Journal of Mathematical Research with Applications
Nov., 2020, Vol. 40, No. 6, pp. 617–627
DOI:10.3770/j.issn:2095-2651.2020.06.006
Http://jmre.dlut.edu.cn

Minimal Rotational Hypersurfaces in Some Non-flat Randers Spaces

Peilong DONG*, **Shan LIU**
School of Mathematical Sciences, Tongji University, Shanghai 200092*, P. R. China*

Abstract The contribution of this paper is second-fold. The first one is to derive the HT-minimal hypersurfaces of rotation in special Randers spaces, which are non-Minkowski but have vanishing flag curvatures. The second one is to characterize the anisotropic minimal rotational hypersurfaces in Funk spaces.

Keywords minimal rotational hypersurfaces; isometric immersion; anisotropic hypersurfaces; Randers spaces

MR(2010) Subject Classification 53B40; 53C60

1. Introduction

A Randers metric is defined as the sum of a Riemannian metric and a 1-form, which was firstly introduced in the research on the general relativity and has been widely applied in many areas of natural science such as biology, physics and psychology, etc. The Randers manifold plays a fundamental role in the Finsler geometry. The simply connected Randers manifolds of constant flag curvature are called the Randers space forms, which were classified by using the Zermelo's navigation method in [1].

Similar to the minimal surface theory in Riemannian geometry, finding any explicit minimal surface in the Finsler space forms is an interesting work. In Finsler geometry, minimal surfaces with respect to the Busemann-Hausdorff measure and the Holmes-Thompson measure are called BH-minimal and HT-minimal surfaces, respectively. As we know, studies on minimal submanifolds in Finsler geometry have made rapid progress in recent years ([2–4], etc). By using the Busemann-Hausdorff volume form, Shen introduced the notion of mean curvature for Finsler submanifolds and obtained some global and local results [5]. Later, He and Shen used the Holmes-Thompson volume form to introduce another notion of mean curvature [6]. By using the Zermelo's navigation method, Yin and He investigated the general (α, β)-spaces with special curvature properties and have constructed minimal surfaces in Randers 3-spaces [7]. Recently, Cui [8] studied the rotationally invariant minimal surfaces in the Bao-Shen's spheres, which are a class of 3-spheres endowed with Randers metrics of constant flag curvature $K = 1$. Moreover, he obtained the nontrivial minimal surfaces in a Finsler 3-sphere with vanishing $\mathbf{S}$-curvature in [9], including the Bao-Shen sphere as a special case.

Received June 24, 2019; Accepted May 28, 2020
* Corresponding author
E-mail address: 1710384@tongji.edu.cn (Peilong DONG); ls2010710@163.com (Shan LIU)

However, all the nontrivial BH-minimal and HT-minimal rotational surfaces in Randers spaces are far from being determined. The purpose of this paper is to discuss minimal rotational hypersurfaces in Randers spaces with constant flag curvatures by the method of Zermelo navigation. We obtain two different kinds of nontrivial minimal rotational hypersurfaces in Randers spaces with $K=0$ and $K=-\frac{1}{4}$, respectively.

There are two different kinds of hypersurfaces in a Finsler manifold. One is the isometric immersion hypersurfaces, which is also a Finsler manifold. The other is the anisotropic hypersurface, which is a Riemannian manifold with the Riemannian metric induced by the Finsler metric along a given normal vector field (see Subsection 2.2 for detail).

This paper is organised as follows. In Section 2, we introduce some definitions and basic concepts in Finsler geometry. In Section 3, we construct the rotational hypersurfaces in a Randers space which is non-Minkowski but has 0 flag curvature (Theorem 3.2-3.3). Moreover, we give the anisotropic minimal rotational hypersurfaces in Funk spaces in Section 4 (Theorem 4.1).

2. Preliminaries

A Finsler metric on M^n is a continuous function $F: TM \to [0,+\infty)$ satisfying: (i) F is smooth on $TM\backslash\{0\}$; (ii) $F(x,\lambda y)=\lambda F(x,y)$ for any $(x,y)\in TM$ and any positive real number λ; (iii) The fundamental form $g:=g_{ij}\mathrm{d}x^i\otimes \mathrm{d}x^j$ is positive definite on $TM\backslash\{0\}$, where $g_{ij}:=\frac{1}{2}[F^2]_{y^iy^j}$. Here and from now on, $F_{y^i}, F_{y^iy^j}$ mean $\frac{\partial F}{\partial y^i}, \frac{\partial^2 F}{\partial y^i\partial y^j}$, and we will use the following convention of index ranges:

$$1\le a,b,\cdots\le n-1,\ 1\le i,j,\cdots\le n,\ 1\le \alpha,\beta,\cdots\le n+p.$$

Einstein summation convention is also used throughout this paper.

The projection $\pi: TM\to M$ gives rise to the pull-back bundle π^*TM and its dual bundle π^*T^*M over $TM\backslash 0$. On π^*TM there exists the unique Chern connection ∇ with $\nabla\frac{\partial}{\partial x^i}=\omega_i^j\frac{\partial}{\partial x^j}=\Gamma^i_{jk}\mathrm{d}x^k\otimes\frac{\partial}{\partial x^j}$ satisfying [10]

$$\omega^i_j\wedge \mathrm{d}x^j=0,$$

$$\mathrm{d}g_{ij}-g_{ik}\omega^k_j-g_{kj}\omega^k_i=2FC_{ijk}\delta y^k,\ \delta y^k:=\frac{1}{F}(\mathrm{d}y^k+y^j\omega^k_j),$$

where $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is called the Cartan tensor.

Let $X=X^i\frac{\partial}{\partial x^i}$ be a vector field. Then the covariant derivative of X along $v=v^i\frac{\partial}{\partial x^i}\in T_xM$ with respect to a reference vector $w\in T_xM\backslash 0$ for the Chern connection is defined by

$$D^w_vX(x):=\{v^j\frac{\partial X^i}{\partial x^j}(x)+{}^b\Gamma^i_{jk}(w)v^jX^k(x)\}\frac{\partial}{\partial x^i}. \tag{2.1}$$

Let $\mathcal{L}: TM\to T^*M$ denote the Legendre transformation, which satisfies $\mathcal{L}(\lambda y)=\lambda\mathcal{L}(y)$ for all $\lambda>0$, $y\in TM$ and [11]

$$\mathcal{L}(y)=F(y)[F]_{y^i}(y)\mathrm{d}x^i,\quad \forall y\in TN\setminus\{0\}, \tag{2.2}$$

$$\mathcal{L}^{-1}(\xi)=F^*(\xi)[F^*]_{\xi_i}(\xi)\frac{\partial}{\partial x^i},\quad \forall\xi\in T^*N\setminus\{0\}, \tag{2.3}$$

where F^* is the dual metric of F. In general, $\mathcal{L}^{-1}(-\xi) \neq -\mathcal{L}^{-1}(\xi)$.

The pull back of the Sasaki-type metric from $TM\backslash\{0\}$ to the projective sphere bundle SM is a Riemannian metric and the volume element $\mathrm{d}V_{SM}$ of SM with respect to it can be expressed as

$$\mathrm{d}V_{SM} = \Omega \mathrm{d}\tau \wedge \mathrm{d}x,$$

where

$$\Omega := \det(\frac{g_{ij}}{F}), \quad \mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

$$\mathrm{d}\tau := \sum_{i=1}^{n} (-1)^{i-1} y^i \mathrm{d}y^1 \wedge \cdots \wedge \widehat{\mathrm{d}y^i} \wedge \cdots \wedge \mathrm{d}y^n.$$

The HT-volume form of an n-dimensional Finsler manifold (M, F) is defined by

$$\mathrm{d}V_M := \sigma(x)\mathrm{d}x, \quad \sigma(x) := \frac{1}{c_{n-1}} \int_{S_xM} \Omega \mathrm{d}\tau,$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$ dimension sphere S^{n-1} and

$$S_xM = \{[y] | y \in T_xM\}.$$

2.1. Isometric immersion submanifolds

An immersion $\phi : (M^n, F) \to (\widetilde{M}^{n+p}, \widetilde{F})$ between Finsler manifolds is called isometric, if $F(x, y) = \widetilde{F}(\phi(x), \mathrm{d}\phi(y))$ for any $(x, y) \in TM \setminus \{0\}$. It is clear that for the isometric immersion ϕ, the induced metric F on M satisfies

$$g_{ij}(x, y) = \widetilde{g}_{\alpha\beta}(\widetilde{x}, \widetilde{y})\phi_i^\alpha \phi_j^\beta,$$

where

$$\tilde{x}^\alpha = \phi^\alpha(x), \ \tilde{y}^\alpha = \phi_i^\alpha y^i, \ \phi_i^\alpha = \frac{\partial \phi^\alpha}{\partial x^i}.$$

An isometric immersion $\phi : (M, F) \to (\widetilde{M}, \widetilde{F})$ is called minimal if any compact domain of M is the critical point of its volume functional with respect to any variation.

Set

$$h^\alpha = \phi_{ij}^\alpha y^i y^j - \phi_k^\alpha G^k + \widetilde{G}^\alpha, \ h_\alpha = \widetilde{g}_{\alpha\beta} h^\beta, \ h := \frac{h^\alpha}{F^2} \frac{\partial}{\partial \widetilde{x}^\alpha}, \ \phi_{ij}^\alpha = \frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j},$$

where G^k and $\widetilde{G}^\alpha$ are the geodesic coefficients of F and $\widetilde{F}$, respectively, h is called the normal curvature.

Let $(\pi^*TM)^\perp$ be the orthogonal complement of π^*TM in $\pi^*(\phi^{-1}T\widetilde{M})$ with respect to $\widetilde{g}$ and denote

$$\nu^* := \{\xi \in \Gamma(\phi^{-1}T^*\widetilde{M}) | \xi(\mathrm{d}\phi X) = 0, \quad \forall X \in \Gamma(TM)\},$$

which is called the normal bundle of ϕ in [5]. Set

$$\mu_\phi := \frac{1}{c_{n-1}\sigma} \Big(\int_{S_xM} \frac{h_\alpha}{F^2} \Omega \mathrm{d}\tau \Big) \mathrm{d}\widetilde{x}^\alpha,$$

which is called the mean curvature form of ϕ. It is known from [6] that $h \in \Gamma((\pi^*TM)^\perp)$, $\mu_\phi \in \nu^*$ and ϕ is minimal if and only if $\mu_\phi = 0$.

We now suppose that $\widetilde{F}=\widetilde{\alpha}+\widetilde{\beta}$ is a Randers metric, where $\widetilde{\alpha}$ is a Riemannian metric and $\widetilde{\beta}$ is a 1-form. There are two ways to study the submanifold theory in Randers manifolds: one way is to use the Riemannian metric and the one form, the other way is to use the Zermelo's navigation method [1]. We take the second way in this paper. By the Zermelo navigation,

$$\widetilde{F}=\frac{\sqrt{\widetilde{\lambda}\widetilde{h}^2+\widetilde{W_0}^2}}{\widetilde{\lambda}}-\frac{\widetilde{W_0}}{\widetilde{\lambda}},\quad \widetilde{W_0}:=\widetilde{W_\alpha}\widetilde{y}^\alpha,$$

which can be characterized by the navigation data $(\widetilde{h},\widetilde{W})$, where $\widetilde{h}=\sqrt{\widetilde{h}_{\alpha\beta}d\widetilde{x}^\alpha d\widetilde{x}^\beta}$ is a Riemannian metric and $\widetilde{W}=\widetilde{W}^\alpha\frac{\partial}{\partial\widetilde{x}^\alpha}$ is a vector field satisfying $\|\widetilde{W}\|_{\widetilde{h}}=\widetilde{W}_\alpha\widetilde{W}^\alpha<1$ at any point $\widetilde{x}\in\widetilde{M}$, and $\widetilde{W}_\alpha=\widetilde{h}_{\alpha\beta}\widetilde{W}^\beta$, $\widetilde{\lambda}:=1-\|\widetilde{W}\|^2_{\widetilde{h}}$.

2.2. Anisotropic hypersurfaces

Let (N,F) be an $(n+1)$-dimensional oriented smooth Finsler manifold and $\phi:M\to(N,F)$ be an n-dimensional immersion. For any $x\in M$, there exist exactly two unit normal vectors $\boldsymbol{n}$ and $\boldsymbol{n}_-$. Let $\boldsymbol{n}$ be a given unit normal vector and set $\hat{g}=\phi^*g_{\boldsymbol{n}}$, we call $(M,\hat{g})$ an oriented anisotropic hypersurface.

For any $X\in T_xM$, define the shape operator $A:T_xM\to T_xM$ by

$$A(X)=-D^{\boldsymbol{n}}_X\boldsymbol{n},\tag{2.4}$$

which is called Weingarten formula. The eigenvalues of A, $\lambda_1,\lambda_2,\ldots,\lambda_m$, and $H_{\boldsymbol{n}}=\sum_{i=1}^m\lambda_i$ are called the principal curvatures and the anisotropic mean curvature with respect to $\boldsymbol{n}$, respectively. If $H_{\boldsymbol{n}}=0$, we call M an anisotropic-minimal hypersurface of (N,F).

3. Minimal rotational hypersurfaces in special Randers spaces with nontrivial Killing fields

Randers space is one of the most important spaces in Finsler geometry. In this section, we will find some minimal rotational hypersurfaces in the special Randers spaces which are non-Minkowski but have 0 flag curvatures.

Denote

$$\widetilde{s}_{\alpha\beta}=\frac{1}{2}(\widetilde{W}_{\alpha|\beta}-\widetilde{W}_{\beta|\alpha}),\ \widetilde{s}^\alpha_0=\widetilde{h}^{\alpha\beta}\widetilde{s}_{\beta\gamma}\widetilde{y}^\gamma,\ \widetilde{s}^\alpha=\widetilde{h}^{\alpha\beta}\widetilde{s}_{\gamma\beta}\widetilde{W}^\gamma,$$

where " | " denotes covariant derivative with respect to $\widetilde{h}$. The following theorem was actually proved in [7].

Theorem 3.1 ([7]) *Let $\phi:(M^n,F)\to(\widetilde{M}^{n+1},\widetilde{F})$ be a hypersurface with $\widetilde{F}=\frac{\sqrt{\widetilde{\lambda}\widetilde{h}^2+\widetilde{W}_0^2}}{\widetilde{\lambda}}-\frac{\widetilde{W}_0}{\widetilde{\lambda}}$. If $\widetilde{h}$ is a Riemannian metric with constant sectional curvature and $\widetilde{W}$ is a Killing vector, then ϕ is minimal if and only if*

$$\widetilde{h}_{\beta\delta}\bar{n}^\delta\int_{\sqrt{\widetilde{\lambda}h^2+W_0^2}=1}[\phi^\beta_{ij}y^iy^j-\frac{\widetilde{s}^\beta(1-W_0)^2}{\widetilde{\lambda}^2}-\frac{2\widetilde{s}^\beta_0(1-W_0)}{\widetilde{\lambda}}]\mathrm{d}\tau=0,$$

where $h=\phi^*\widetilde{h}$, $W_0=\phi^*\widetilde{W}_0$, $\bar{\boldsymbol{n}}$ is a unit normal vector field with respect to $\widetilde{h}$, $\widetilde{\lambda}=1-\|\widetilde{W}\|^2_{\widetilde{h}}$.

Now we are in the position to prove the following theorems.

Theorem 3.2 *For* $n = 2m$, *let* $(\widetilde{M}^{n+1}, \widetilde{F})$ *be a Randers manifold, where* $\widetilde{M}^{n+1} := \{(\widetilde{x}^1, \dots, \widetilde{x}^{n+1}) \in \mathbb{R}^{n+1} | (\widetilde{x}^1)^2 + \cdots + (\widetilde{x}^n)^2 < \frac{1}{\varepsilon^2},\ \varepsilon = \text{const}\}$ *and the navigation data* $(\widetilde{h}, \widetilde{W})$ *of* $\widetilde{F}$ *is given by*

$$\widetilde{h} = \sqrt{\sum_{\alpha=1}^{n+1} (\widetilde{y}^\alpha)^2}, \quad \widetilde{W} = (\varepsilon\widetilde{x}^2, -\varepsilon\widetilde{x}^1, \varepsilon\widetilde{x}^4, -\varepsilon\widetilde{x}^3, \dots, \varepsilon\widetilde{x}^{2m}, -\varepsilon\widetilde{x}^{2m-1}, 0). \tag{3.1}$$

Then there exists minimal rotational hypersurfaces in $\widetilde{M}^{n+1}$ *which can be expressed as*

$$\phi = (uf^1(\theta), \dots, uf^n(\theta), \pm \int \frac{\sqrt{C}(\varepsilon^2 u^2 - 1)^{\frac{n+1}{2}}}{\sqrt{u^{2(n-1)^2} - C(\varepsilon^2 u^2 - 1)^{n+1}}} \mathrm{d}u), \tag{3.2}$$

where $\sum (f^i(\theta))^2 = 1$, *and* C *is an arbitrary non-negative constant.*

Proof Let $\phi : M^n \to (\widetilde{M}^{n+1}, \widetilde{F})$ be a rotational hypersurface defined by

$$\phi(\theta, u) = (uf^1(\theta), \dots, uf^n(\theta), g(u)), \quad \sum_{i=1}^n (f^i(\theta))^2 = 1, \tag{3.3}$$

where $g(u)$ will be determined, $x^a = \theta^a, x^n = u$. A direct calculation gives

$$\begin{aligned}
&\sum_{i=1}^n f^i f^i_a = 0, \quad \sum_{i=1}^n (f^i f^i_{ab} + f^i_a f^i_b) = 0;\\
&(\phi^\alpha_a)_{1\times(n+1)} = (uf^i_a, 0), \quad (\phi^\alpha_n)_{1\times(n+1)} = (f^i, g');\\
&(\phi^\alpha_{ab})_{1\times(n+1)} = (uf^i_{ab}, 0), \quad (\phi^\alpha_{an})_{1\times(n+1)} = (f^i_a, 0), \quad (\phi^\alpha_{nn})_{1\times(n+1)} = (0, g''),
\end{aligned}$$

where ϕ^α_a denotes differentiating ϕ^α with respect to θ^a, and ϕ^α_n denotes differentiating ϕ^α with respect to u. A unit normal vector field with respect to $\widetilde{h}$ is given by

$$\bar{\boldsymbol{n}} = \frac{1}{\sqrt{1 + (g')^2}} (g' f^i, -1). \tag{3.4}$$

A straightforward computation shows that

$$h_{ab} = \sum_i u^2 f^i_a f^i_b, \quad h_{an} = 0, \quad h_{nn} = 1 + (g')^2, \tag{3.5}$$

$$B_{ab} = -\frac{g'}{u\sqrt{1 + (g')^2}} h_{ab}, \quad B_{an} = 0, \quad B_{nn} = -\frac{g''}{\sqrt{1 + (g')^2}}, \tag{3.6}$$

where $h_{ij} = \langle \phi_i, \phi_j \rangle_{\widetilde{h}}$, $B_{ij} = \langle \phi_{ij}, \bar{\boldsymbol{n}} \rangle_{\widetilde{h}}$.

Take a local coordinate system (u^i, v^i) such that at x, we have

$$W_0 = b_0 v^n, \quad \widetilde{\lambda} h^2 + W_0^2 = \delta_{ij} v^i v^j,$$

where b_0 will be determined, $\widetilde{\lambda} \circ \phi = 1 - \|\widetilde{W}\|^2_{\widetilde{h}} = 1 - \sum (\varepsilon \tilde{x}^i)^2 = 1 - \varepsilon^2 u^2$. Then we obtain a coordinate transformation $(x^i, y^i) \to (u^i, v^i)$ given by

$$\frac{\partial}{\partial x^i} = p^j_i \frac{\partial}{\partial u^j}, \quad \mathrm{d}x^i = q^i_j \mathrm{d}u^j,$$

where $(q^j_i) = (p^j_i)^{-1}$.

From $y = y^i \frac{\partial}{\partial x^i} = v^j \frac{\partial}{\partial u^j}$, we get $y^i p_i^j = v^j$, $y^i = v^j q_j^i$. Notice that $W_0 = W_i y^i = b_0 v^n$, we have $\frac{W_i}{b_0} y^i = v^n$, and then

$$b_0 p_i^n = W_i. \tag{3.7}$$

Set $\widetilde{\lambda} h^2 + W_0^2 = a_{ij} y^i y^j$, then $a_{ij} = \widetilde{\lambda} h_{ij} + W_i W_j$. By the above equations, we have $q_l^i = \delta_{kl} p_j^k a^{ij}$. Let $l = n$. Then $q_n^i = p_j^n a^{ij}$. Since $\langle \widetilde{W}, \bar{\boldsymbol{n}} \rangle_{\widetilde{h}} = 0$, it is easy to obtain $1 - \|W\|_h^2 = \widetilde{\lambda}$, which implies $1 - \widetilde{\lambda} = \|W\|_h^2 = h^{\overline{ij}} W_{\bar{i}} W_{\bar{j}} = \frac{\widetilde{\lambda} b_0^2}{1 - b_0^2}$, thus we have

$$b_0^2 = 1 - \widetilde{\lambda} = \varepsilon^2 u^2 \tag{3.8}$$

where $h_{\overline{ij}} = h_{kl} q_i^k q_j^l$. With (3.7), we obtain

$$b_0 q_n^i = W_j \frac{1}{\widetilde{\lambda}} (h^{ij} - W^i W^j) = W^i.$$

Let $\rho : T\widetilde{M} \to d\phi TM$ be an orthogonal projection with respect to $\widetilde{h}$. Then

$$b_0 q_n^i \phi_i^\beta = W^i \phi_i^\beta = h^{ij} W_j \phi_i^\beta = h^{ij} \widetilde{h}_{\alpha\gamma} \phi_j^\alpha \phi_i^\beta \widetilde{W}^\gamma = \rho_\gamma^\beta \widetilde{W}^\gamma = \widetilde{W}^\beta. \tag{3.9}$$

By

$$\widetilde{\lambda} h^2 + W_0^2 = \widetilde{\lambda} h_{\overline{ij}} v^i v^j + b_0^2 (v^n)^2 = \delta_{ij} v^i v^j,$$

we get

$$(h_{\overline{ij}}) = \begin{pmatrix} \frac{1}{\widetilde{\lambda}} & & & \\ & \ddots & & \\ & & \frac{1}{\widetilde{\lambda}} & \\ & & & \frac{1 - b_0^2}{\widetilde{\lambda}} \end{pmatrix},$$

and then

$$(h^{\overline{ij}}) = \begin{pmatrix} \widetilde{\lambda} & & & \\ & \ddots & & \\ & & \widetilde{\lambda} & \\ & & & \frac{\widetilde{\lambda}}{1 - b_0^2} \end{pmatrix}.$$

Set $\phi_{\bar{j}}^\alpha = \frac{\partial \phi^\alpha}{\partial u^j}$, then

$$\phi_{\bar{j}}^\alpha = \phi_i^\alpha q_j^i, \quad \phi_{\overline{ij}}^\alpha = \phi_{kl}^\alpha q_i^k q_j^l + \phi_k^\alpha (q_i^k)_{\bar{j}},$$

therefore, by (3.4)

$$\bar{n}^\alpha \phi_{\overline{ij}}^\alpha = \bar{n}^\alpha \phi_{kl}^\alpha q_i^k q_j^l. \tag{3.10}$$

It follows from (3.5) that

$$(h_{ij}) = \begin{pmatrix} (h_{ab}) & \\ & 1 + (g')^2 \end{pmatrix}, \tag{3.11}$$

then

$$(h^{ij}) = \begin{pmatrix} (h^{ab}) & \\ & \frac{1}{1 + (g')^2} \end{pmatrix}. \tag{3.12}$$

Note that $W_n = \phi_n^\alpha \widetilde{W}_\alpha = 0$, we have $W^n = h^{ni} W_i = 0$. Then with (3.6), we have the following

$$\begin{aligned} B_{kl}(h^{kl} - W^k W^l) &= B_{ab}(h^{ab} - W^a W^b) + B_{an}(h^{an} - W^a W^n) + B_{nn}(h^{nn} - W^n W^n) \\ &= -\frac{g'}{u\sqrt{1+(g')^2}} h_{ab}(h^{ab} - W^a W^b) - \frac{g''}{\sqrt{1+(g')^2}} \frac{1}{1+(g')^2} \\ &= -\frac{1}{u\sqrt{1+(g')^2}} [g'(n-1)^2 - g'(1-\widetilde{\lambda}) + \frac{ug''}{1+(g')^2}]. \end{aligned} \tag{3.13}$$

In our case, by Theorem 3.1, ϕ is minimal if and only if

$$\bar{n}^\alpha \int_{|v|^2=1} [\phi_{\overline{ij}}^\alpha v^i v^j - \frac{\widetilde{s}^\alpha (1-b_0 v^n)^2}{\widetilde{\lambda}^2} - \frac{2\widetilde{s}_0^\alpha (1-b_0 v^n)}{\widetilde{\lambda}}] \mathrm{d}\overline{\tau} = 0, \tag{3.14}$$

where $\mathrm{d}\overline{\tau} = \sum (-1)^{i-1} v^i \mathrm{d}v^1 \wedge \cdots \wedge \widehat{\mathrm{d}v^i} \wedge \cdots \wedge dv^n = \sqrt{\det(a_{ij})}\mathrm{d}\tau$. By (3.1), a direct computation yields

$$\widetilde{s}_{12} = \widetilde{s}_{34} = \cdots = \widetilde{s}_{(2m-1)(2m)} = \varepsilon, \ \ \widetilde{s}_{21} = \widetilde{s}_{43} = \cdots = \widetilde{s}_{(2m)(2m-1)} = -\varepsilon, \ \text{ other } \widetilde{s}_{\alpha\beta} = 0; \tag{3.15}$$

$$\widetilde{s}^i = \varepsilon^2 u f^i, \quad \widetilde{s}^{n+1} = 0. \tag{3.16}$$

For all i, we have $\int_{|v|^2=1} (v^i)^2 \mathrm{d}\overline{\tau} = \frac{c_{n-1}}{n}$, plugging (3.8)–(3.13), (3.15) and (3.16) into (3.14) yields

$$\begin{aligned} &\bar{n}^\alpha \int_{|v|^2=1} [\phi_{\overline{ii}}^\alpha (v^i)^2 - \frac{\widetilde{s}^\alpha (1-b_0 v^n)^2}{\widetilde{\lambda}^2} - \frac{2\widetilde{s}_\beta^\alpha v^j q_j^i \phi_i^\beta (1-b_0 v^n)}{\widetilde{\lambda}}] \mathrm{d}\overline{\tau} \\ &= \bar{n}^\alpha \int_{|v|^2=1} [\phi_{kl}^\alpha q_i^k q_i^l (v^i)^2 - \frac{\widetilde{s}^\alpha}{\widetilde{\lambda}^2}(1 + b_0^2 (v^n)^2) + \frac{2}{\widetilde{\lambda}} \widetilde{s}_\beta^\alpha b_0 q_n^i \phi_i^\beta (v^n)^2] \mathrm{d}\overline{\tau} \\ &= \bar{n}^\alpha (\phi_{kl}^\alpha a^{kl} - \frac{(n+b_0^2)\widetilde{s}^\alpha}{\widetilde{\lambda}^2} + \frac{2}{\widetilde{\lambda}} \widetilde{s}_{\alpha\beta} \widetilde{W}^\beta) \frac{c_{n-1}}{n} \\ &= \frac{1}{\widetilde{\lambda}} [B_{kl}(h^{kl} - W^k W^l) - \frac{1}{\widetilde{\lambda}}(n+b_0^2) \bar{n}^\alpha \widetilde{s}^\alpha + 2\bar{n}^\alpha \widetilde{s}_{\alpha\beta} \widetilde{W}^\beta] \frac{c_{n-1}}{n} \\ &= -\frac{1}{\widetilde{\lambda} u\sqrt{1+(g')^2}} [(n-1)^2 g' - \varepsilon^2 u^2 g' + \frac{ug''}{1+(g')^2} + (n+b_0^2) \frac{\varepsilon^2 u^2 g'}{1-\varepsilon^2 u^2} + 2\varepsilon^2 u^2 g'] \frac{c_{n-1}}{n} \\ &= -\frac{1}{\widetilde{\lambda} u\sqrt{1+(g')^2}} [\frac{ug''}{1+(g')^2} + (n-1)^2 g' + \varepsilon^2 u^2 g' + \frac{(n+\varepsilon^2 u^2)\varepsilon^2 u^2 g'}{1-\varepsilon^2 u^2}] \frac{c_{n-1}}{n} = 0. \end{aligned} \tag{3.17}$$

When $g' \equiv 0$, then $g = \text{const}$, M^n appears to be a hyperplane and (3.14) holds on absolutely. Otherwise, suppose $g'(u) \neq 0$, we can choose a neighborhood U about this u such that $g' \neq 0$ on U, then we obtain

$$\frac{g''}{g'(1+(g')^2)} = \frac{(3n-n^2)\varepsilon^2 u^2 + (n-1)^2}{u(\varepsilon^2 u^2 - 1)}.$$

By a straightforward computation, we obtain

$$(g')^2 = \frac{C(\varepsilon^2 u^2 - 1)^{n+1}}{u^{2(n-1)^2} - C(\varepsilon^2 u^2 - 1)^{n+1}},$$

where C is an arbitrary non-negative constant. Therefore

$$g = \pm \int \frac{\sqrt{C}(\varepsilon^2 u^2 - 1)^{\frac{n+1}{2}}}{\sqrt{u^{2(n-1)^2} - C(\varepsilon^2 u^2 - 1)^{n+1}}} \mathrm{d}u. \ \ \square \tag{3.18}$$

Theorem 3.3 *For $n = 2m-1$, let $(\widetilde{M}^{n+1}, \widetilde{F})$ be a Randers manifold, where $\widetilde{M}^{n+1} := \{(\widetilde{x}^1, \dots, \widetilde{x}^{n+1}) \in \mathbb{R}^{n+1} \mid (\widetilde{x}^1)^2 + \cdots + (\widetilde{x}^{n-1})^2 < \frac{1}{\varepsilon^2},\ \varepsilon = \text{const}\}$ and the navigation data $(\widetilde{h}, \widetilde{W})$ of $\widetilde{F}$ is given by*

$$\widetilde{h} = \sqrt{\sum_{\alpha=1}^{n+1} (\widetilde{y}^\alpha)^2}, \quad \widetilde{W} = (\varepsilon\widetilde{x}^2, -\varepsilon\widetilde{x}^1, \varepsilon\widetilde{x}^4, -\varepsilon\widetilde{x}^3, \dots, \varepsilon\widetilde{x}^{2m-2}, -\varepsilon\widetilde{x}^{2m-3}, 0, 0). \tag{3.19}$$

Then there exists no nontrivial minimal rotational hypersurface generated around the axis in the direction of $\widetilde{x}^{n+1}$ in $(\widetilde{M}^{n+1}, \widetilde{F})$.

Proof The proof is similar to the proof of Theorem 3.2. We can get immediately

$$\widetilde{s}_{12} = \widetilde{s}_{34} = \cdots = \widetilde{s}_{(2m-3)(2m-2)} = \varepsilon, \quad \widetilde{s}_{21} = \widetilde{s}_{43} = \cdots = \widetilde{s}_{(2m-2)(2m-3)} = -\varepsilon, \quad \text{other } \widetilde{s}_{\alpha\beta} = 0;$$

$$\widetilde{s}^a = \varepsilon^2 u f^a, \quad \widetilde{s}^n = \widetilde{s}^{n+1} = 0; \quad \widetilde{\lambda} \circ \phi = 1 - \varepsilon^2 u^2 (1 - (f^n)^2).$$

Then we yield

$$\begin{aligned}
&\bar{n}^\alpha \int_{|v|^2=1} [\phi^\alpha_{ii}(v^i)^2 - \frac{\widetilde{s}^\alpha(1-b_0v^n)^2}{\widetilde{\lambda}^2} - \frac{2\widetilde{s}^{\alpha\beta}v^j q^i_j \phi^\beta_i (1-b_0v^n)}{\widetilde{\lambda}}] \mathrm{d}\overline{\tau} \\
&= \frac{1}{\widetilde{\lambda}}[B_{kl}(h^{kl} - W^kW^l) - \frac{1}{\widetilde{\lambda}}(n+b_0^2)\bar{n}^\alpha \widetilde{s}^\alpha + 2\bar{n}^\alpha \widetilde{s}_{\alpha\beta}\widetilde{W}^\beta] \frac{c_{n-1}}{n} \\
&= -\frac{1}{\widetilde{\lambda}u\sqrt{1+(g')^2}}[(n-1)^2 g' - b_0^2 g' + \frac{ug''}{1+(g')^2} + \frac{(n+b_0^2)}{\widetilde{\lambda}} g'\varepsilon^2u^2(1-(f^n)^2) + \\
&\quad 2g'\varepsilon^2u^2(1-(f^n)^2)]\frac{c_{n-1}}{n} \\
&= -\frac{1}{\widetilde{\lambda}u\sqrt{1+(g')^2}}[\frac{ug''}{1+(g')^2} - \frac{(3n-n^2)b_0^2 + (n-1)^2}{b_0^2 - 1} g']\frac{c_{n-1}}{n} = 0,
\end{aligned} \tag{3.20}$$

where $b_0^2 = \varepsilon^2 u^2 (1 - (f^n)^2)$. As we know, a hyperplane is minimal, so we only consider the nontrivial case, that is $g'(u) \neq 0$, then we obtain

$$\frac{ug''}{g'(1+(g')^2)} = \frac{(3n-n^2)b_0^2 + (n-1)^2}{b_0^2 - 1}.$$

Denote $\frac{ug''}{g'(1+(g')^2)} = \psi(u)$, we have the following

$$(\psi + n^2 - 3n)\varepsilon^2 u^2 (f^n)^2 = \psi(\varepsilon^2 u^2 - 1) + (n^2 - 3n)\varepsilon^2 u^2 - (n-1)^2,$$

it stands on only when

$$\begin{cases} \psi = 3n - n^2, \\ \psi(\varepsilon^2u^2 - 1) + (n^2 - 3n)\varepsilon^2 u^2 - (n-1)^2 = 0, \end{cases}$$

that is a contradiction since $n > 2$. Hence there is no nontrivial solution of g. $\square$

4. Anisotropic minimal rotational hypersurfaces in a Funk space

Let $\phi : M \to (\mathbb{B}^{n+1}, F)$ be an n-dimensional immersion in a Funk space, where $F = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \epsilon\langle x,y\rangle}{1-|x|^2}$. It is well known that the Funk space has constant $\mathbf{S}$-curvature $\mathbf{S} =$

$\frac{\epsilon}{2}(n+1)F$, constant flag curvature $K=-\frac{1}{4}$ and geodesic coefficients $G^\alpha=\frac{\epsilon}{2}Fy^\alpha$. Then we can get the following theorem.

Theorem 4.1 *Let $(\mathbb{B}^{n+1},F)$ be a Funk space and the navigation data (h,W) of F is given by*

$$h=\sqrt{\langle y,y\rangle},\ W=-\epsilon x,\quad \forall x\in\mathbb{B}^{n+1},\ y\in\mathbb{R}^{n+1}. \tag{4.1}$$

Then the anisotropic minimal rotational hypersurfaces in $(\mathbb{B}^{n+1},F)$ must be

$$\phi(\theta,u)=(uf^1(\theta),\dots,uf^n(\theta),\int\sqrt{\frac{(\epsilon u^n+C_1)^2}{4u^{2(n-1)}-(\epsilon u^n+C_1)^2}}\mathrm{d}u+C_2), \tag{4.2}$$

where $\sum(f^i(\theta))^2=1$, and C_1,C_2 are arbitrary constants.

Proof Let $\phi:M^n\to(\mathbb{B}^{n+1},F)$ be an anisotropic rotational hypersurface, which can be expressed as

$$x=\phi(\theta,u)=(uf^1(\theta),\dots,uf^n(\theta),g(u)), \tag{4.3}$$

where $x=(x^\alpha)$, $\theta=(\theta^1,\dots,\theta^{n-1})$, and $g(u)$ will be determined. Then we can write

$$\begin{cases} f^1(\theta)=\cos\theta^{n-1}\cos\theta^{n-2}\cdots\cos\theta^1,\\ f^1(\theta)=\cos\theta^{n-1}\cos\theta^{n-2}\cdots\sin\theta^1,\\ \cdots\\ f^1(\theta)=\sin\theta^{n-1}.\end{cases}$$

A direct calculation gives

$$\begin{aligned}&\sum_{i=1}^n f^if^i_a=0,\quad \sum_{i=1}^n(f^if^i_{ab}+f^i_af^i_b)=0,\quad \sum_{i=1}^n f^i_af^i_b=r\delta_{ab};\\ &(\phi^\alpha_a)_{1\times(n+1)}=(uf^i_a,0),\quad (\phi^\alpha_n)_{1\times(n+1)}=(f^i,g');\\ &(\phi^\alpha_{ab})_{1\times(n+1)}=(uf^i_{ab},0),\quad (\phi^\alpha_{an})_{1\times(n+1)}=(f^i_a,0),\quad (\phi^\alpha_{nn})_{1\times(n+1)}=(0,g'').\end{aligned}$$

$$r=\begin{cases}1, & a=b=n-1,\\ \cos^2\theta^{n-1}\cdots\cos^2\theta^{a+1}, & a=b=1,\dots,n-2.\end{cases} \tag{4.4}$$

Then

$$\bar{h}_{ab}=\sum_i u^2r\delta_{ab},\quad \bar{h}_{an}=0,\quad \bar{h}_{nn}=1+(g')^2; \tag{4.5}$$

$$\bar{h}^{ab}=\sum_i\frac{1}{u^2r}\delta_{ab},\quad \bar{h}^{an}=0,\quad \bar{h}^{nn}=\frac{1}{1+(g')^2}; \tag{4.6}$$

$$\bar{B}_{ab}=-\frac{g'}{u\sqrt{1+(g')^2}}\bar{h}_{ab},\quad \bar{B}_{an}=0,\quad \bar{B}_{nn}=-\frac{g''}{\sqrt{1+(g')^2}}, \tag{4.7}$$

where $\bar{h}_{ij}=\langle\phi_i,\phi_j\rangle$ and $\bar{B}_{ij}=\langle\phi_{ij},\bar{\boldsymbol{n}}\rangle$ are the components of fundamental tensor and the second fundamental form of M in (N,h), respectively.

Let $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$ be the unit normal vector field of M with respect to F and h, respectively, where $\bar{\boldsymbol{n}}$ is given by (3.4). Using (2.3), $F^*=h^*+W^*=\sqrt{\langle\xi,\xi\rangle}-\epsilon x^\alpha\xi_\alpha$, and setting $\xi_\alpha=\frac{\bar{\boldsymbol{n}}^\alpha}{F^*(\bar{\boldsymbol{n}})}$, we have

$$[F^*]_{\xi_\alpha} = \frac{\xi_\alpha}{h^*} - \epsilon x^\alpha = \frac{\bar{\boldsymbol{n}}^\alpha}{h^* F^*(\bar{\boldsymbol{n}})} - \epsilon x^\alpha = \bar{\boldsymbol{n}}^\alpha - \epsilon x^\alpha, \tag{4.8}$$

$$\boldsymbol{n} = \boldsymbol{n}^\alpha \frac{\partial}{\partial x^\alpha} = [F^*]_{\xi_\alpha}(\xi)\frac{\partial}{\partial x^\alpha} = \bar{\boldsymbol{n}} - \epsilon x. \tag{4.9}$$

According to [12], for the Funk space

$$N^\alpha_\beta = \frac{\partial G^\alpha}{\partial y^\beta} = \frac{\epsilon}{2}(F_{y^\beta} y^\alpha + F\delta^\alpha_\beta). \tag{4.10}$$

Note that $F(\boldsymbol{n}) = 1$, so

$$\begin{aligned} D^{\boldsymbol{n}}_X \boldsymbol{n} =& X^i(\boldsymbol{n}^\alpha_{u^i} + N^\alpha_\beta(\boldsymbol{n})\phi^\beta_i)\frac{\partial}{\partial x^\alpha} \\ =& X^i(\bar{\boldsymbol{n}}^\alpha_{u^i} - \epsilon x^\alpha_{u^i} + \frac{\epsilon}{2}(F_{y^\beta} y^\alpha + F\delta^\alpha_\beta)\phi^\beta_i)\frac{\partial}{\partial x^\alpha} \\ =& X^i(\bar{\boldsymbol{n}}^\alpha_{u^i} - \epsilon \phi^\alpha_i + \frac{\epsilon}{2}\phi^\alpha_i)\frac{\partial}{\partial x^\alpha} \\ =& \nabla^h_X \bar{\boldsymbol{n}} - \frac{\epsilon}{2}\mathrm{d}\phi X, \end{aligned}$$

where $u^a = \theta^a, u^n = u$. Using the above equation and Weingarten formula (2.4), we can get

$$\mathrm{d}\phi A(X) = \frac{\varepsilon}{2}\mathrm{d}\phi X + \mathrm{d}\phi \bar{A}(X). \tag{4.11}$$

When $X = \frac{\partial}{\partial \theta^a}$ or $\frac{\partial}{\partial u}$, we have

$$\phi^\alpha_a B^a_b = \frac{\varepsilon}{2}\phi^\alpha_b + \phi^\alpha_a \bar{B}^a_b,\ \phi^\alpha_n B^n_n = \frac{\varepsilon}{2}\phi^\alpha_n + \phi^\alpha_n \bar{B}^n_n, \tag{4.12}$$

i.e.,

$$\phi^\alpha_j B^j_i = \frac{\epsilon}{2}\phi^\alpha_i + \phi^\alpha_j \bar{B}^j_i, \tag{4.13}$$

where $\bar{B}^i_j = \bar{h}^{ik}\bar{B}_{kj},\ B^i_j = \hat{g}^{ik}B_{kj}$. Then we can get

$$H = \bar{H} + \frac{\epsilon}{2}, \tag{4.14}$$

where $\bar{H} = \frac{1}{n}\bar{h}^{ij}\bar{B}_{ij}$ is the mean curvature. In order to get anisotropic minimal rotational hypersurfaces, $H = 0$. Its expression is

$$\frac{(n-1)g'}{u\sqrt{1+(g')^2}} + \frac{g''}{(1+(g')^2)^{\frac{3}{2}}} - \frac{\epsilon n}{2} = 0, \tag{4.15}$$

this is

$$\frac{(n-1)g'((1+(g')^2) + ug''}{(1+(g'))^{\frac{3}{2}}} - \frac{\epsilon n u}{2} = 0.$$

This is an ODE equation. We can use the method of constant variation to solve it. Therefore,

$$g = \int \sqrt{\frac{(\epsilon u^n + C_1)^2}{4u^{2(n-1)} - (\epsilon u^n + C_1)^2}}\mathrm{d}u + C_2, \tag{4.16}$$

where C_1, C_2 are arbitrary constants. □

Acknowledgements We thank the referees for their time and comments.

References

[1] D. BAO, C. ROBLES, Zhongmin SHEN. *Zermelo navigation on Riemannian manifolds*. J. Differ. Geom., 2004, **66**(3): 377–435.

[2] Ningwei CUI, Yibing SHEN. *Nontrivial minimal surfaces in a hyperbolic Randers space*. Math. Nachr., 2017, **290**(4): 570–582.

[3] Qun HE, Yibing SHEN. *On Bernstein type theorems in Finsler spaces with the volume form induced from the projective sphere bundle*. Proc. Amer. Math. Soc., 2006, **134**(3): 871–880.

[4] M. SOUZA, K. TENENBLAT. *Minimal surfaces of rotation in a Finsler space with a Randers metric*. Math. Ann., 2003, **325**(4): 625–642.

[5] Zhongmin SHEN. *On Finsler geometry of submanifolds*. Math. Ann., 1998, **311**(3): 549–576.

[6] Qun HE, Yibing SHEN. *On the mean curvature of Finsler submanifolds*. Chin. Ann. Math., 2006, **27**(5): 663–674.

[7] Songting YIN, Qun HE, Dinghe XIE. *Minimal submanifolds in general* (α,β)*-spaces*. Ann. Polon. Math., 2013, **108**(1): 43–59.

[8] Ningwei CUI. *Nontrivial minimal surfaces in a class of Finsler 3-spheres*. Internat. J. Math., 2014, **25**(4): 17.

[9] Ningwei CUI. *Nontrivial minimal surfaces in a class of Finsler spheres of vanishing* S*-curvature*. Nonlinear Anal., 2015, **129**: 143–159.

[10] D. BAO, S. S. CHERN, Zhongmin SHEN. *An introduction to Riemann-Finsler Geometry*. Springer-Verlag, New York, 2000

[11] Zhongmin SHEN. *Lectures on Finsler Geometry*. World Scientific Publishing Co., Singapore, 2001.

[12] Qun HE, Songting YIN, Yibing SHEN. *Isoparametric hypersurfaces in Funk spaces*. Sci. China Math., 2017, **60**(12): 2447–2464.