \leq 4} B_q \Big\} \backslash \Big\{ \bigcup_{1 \leq p \leq 4} A_p \Big\},$$

$$S_4 = \bigcup_{\substack{1 \leq p \leq 4 \\ 1 \leq q \leq 4 \\ p \neq q}} \{A_p \bigcap B_q\}.$$

So we have

$$\bigcup_{1 \leq j \leq 4} S_j = \bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q.$$

Then the poles of H possibly only come from the set $\bigcup_{1 \leq j \leq 4} S_j \bigcup S_0$. Since b_1, b_2, b_3, b_4 are four distinct elements in $\{S(F) \cap S(G)\} \cup \{\infty\}$, the contribution of S_0 to $N(r, H)$ is at most $S(r, F) + S(r, G)$. So we only need to investigate the sets of S_j $(j = 1, 2, 3, 4)$ which contribute to $N(r, H)$. Now we will divide four steps to estimate $N(r, H)$.

Case 1. Let z_{11} be a zero of F of order p_1 and G of order q_1 but neither a zero of $a, a-1$ nor a pole of a. By (3.9) we know that z_{11} is a zero of Q of order at least $p_1 + q_1 - 1$. From (3.8) we know that z_{11} is not a pole of H. Let z_{12} be a pole of F and G, z_{13} be a zero of $F-1$ and $G-1$, z_{14} be a zero of $F-a$ and $G-a$ but them be neither zeros of $a, a-1$ nor poles of a. Similarly, we also obtain that they are not poles of H.

Case 2. Let z_{21} be a zero of F but neither zero of $G, G-1, G-a, a, a-1$ nor pole of G, a. Then from (3.7) we know that z_{21} is a pole of H of order at most 1. Let z_{22} be a pole of F, z_{23}

be a zero of $F-1$, z_{24} be a zero of $F-a$ but them be neither zeros of $G, G-1, G-a, a, a-1$ nor poles of G, a. Similarly, from (3.7) we know that they are poles of H of order at most 1.

Case 3. Let z_{31} be a zero of G but neither zero of $F, F-1, F-a, a, a-1$ nor pole of F, a. Then from (3.7) we know that z_{31} is a pole of H of order at most 1. Let z_{32} be a pole of G, z_{33} be a zero of $G-1$, z_{34} be a zero of $G-a$ but them be neither zeros of $F, F-1, F-a, a, a-1$ nor poles of F, a. Similarly, from (3.7) we know that they are poles of H of order at most 1.

Case 4. Let z_{41} be a zero of F and a pole of G or a zero of $G-1$ or $G-a$ but neither a zero of $a, a-1$ nor a pole of a, by Lemma 2.3 and (3.7) we know that z_{41} is a pole of H of order at most 2. Let z_{42} be a pole of F and a zero of G or $G-1$ or $G-a$, z_{43} be a zero of $F-1$ and a zero of G or $G-a$ or a pole of G, z_{44} be a zero of $F-a$ and a zero of G or $G-1$ or a pole of G but them be neither zeros of $a, a-1$ nor poles of a. Similarly, we know that they are poles of H of order at most 2.

In summary, from Case 1 to Case 4 we have

$$N(r,H) \le \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r,F) + S(r,G). \tag{3.17}$$

By (3.16) and (3.17) we obtain

$$\overline{N}_0(r, b, F, G) \le \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r,F) + S(r,G).$$

That is

$$\overline{N}_0(r, a_5, f, g) \le \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r,f) + S(r,g). \tag{3.18}$$

We notice that

$$\overline{N}(r, \frac{1}{f-a_5}) + \overline{N}(r, \frac{1}{g-a_5}) = 2\overline{N}_0(r, a_5, f, g) + \overline{N}_{12}(r, a_5, f, g). \tag{3.19}$$

By (3.18) and (3.19) we have

$$\begin{aligned} &\overline{N}(r, \frac{1}{f-a_5}) + \overline{N}(r, \frac{1}{g-a_5}) \\ &\le 2\sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_5, f, g) + S(r,f) + S(r,g). \end{aligned} \tag{3.20}$$

Similarly, we obtain

$$\begin{aligned} &\overline{N}(r, \frac{1}{f-a_j}) + \overline{N}(r, \frac{1}{g-a_j}) \\ &\le 2\sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_j, f, g) + S(r,f) + S(r,g), \quad j = 1,2,3,4. \end{aligned} \tag{3.21}$$

By Lemma 2.1 we have

$$(3-\varepsilon)T(r,f) + (3-\varepsilon)T(r,g)$$

$$\leq \sum_{j=1}^{5} \overline{N}(r, \frac{1}{f-a_j}) + \sum_{j=1}^{5} \overline{N}(r, \frac{1}{g-a_j}) + S(r,f) + S(r,g). \tag{3.22}$$

By (3.20) - (3.22) we obtain

$$(\frac{1}{3}-\varepsilon)[T(r,f)+T(r,g)] \leq \sum_{j=1}^{5} \overline{N}_{12}(r,a_j,f,g) + S(r,f) + S(r,g) \tag{3.23}$$

where ε is a sufficiently small positive number, which may be different at different places. □

Theorem 3.2 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a_j $(j = 1,2,\ldots,5)$ be five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$, k_j $(j=1,2,\ldots,5)$ be five positive integers or ∞ and satisfying $k_1 \geq k_2 \geq \cdots \geq k_5$. If*

$$\overline{E}_{k_j)}(a_j,f) = \overline{E}_{k_j)}(a_j,g), \quad j=1,2,\ldots,5$$

and

$$\sum_{j=1}^{5} \frac{1}{k_j+1} < \frac{1}{3}, \tag{3.24}$$

then we have $f(z) \equiv g(z)$.

Proof Firstly, we suppose that $f(z) \not\equiv g(z)$. If $k_1 = k_2 = \cdots = k_5 = \infty$, then we have $\overline{E}(a_j,f) = \overline{E}(a_j,g)$ $(j=1,2,\ldots,5)$. So we obtain

$$\overline{N}_{12}(r,a_j,f,g) = 0, \quad j=1,2,\ldots,5. \tag{3.25}$$

By (3.23) and (3.25) we have

$$(\frac{1}{3}-\varepsilon)[T(r,f)+T(r,g)] = S(r,f)+S(r,g).$$

This is a contradiction. So we suppose that k_j $(j=1,2,\ldots,5)$ are five positive integers. By $\overline{E}_{k_j)}(a_j,f) = \overline{E}_{k_j)}(a_j,g)$ $(j=1,2,\ldots,5)$, we can obtain

$$\begin{aligned}\overline{N}_{12}(r,a_j,f,g) \leq & \overline{N}_{(k_j+1}(r,\frac{1}{f-a_j}) + \overline{N}_{(k_j+1}(r,\frac{1}{g-a_j}) + \\ & S(r,f)+S(r,g), \quad j=1,2,\ldots,5.\end{aligned} \tag{3.26}$$

By (3.23) and (3.26) we have

$$\begin{aligned}(\frac{1}{3}-\varepsilon)[T(r,f)+T(r,g)] \leq & \sum_{j=1}^{5} \overline{N}_{(k_j+1}(r,\frac{1}{f-a_j}) + \sum_{j=1}^{5} \overline{N}_{(k_j+1}(r,\frac{1}{g-a_j}) + \\ & S(r,f)+S(r,g).\end{aligned} \tag{3.27}$$

By Lemma 2.2 we obtain

$$\sum_{j=1}^{5} \overline{N}_{(k_j+1}(r,\frac{1}{f-a_j}) \leq \sum_{j=1}^{5} \frac{1}{k_j+1} T(r,f) + S(r,f), \tag{3.28}$$

$$\sum_{j=1}^{5} \overline{N}_{(k_j+1}(r,\frac{1}{g-a_j}) \leq \sum_{j=1}^{5} \frac{1}{k_j+1} T(r,g) + S(r,g). \tag{3.29}$$

By (3.27)–(3.29) we have

$$(\frac{1}{3}-\varepsilon)[T(r,f)+T(r,g)] \le \sum_{j=1}^{5}\frac{1}{k_j+1}T(r,f)+\sum_{j=1}^{5}\frac{1}{k_j+1}T(r,g)+ S(r,f)+S(r,g).$$

This is

$$\Big(\frac{1}{3}-\sum_{j=1}^{5}\frac{1}{k_j+1}-\varepsilon\Big)[T(r,f)+T(r,g)] \le S(r,f)+S(r,g)$$

where ε is a sufficiently small positive number, which may be different at different places. By the assumption (3.24) we have $f(z)\equiv g(z)$. $\square$

Corollary 3.3 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a_j ($j=1,2,\ldots,5$) be five distinct elements in $\{S(f)\cap S(g)\}\cup\{\infty\}$. If they share a_j ($j=1,2,\ldots,5$) IM, then $f(z)\equiv g(z)$.*

References

[1] Hongxun YI, Chongjun YANG. *Uniqueness Theory of Meromorphic Function*. Science Press, Beijing, 1995. (in Chinese)

[2] W. K. HAYMAN. *Meromorphic Functions*. Oxford University Press, Oxford, 1964.

[3] R. NEVANLINNA. *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthiers-Villars, Paris, 1929.

[4] Qingde ZHANG. *A uniqueness theorem for meromrophic functions with respect to slowly growing functions*. Acta Math. Sin., 1993, **36**(6): 826–833.

[5] K. ISHIZAKI, N. TODA. *Unicity theorems for meromrophic functions sharing four small functions*. Kodai Math. J., 1998, **21**: 350–371.

[6] Hongxun YI, Yuhua LI. *Meromrophic functions share four small functions*. Chinese Ann. Math., Ser. A, 2001, **22**(3): 271–278.

[7] Hongxun YI. *On one problem of uniqueness of meromrophic functions concerning small functions*. P. Am. Math. Soc., 2002, **130**(6): 1689–1697.

[8] Yuhua LI, Jianyong QIAO. *The uniqueness of meromrophic functions concerning small functions*. Sci. China Ser. A, 2000, **43**(6): 581–590.

[9] K. YAMANOI. *The second main theorem for small functions and related problems*. Acta Math., 2004, **192**: 225–294.