Journal of Mathematical Research with Applications
Nov., 2020, Vol. 40, No. 6, pp. 628–646
DOI:10.3770/j.issn:2095-2651.2020.06.007
Http://jmre.dlut.edu.cn

A Convergent Family of Linear Hermite Barycentric Rational Interpolants

Ke JING[1], Yezheng LIU[2], Ning KANG[3,*], Gongqin ZHU[4]

1. *School of Applied Mathematics, Nanjing University of Finance and Economics, Jiangsu* 210023*, P. R. China;*
2. *School of Management, Hefei University of Technology, Anhui* 230009*, P. R. China;*
3. *School of Economics, Nanjing University of Finance and Economics, Jiangsu* 210023*, P. R. China;*
4. *School of Mathematics, Hefei University of Technology, Anhui* 230009*, P. R. China*

Abstract It is well-known that Hermite rational interpolation gives a better approximation than Hermite polynomial interpolation, especially for large sequences of interpolation points, but it is difficult to solve the problem of convergence and control the occurrence of real poles. In this paper, we establish a family of linear Hermite barycentric rational interpolants r that has no real poles on any interval and in the case $k = 0, 1, 2$, the function $r^{(k)}(x)$ converges to $f^{(k)}(x)$ at the rate of $O(h^{3d+3-k})$ as $h \to 0$ on any real interpolation interval, regardless of the distribution of the interpolation points. Also, the function $r(x)$ is linear in data.

Keywords linear Hermite rational interpolation; convergence rate; Hermite interpolation; barycentric form; higher order derivative

MR(2010) Subject Classification 65D05; 41A05; 41A20; 41A25

1. Introduction

Hermite interpolation consists of choosing a sequence of interpolation points

$$a = x_0 < x_1 < \cdots < x_n = b \tag{1.1}$$

to fit to f the Hermite interpolation polynomial p of degree at most $\sum_{i=0}^{n} s_i - 1$ at these interpolation points [1], i.e., with

$$p^{(k)}(x_i) = f^{(k)}(x_i), \quad k = 0, 1, \ldots, s_i - 1;\ i = 0, 1, \ldots, n.$$

However, it is well-known that $p^{(k)}$ may not be a good approximation to $f^{(k)}$ $(k = 0, 1, \ldots, s_i - 1)$ and it can exhibit Runge's phenomenon for large sequences of interpolation points. If we choose the distribution of the interpolation points x_i freely, one remedy is to use various kinds of Chebyshev points [2].

Received December 17, 2019; Accepted April 23, 2020
Supported by the National Natural Science Foundation of China (Grant No. 11601224), the Science Foundation of Ministry of Education of China (Grant No. 18YJC790069), the Natural Science Foundation of Jiangsu Higher Education Institutions of China (Grant No. 18KJD110007) and the National Statistical Science Research Project of China (Grant No. 2018LY28).
* Corresponding author
E-mail address: 9120171058@nufe.edu.cn (Ning KANG)

Instead, if the interpolation points x_i are prescribed, then we need to search for other kinds of interpolants. On the one hand, a popular alternative was to use osculatory rational interpolants. For example, Salzer transformed the nonlinear problem of osculatory rational interpolants into a linear problem [3]. By using continued fractions [4–6], Wuytack investigated a new algorithm for osculatory rational interpolants. By using the same techniques as Wuytack, Claessens presented an algorithm for osculatory rational interpolants which did not suffer from the drawback of computing the intermediate auxiliary quantities [7]. Wang generalized the Thiele-Werner-type osculatory rational interpolants to the case of vector-valued osculatory rational interpolants [8]. By using the Padé approximant and the Newton interpolation formula, an algorithm for vector-valued osculatory rational interpolants was investigated in [9, 10]. On the other hand, some researchers focused on the linear Hermite rational interpolants, among them, Schneider and Werner [11] suggested that every Hermite rational interpolant may be written in barycentric form

$$r(x) = \frac{\sum_{i=0}^{n} \sum_{k=0}^{s_i-1} \frac{w_{i,k}}{(x-x_i)^{k+1}} \sum_{j=0}^{k} \frac{f^{(j)}(x_i)}{j!}(x-x_i)^j}{\sum_{i=0}^{n} \sum_{k=0}^{s_i-1} \frac{w_{i,k}}{(x-x_i)^{k+1}}},$$

for some barycentric weights $w_{i,k}$. Thus it suffices to choose the so-called barycentric weights $w_{i,k}$ in order to specify r. The idea of searching for barycentric weights $w_{i,k}$ gives an interpolant r that has good convergence properties and no real poles. Schneider and Werner [11] showed that this interpolant r had no real poles only in the interpolation interval $[a, b]$, when the rational function is not reduced and the barycentric weights $w_{i,k}$ satisfy the following conditions:

$$\operatorname{sign} w_{i,s_i-1} = (-1)^{s_{i+1}} \operatorname{sign} w_{i+1,s_{i+1}-1}, \quad i = 0, 1, \ldots, n-1.$$

Extending the work in [12], Floater and Hormann [13] has presented a family of barycentric rational interpolants which has no real poles for all $x \in \mathbb{R}$ and convergence rate $O(h^{d+1})$. Berrut [14] has shown $O(h^{d+1-k})$ convergence of the k-th derivative of the family of barycentric rational interpolants for $k = 1$ and 2. In order to derive linear Hermite barycentric rational interpolants without real poles in $\mathbb{R}$ and arbitrarily high convergence rates, the Floater-Hormann rational interpolants have been generalized in three ways. Jing et al. [15] focused on the special case $s_i = 2$, which shows that these interpolants not only possess the convergence rate of $O(h^{2d+1-k})$ of k-th derivative function for $k = 0, 1$, as $h \to 0$, but also are linear in the data and free of real poles in $\mathbb{R}$. Floater and Schulz [16] derived a Hermite version of the Floater-Hormann interpolants by considering multiple interpolation points. Cirillo and Hormann [17, 18] proposed an iterative approach to the barycentric rational Hermite interpolants, which only possess the convergence of interpolation function itself, but the convergence of its derivative function is not studied.

In order to obtain linear Hermite rational interpolants without real poles in $\mathbb{R}$ and arbitrarily high convergence rates for larger s_i and k, the purpose of this paper is to report that there is a family of linear Hermite rational interpolants for the special case $s_i = 3$ and $k = 0, 1, 2$, which is given by the following formula

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \tag{1.2}$$

p_i denotes the Hermite interpolation polynomial of degree at most $3d+2$ that satisfies the interpolation conditions $p_i^{(k)}(x_i) = f^{(k)}(x_i)$, $k = 0, 1, 2$ at the subset $\{x_i, \dots, x_{i+d}\}$ of (1.1), and

$$\lambda_i(x) = \frac{(-1)^{3i}}{(x - x_i)^3 \cdots (x - x_{i+d})^3}.$$

This construction gives a family of linear Hermite rational interpolants, none of which has any real poles in $\mathbb{R}$ for each $d = 0, 1, \dots, n$. Furthermore, the interpolation function r converges to f at the rate of $O(h^{3d+3})$ as $h \to 0$ (under a bounded mesh ratio condition, if $d = 0$), as long as $f(x) \in C^{3d+4}[a, b]$; for fixed $d \geq 1$, the first derivative of interpolation function r' converges to f' at the rate of $O(h^{3d+2})$ as $h \to 0$ (under a bounded mesh ratio condition, if $d = 1$), as long as $f(x) \in C^{3d+5}[a, b]$; for fixed $d \geq 3$, the second derivative of interpolation function r'' converges to f'' at the rate of $O(h^{3d+1})$ as $h \to 0$ (under a bounded mesh ratio condition), as long as $f(x) \in C^{3d+6}[a, b]$, where

$$h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i). \tag{1.3}$$

The organization of this paper is as follows. In Section 2, we present the linear Hermite rational interpolation functions without real poles satisfying the Hermite interpolation conditions. Section 3 describes convergence order of the linear Hermite rational interpolation functions. In Section 4, the barycentric form of the linear Hermite rational interpolation functions is discussed. While Section 5 demonstrates the efficiency of the proposed method. In Section 6, our conclusions are presented.

2. Absence of poles and satisfying the interpolation condition

An important property of the interpolant in (1.2) is that it not only has no real poles for all $x \in \mathbb{R}$ but also satisfies the interpolation condition $r^{(k)}(x_i) = f^{(k)}(x_i)$ $(k = 0, 1, 2; i = 0, 1, \dots, n)$. In order to establish this, we multiply the numerator and denominator in (1.2) by the product, respectively,

$$(-1)^{3n-3d}(x - x_0)^3 \cdots (x - x_n)^3$$

and give

$$r(x) = \frac{\sum_{i=0}^{n-d} \mu_i(x) p_i(x)}{\sum_{i=0}^{n-d} \mu_i(x)}, \tag{2.1}$$

where

$$\mu_i(x) = \prod_{j=0}^{i-1} (x - x_j)^3 \prod_{k=i+d+1}^{n} (x_k - x)^3. \tag{2.2}$$

By the definition of μ_i in (2.2), we now show that r has no real poles for all $x \in \mathbb{R}$.

Theorem 2.1 *For every admissible $d, 0 \le d \le n$, the rational function r in (2.1) has no real poles in $\mathbb{R}$ and satisfies the following interpolation conditions $r^{(k)}(x_i) = f^{(k)}(x_i)$ $(k = 0, 1, 2; i = 0, 1, \ldots, n)$.*

Proof We will show that the denominator of r in (2.1),

$$s(x) = \sum_{i=0}^{n-d} \mu_i(x)$$

is positive for all $x \in \mathbb{R}$ and use the following index sets

$$I = \{0, 1, \ldots, n-d\}, \quad J_\alpha = \{i \in I : \alpha - d \le i \le \alpha\}.$$

and first consider the case that $x = x_\alpha$ for some $\alpha, 0 \le \alpha \le n$. Then from (2.2), we can derive that $\mu_i(x_\alpha) = 0$ for all $i \in I \backslash J_\alpha$ and $\mu_i(x_\alpha) > 0$ for all $i \in J_\alpha$. Hence,

$$s(x_\alpha) = \sum_{i \in I} \mu_i(x_\alpha) = \sum_{i \in J_\alpha} \mu_i(x_\alpha) > 0.$$

Next suppose that $x \in (x_\alpha, x_{\alpha+1})$ for some $\alpha, 0 \le \alpha \le n-1$, then let

$$I_1 = \{i \in I : i \le \alpha - d\}, \; I_2 = \{i \in I : \alpha - d + 1 \le i \le \alpha\}, \; I_3 = \{i \in I : \alpha + 1 \le i\}. \tag{2.3}$$

We also divide the sum $s(x)$ into three parts as in [13],

$$s(x) = s_1(x) + s_2(x) + s_3(x), \quad \text{with} \quad s_k(x) = \sum_{i \in I_k} \mu_i(x).$$

We will show that the partial sum $s_k(x) > 0$ for each $k = 1, 2, 3$, if I_k is non-empty. Since $s_k(x) = 0$, if I_k is empty, and since one of I_1, I_2, I_3 is non-empty at least, it then follows that $s(x)$ is positive for all $x \in \mathbb{R}$.

To this end, we first consider the partial sum s_2. If $d \ge 1$, then I_2 is non-empty and from (2.2), we derive that $\mu_i(x) > 0$ for all $i \in I_2$, therefore, $s_2(x)$ is positive for all $x \in \mathbb{R}$. If $d = 0$, then I_2 is empty.

Next, we consider the partial sum s_3. If $\alpha \ge n - d$, then I_3 is empty. Otherwise, $\alpha \le n - d - 1$, I_3 is non-empty and

$$s_3(x) = \mu_{\alpha+1}(x) + \mu_{\alpha+2}(x) + \cdots + \mu_{n-d}(x).$$

Using (2.2), we derive that $\mu_{\alpha+1}(x) > 0, \mu_{\alpha+2}(x) < 0, \mu_{\alpha+3}(x) > 0$, and so on. Moreover, by the definition of $\mu_i(x)$, we can further show that the terms in $s_3(x)$ are decreasing in absolute value, i.e.,

$$|\mu_{\alpha+1}(x)| > |\mu_{\alpha+2}(x)| > |\mu_{\alpha+2}(x)| > \cdots .$$

To derive this, suppose $i \ge \alpha + 1$ and compare the expression of μ_{i+1},

$$\mu_{i+1}(x) = \prod_{j=0}^{i} (x - x_j)^3 \prod_{k=i+d+2}^{n} (x_k - x)^3,$$

with that of μ_i. Since

$$(x_{i+d+1} - x)^3 > (x_i - x)^3,$$

it follows that $|\mu_i(x)| > |\mu_{i+1}(x)|$. Hence, by expression $s_3(x)$ in the form

$$s_3(x) = (\mu_{\alpha+1}(x) + \mu_{\alpha+2}(x)) + (\mu_{\alpha+3}(x) + \mu_{\alpha+4}(x)) + \cdots,$$

we derive $s_3(x) > 0$.

Using similar arguments, we will show that $s_1(x) > 0$. If I_1 is non-empty, then we can express s_1 as

$$s_1(x) = (\mu_{\alpha-d}(x) + \mu_{\alpha-d-1}(x)) + (\mu_{\alpha-d-2}(x) + \mu_{\alpha-d-3}(x)) + \cdots.$$

We have shown that $s(x) > 0$ for all $x \in [x_0, x_n]$. In the end, using similar arguments, we can derive that $s(x) > 0$ for $x < x_0$ from writing it as

$$s(x) = (\mu_0(x) + \mu_1(x)) + (\mu_2(x) + \mu_3(x)) + \cdots,$$

and for $x > x_n$ by writing it as

$$s(x) = (\mu_{n-d}(x) + \mu_{n-d-1}(x)) + (\mu_{n-d-2}(x) + \mu_{n-d-3}(x)) + \cdots.$$

The above mentioned has established that r has no real poles. It is now easy to check that $r(x)$ satisfies the interpolation conditions $r^{(k)}(x_i) = f^{(k)}(x_i)$ $(k = 0, 1, 2; i = 0, 1, \ldots, n)$. Indeed, if $x = x_\alpha$ in (2.1) for some α with $0 \le \alpha \le n$, then $p_i^{(k)}(x_\alpha) = f^{(k)}(x_\alpha)$ for all $i \in J_\alpha$ with $k = 0, 1, 2$, and recalling that $\mu_i(x_\alpha) > 0$ for all $i \in J_\alpha$, otherwise $\mu(x_\alpha) = 0$,

$$r(x_\alpha) = \frac{\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha) p_i(x_\alpha)}{\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)} = f(x_\alpha)\frac{\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)}{\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)} = f(x_\alpha),$$

$$r'(x_\alpha) = \frac{\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha) \sum\limits_{i\in J_\alpha} \big[\mu_i'(x_\alpha) p_i(x_\alpha) + \mu_i(x_\alpha) p_i'(x_\alpha)\big] - \sum\limits_{i\in J_\alpha} \mu_i(x_\alpha) p_i(x_\alpha) \sum\limits_{i\in J_\alpha} \mu_i'(x_\alpha)}{\big(\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)\big)^2}$$

$$= \frac{\big(\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)\big)^2 f'(x_\alpha)}{\big(\sum\limits_{i\in J_\alpha} \mu_i(x_\alpha)\big)^2} = f'(x_\alpha).$$

Using similar arguments, we derive that $r''(x_\alpha) = f''(x_\alpha)$. $\Box$

3. Error at intermediate points

Next we will deal with the convergence order of the linear Hermite rational interpolant. Here we treat the case $d \ge 1$, the advantage of which is that the index set I_2 in (2.3) is non-empty, since we can use the partial sum $s_2(x)$ to get an error bound. Let $\|f\| = \max_{x\in[a,b]} |f(x)|$.

Theorem 3.1 *Suppose $1 \le d \le n$, $f \in C^{3d+4}[a, b]$, and h be as in (1.3). Then, if $n - d$ is odd*

$$\|f - r\| = \|e\| \le 3(b-a)(d+1)(d!)^3 h^{3d+3} \frac{\|f^{(3d+4)}\|}{(3d+4)!} = K h^{3d+3};$$

if $n - d$ is even,

$$\|f - r\| = \|e\| \le (d!)^3 h^{3d+3} \big(3(b-a)(d+1)\frac{\|f^{(3d+4)}\|}{(3d+4)!} + \frac{\|f^{(3d+3)}\|}{(3d+3)!}\big) = K h^{3d+3}.$$

In the above estimates, the constant K only depends on f, d and the interpolation interval $[a, b]$.

Proof For any point $x \in [a, b]$, we can express the error as

$$e(x) = f(x) - r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x)(f(x) - p_i(x))}{\sum_{i=0}^{n-d} \lambda_i(x)}.$$

Using the Hermite interpolation error formula [19, Chap. VI],

$$f(x) - p_i(x) = f[(x_i)^3, \dots, (x_{i+d})^3, x](x - x_i)^3 \cdots (x - x_{i+d})^3, \tag{3.1}$$

we thus get

$$f(x) - r(x) = \frac{\sum_{i=0}^{n-d} (-1)^{3i} f\left[(x_i)^3, \dots, (x_{i+d})^3, x\right]}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)}, \tag{3.2}$$

where

$$A(x) = \sum_{i=0}^{n-d} (-1)^{3i} f[(x_i)^3, \dots, (x_{i+d})^3, x]$$

and

$$B(x) = \sum_{i=0}^{n-d} \lambda_i(x).$$

Next, we first consider the numerator in (3.2). In order to avoid a bound which depends on n, we go to divided differences of higher order by employing the oscillating signs. By combining the first and second terms, the third and fourth terms, and so on, we can express it as

$$\sum_{i=0,\, i \text{ even}}^{n-d-1} (x_i - x_{i+d+1})\big(f[(x_i)^3, \dots, (x_{i+d})^3, x_{i+d+1}, x] + f[(x_i)^2, (x_{i+1})^3, \dots, (x_{i+d})^3, (x_{i+d+1})^2, x] + f[x_i, (x_{i+1})^3, \dots, (x_{i+d+1})^3, x]\big),$$

if $n - d$ is odd, and

$$\sum_{i=0,\, i \text{ even}}^{n-d-2} (x_i - x_{i+d+1})\big(f[(x_i)^3, \dots, (x_{i+d})^3, x_{i+d+1}, x] + f[(x_i)^2, (x_{i+1})^3, \dots, (x_{i+d})^3, (x_{i+d+1})^2, x] + f[x_i, (x_{i+1})^3, \dots, (x_{i+d+1})^3, x]\big) + f[(x_{n-d})^3, \dots, (x_n)^3, x],$$

if $n - d$ is even. In view of

$$\sum_{i=0}^{n-d-1} (x_{i+d+1} - x_i) = \sum_{i=0}^{n-d-1} \sum_{k=i}^{i+d} (x_{k+1} - x_k) \le (d+1) \sum_{k=0}^{n-1} (x_{k+1} - x_k) = (d+1)(b-a),$$

it follows that

$$\Big| \sum_{i=0}^{n-d} (-1)^{3i} f[(x_i)^3, \dots, (x_{i+d})^3, x] \Big| \le 3(b-a)(d+1) \frac{\|f^{(3d+4)}\|}{(3d+4)!} \tag{3.3}$$

if $n-d$ is odd, and

$$\Big|\sum_{i=0}^{n-d}(-1)^{3i}f[(x_i)^3,\ldots,(x_{i+d})^3,x]\Big|\le 3(b-a)(d+1)\frac{\|f^{(3d+4)}\|}{(3d+4)!}+\frac{\|f^{(3d+3)}\|}{(3d+3)!}\tag{3.4}$$

if $n-d$ is even.

Next, considering the denominator in (3.2), we determine the open subinterval $(x_\alpha, x_{\alpha+1})$ for some α with $0\le\alpha\le n-1$. Since $d\ge1$, the index set I_2 is non-empty, so let j be any member of I_2. Then

$$s(x)\ge s_2(x)\ge\mu_j(x)>0,$$

and so, by the definition of μ_i, we derive that

$$\Big|\sum_{i=0}^{n-d}\lambda_i(x)\Big|=\frac{s(x)}{\prod_{i=0}^n|(x-x_i)^3|}\ge\frac{\mu_j(x)}{\prod_{i=0}^n|(x-x_i)^3|}=|\lambda_j(x)|=\frac{1}{|(x-x_j)^3\cdots(x-x_{j+d})^3|}.$$

To increase readability, the following notations are introduced: $d_i=|x-x_i|$, $d_{i,k}=|x_i-x_k|$ for $x\in[a,b]$.

Since $x_j\le x_\alpha<x<x_{\alpha+1}\le x_{j+d}$, we have

$$|(x-x_j)^3\cdots(x-x_{j+d})^3|=d_j^3\cdots d_{j+d}^3\le Kh^{3d+3},$$

hence

$$\Big|\sum_{i=0}^{n-d}\lambda_i(x)\Big|\ge\frac{1}{Kh^{3d+3}}.\tag{3.5}$$

Combined with (3.3) and (3.4), we derive the result from (3.5). □

Thus for $d\ge1, r$ converges to f at the rate of $O(h^{3d+3})$ as $h\to0$, as long as $f\in C^{3d+4}[a,b]$. In the remaining case $d=0$, we establish r converges to f at the rate of $O(h^3)$ but under a bounded mesh ratio

$$\beta=\max\Big\{\max_{1\le i\le n-1}\frac{d_{i,i+1}}{d_{i,i-1}},\max_{0\le i\le n-2}\frac{d_{i+1,i}}{d_{i+1,i+2}}\Big\}.\tag{3.6}$$

Theorem 3.2 *Suppose that $d=0$, $f\in C^4[a,b]$ and h be as in (1.3). If n is odd, then*

$$\|f-r\|=\|e\|\le3(b-a)h^3\frac{\|f^{(4)}\|}{4!(1-\beta^3)}=\frac{Kh^3}{1-\beta^3}.$$

If n is even, then

$$\|f-r\|=\|e\|\le h^3\frac{(3(b-a)\frac{\|f^{(4)}\|}{4!}+\frac{\|f^{(3)}\|}{3!})}{(1-\beta^3)}=\frac{Kh^3}{1-\beta^3}.$$

Proof For $d=0$, the Hermite interpolation error formula (3.1) also remains valid and reduces to

$$\Big|\sum_{i=0}^{n}(-1)^{3i}f[(x_i)^3,x]\Big|\le3(b-a)\frac{\|f^{(4)}\|}{4!}$$

if n is odd, and

$$\Big|\sum_{i=0}^{n}(-1)^{3i}f[(x_i)^3,x]\Big|\le3(b-a)\frac{\|f^{(4)}\|}{4!}+\frac{\|f^{(3)}\|}{3!}$$

if n is even. Thus we only need to show that the denominator in (3.2) satisfies the lower bound

$$\Big|\sum_{i=0}^{n}\lambda_i(x)\Big| \ge \frac{1-\beta^3}{h^3}. \tag{3.7}$$

Since $d=0$, the index set I_2 in (2.3) is empty, and we must turn to $s_1(x)$ and $s_3(x)$. Suppose first that $\alpha = n-1$, then

$$s(x) \ge s_3(x) = \mu_n(x),$$

and so

$$\Big|\sum_{i=0}^{n}\lambda_i(x)\Big| \ge |\lambda_n(x)| = \frac{1}{(x_n-x)^3} \ge \frac{1}{h^3}.$$

Similarly, if $\alpha = 0$, we have

$$s(x) \ge s_1(x) = \mu_0(x),$$

and so

$$\Big|\sum_{i=0}^{n}\lambda_i(x)\Big| \ge |\lambda_0(x)| = \frac{1}{(x-x_0)^3} \ge \frac{1}{h^3},$$

which has proved (3.7). Otherwise, $1 \le \alpha \le n-2$ and we get a bound from s_1 and s_3. On one hand, using s_3, we have

$$s(x) \ge s_3(x) \ge \mu_{\alpha+1}(x) + \mu_{\alpha+2}(x),$$

and then

$$\begin{aligned}\Big|\sum_{i=0}^{n}\lambda_i(x)\Big| &\ge |\lambda_{\alpha+1}(x)+\lambda_{\alpha+2}(x)| = \frac{1}{(x_{\alpha+1}-x)^3} - \frac{1}{(x_{\alpha+2}-x)^3}\\ &= \frac{d_{\alpha+2}^3 - d_{\alpha+1}^3}{d_{\alpha+1}^3 d_{\alpha+2}^3} = \frac{1-d_{\alpha+1}^3/d_{\alpha+2}^3}{d_{\alpha+1}^3}.\end{aligned}$$

On the other hand, using s_1 and similar arguments to the above, we have

$$\Big|\sum_{i=0}^{n}\lambda_i(x)\Big| \ge \frac{d_{\alpha-1}^3 - d_{\alpha}^3}{d_{\alpha}^3 d_{\alpha-1}^3} = \frac{1-d_{\alpha}^3/d_{\alpha-1}^3}{d_{\alpha}^3}.$$

Taking the maximum of these two lower bounds, we can derive (3.7). □

We now consider the convergence rate of the derivative at any point $x \in [a,b]$. For the first derivative, we derive that r' converges to f' at the rate of $O(h^{3d+2})$ as $h \to 0$, but under the stricter condition that $f(x) \in C^{3d+5}[a,b]$; and for the second derivative, we derive that r'' converges to f'' at the rate of $O(h^{3d+1})$ as $h \to 0$, but under the stricter condition that $f(x) \in C^{3d+6}[a,b]$.

Lemma 3.3 *If $f \in C^{3d+4+k}[a,b]$ for $k \in \mathbb{N}$, then*

$$|A^{(k)}(x)| \le K,\ x \in [a,b].$$

Proof The case $s_i = 1$ has been treated in [14]. For $s_i = 3$ and using the derivative formula for Newton divided differences [1, 14], we have

$$A^{(k)}(x) = k!\sum_{i=0}^{n-d}(-1)^i f[(x_i)^{3i},\ldots,(x_{i+d})^{3i},(x)^{k+1}],$$

then, using similar arguments as in [14], we derive that $A^{(k)}/k!$ equals

$$\sum_{i=0,\, i\ even}^{n-d-1}(x_i-x_{i+d+1})\big(f[(x_i)^3,\ldots,(x_{i+d})^3,x_{i+d+1},(x)^{k+1}]+ f[(x_i)^2,(x_{i+1})^3,\ldots,(x_{i+d})^3,(x_{i+d+1})^2,(x)^{k+1}]+ f[x_i,(x_{i+1})^3,\ldots,(x_{i+d+1})^3,(x)^{k+1}]\big),$$

if $n-d$ is odd, and

$$\sum_{i=0,\, i\ even}^{n-d-2}(x_i-x_{i+d+1})\big(f[(x_i)^3,\ldots,(x_{i+d})^3,x_{i+d+1},(x)^{k+1}]+ f[(x_i)^2,(x_{i+1})^3,\ldots,(x_{i+d})^3,(x_{i+d+1})^2,x^{k+1}]+ f[x_i,(x_{i+1})^3,\ldots,(x_{i+d+1})^3,(x)^{k+1}]\big)+f[(x_{n-d})^3,\ldots,(x_n)^3,(x)^{k+1}],$$

if $n-d$ is even. Using similar arguments as in [14], we can derive result. □

Theorem 3.4 *Suppose $d\ge 2$ and $f\in C^{3d+5}[a,b]$. Then*

$$\|f'-r'\|=\|e'\|\le Kh^{3d+2}. \tag{3.8}$$

Proof Due to the continuity of e', it is sufficient to let $x\in(x_j,x_{j+1})$ and to derive (3.8), independently of j, we differentiate (3.2),

$$e'(x)=\frac{A'(x)}{B(x)}-A(x)\frac{B'(x)}{B^2(x)}. \tag{3.9}$$

In the proof of (3.5), we have shown that

$$|B(x)|=\Big|\sum_{i=0}^{n-d}\lambda_i(x)\Big|\ge\frac{1}{Kh^{3d+3}},\quad \forall x\in[a,b],$$

and so it follows from Lemma 3.3 that

$$\frac{|A'(x)|}{|B(x)|}\le Kh^{3d+3}.$$

Since $|A(x)|\le K$, we only need to show that

$$\frac{|B'(x)|}{|B^2(x)|}\le Kh^{3d+2}. \tag{3.10}$$

We use the index sets in (2.3). Now

$$|B'(x)|=\Big|\sum_{i\in I}\sum_{m=0}^{d}\frac{3(-1)^{3i+1}}{(x-x_i)^3\cdots(x-x_{i+d})^3(x-x_{i+m})}\Big| \le 3\sum_{m=0}^{d}(M_{m,1}+M_{m,2}+M_{m,3}), \tag{3.11}$$

where we set

$$M_{m,k}=\Big|\sum_{i\in I_k}\frac{(-1)^{3i+1}}{(x-x_i)^3\cdots(x-x_{i+d})^3(x-x_{i+m})}\Big|,\quad k=1,2,3.$$

The terms in $M_{m,1}$ and $M_{m,3}$ are oscillating in sign and increasing, respectively decreasing in absolute value and so

$$M_{m,1} \le \frac{1}{d_{\alpha-d}^3 \cdots d_\alpha^3 d_{\alpha-d+m}} \quad \text{and} \quad M_{m,3} \le \frac{1}{d_{\alpha+1}^3 \cdots d_{\alpha+d+1}^3 d_{\alpha+1+m}}.$$

In the proof of (3.5), we have shown that

$$|B(x)| \ge |\lambda_i(x)|, \quad \forall i \in I_2. \tag{3.12}$$

Next, using (3.12) with $i = \alpha - d + 1$, we divide $M_{m,1}$ by $|B^2(x)|$, so that

$$\frac{M_{m,1}}{|B^2(x)|} \le \frac{d_{\alpha-d+1}^6 \cdots d_{\alpha+1}^6}{d_{\alpha-d}^3 \cdots d_\alpha^3 d_{\alpha-d+m}} = \frac{d_{\alpha-d+1}^3 \cdots d_\alpha^3 d_{\alpha+1}^6}{d_{\alpha-d}^3 d_{\alpha-d+m}};$$

since $d_\alpha / d_{\alpha-d+m} \le 1$ for $m = 0, \ldots, d$ and $d_{\alpha-d+1}/d_{\alpha-d} \le 1$, we derive that

$$\frac{M_{m,1}}{|B^2(x)|} \le d_{\alpha-d+2}^3 \cdots d_{\alpha-1}^3 d_\alpha^2 d_{\alpha+1}^6 \le K h^{3d+2}.$$

Similarly,

$$\frac{M_{m,3}}{|B^2(x)|} \le K h^{3d+2}.$$

Finally, we claim that $M_{m,2}/|B^2(x)|$ is also bounded. In fact, since $d \ge 2, I_2$ is non-empty. We choose the same $i \in I_2$ in (3.12) as in each term of $M_{m,2}$, it follows that

$$\frac{M_{m,2}}{|B^2(x)|} \le \sum_{i \in I_2} \frac{d_i^6 \cdots d_{i+d}^6}{d_i^3 \cdots d_{i+d}^3 d_{i+m}} = \sum_{i \in I_2} d_i^3 \cdots d_{i+m-1}^3 d_{i+m}^2 d_{i+m+1}^3 \le K h^{3d+2}.$$

Thus we derive the result (3.10) from (3.11). □

In the remaining case $d = 1$, we establish r' converges to f' at the rate of $O(h^5)$ but under a bounded mesh ratio β in (3.6).

Theorem 3.5 *If $d = 1$ and $f \in C^8[a,b]$, then*

$$\|f' - r'\| = \|e'\| \le K(2\beta + 1)h^5.$$

Proof Again considering (3.9), we determine the open subinterval $(x_\alpha, x_{\alpha+1})$. For $d = 1$, $|A(x)|, |A'(x)|$ and $|B(x)|$ are bounded from the previous theorem and lemma, so $|B'(x)|/|B^2(x)|$ is bounded for $d = 1$ and $I_2 = \{\alpha\}$. Using similar arguments as in previous lemma and theorem, we derive

$$\begin{aligned}
\frac{|B'(x)|}{|B^2(x)|} &\le \sum_{m=0}^{1} \Big[\frac{1}{|B^2(x)|} \Big| 3 \sum_{i \in I} \frac{1}{(x-x_i)^3 (x-x_{i+1})^3 (x - x_{i+m})} \Big| \Big] \\
&\le 3 \sum_{m=0}^{1} \Big(\frac{d_\alpha^6 d_{\alpha+1}^6}{d_{\alpha-1}^3 d_\alpha^3 d_{\alpha-1+m}} + \frac{d_\alpha^6 d_{\alpha+1}^6}{d_\alpha^3 d_{\alpha+1}^3 d_{\alpha+m}} + \frac{d_\alpha^6 d_{\alpha+1}^6}{d_{\alpha+1}^3 d_{\alpha+2}^3 d_{\alpha+1+m}} \Big) \\
&= 3 \Big(\frac{d_\alpha^3 d_{\alpha+1}^6}{d_{\alpha-1}^4} + \frac{d_\alpha^2 d_{\alpha+1}^6}{d_{\alpha-1}^3} + d_\alpha^2 d_{\alpha+1}^3 + d_\alpha^3 d_{\alpha+1}^2 + \frac{d_\alpha^6 d_{\alpha+1}^2}{d_{\alpha+2}^3} + \frac{d_\alpha^6 d_{\alpha+1}^3}{d_{\alpha+2}^4} \Big) \\
&\le 3 \Big(2 \frac{d_{\alpha+1}^6}{d_{\alpha-1}} + d_\alpha^2 d_{\alpha+1}^3 + d_\alpha^3 d_{\alpha+1}^2 + 2 \frac{d_\alpha^6}{d_{\alpha+2}} \Big) \\
&\le K(2\beta + 1) h^5. \quad \square
\end{aligned}$$

Lemma 3.6 *If $d \geq 1$ and $x \in [a,b]$, then*

$$\frac{1}{|\tilde{B}(x)|} = \frac{1}{|B(x)\psi(x)|} = \frac{1}{|B(x)(x-x_\alpha)^3(x-x_{\alpha+1})^3|} \leq Kh^{3d-3}.$$

Proof We continue to consider the open subinterval $(x_\alpha, x_{\alpha+1})$ and define $\tilde{B}(x)$ as

$$\tilde{B}(x) = \psi(x)\sum_{i=0}^{n-d} \lambda_i(x),$$

where

$$\psi(x) = (x-x_\alpha)^3(x-x_{\alpha+1})^3.$$

Since $|B(x)| \geq |\lambda_i(x)|$ for any $i \in I_2$, it follows from the definition of $\lambda_i(x)$ that

$$\frac{1}{\tilde{B}(x)} \leq \frac{\prod_{k=i}^{i+d} d_k^3}{d_\alpha^3 d_{\alpha+1}^3} \leq Kh^{3d-3}, \quad \forall i \in I_2, \tag{3.13}$$

which also holds at the interpolation points (1.1). □

Theorem 3.7 *If $d \geq 3$ and $f \in C^{3d+6}[a,b]$, then*

$$\|f'' - r''\| = \|e''\| \leq K(1+\beta)h^{3d+1}.$$

Proof We continue to consider the open subinterval $(x_\alpha, x_{\alpha+1})$ and express the error e in (3.1) as

$$e(x) = \psi(x)\tilde{e}(x),$$

where

$$\tilde{e}(x) = \frac{A(x)}{\tilde{B}(x)} \quad \text{and} \quad \tilde{B}(x) = \psi(x)B(x).$$

Now, by the Leibniz rule, we derive the following:

$$\begin{aligned} e''(x) =& \sum_{i=0}^{2} \tbinom{2}{i} \psi^{(2-i)}\tilde{e}^{(i)}(x) \\ =& 2\psi''(x)\frac{A(x)}{\tilde{B}(x)} + 2\psi'(x)\Big(\frac{A'(x)}{\tilde{B}(x)} - A(x)\frac{\tilde{B}'(x)}{\tilde{B}^2(x)}\Big) + \\ & \psi(x)\Big(\frac{A''(x)}{\tilde{B}(x)} - 2A'(x)\frac{\tilde{B}'(x)}{\tilde{B}^2(x)} + 2A(x)\frac{\tilde{B}'^2(x)}{\tilde{B}^3(x)} - A(x)\frac{\tilde{B}''(x)}{\tilde{B}^2(x)}\Big). \end{aligned} \tag{3.14}$$

We deduce from Lemma 3.3 that every factor $A^{(k)}(x)$ is bounded.

The following factors are also bounded:

$$N_1(x) = \frac{\tilde{B}'(x)}{\tilde{B}^2(x)}, \ N_2(x) = \psi(x)\frac{\tilde{B}'(x)}{\tilde{B}(x)}, \ N_3(x) = \psi(x)\frac{\tilde{B}''(x)}{\tilde{B}^2(x)}.$$

We also divide the sum $\tilde{B}(x)$ into five parts as in [14]:

$$\tilde{B}(x) = \psi(x)\Big(\sum_{i=0}^{\alpha-d-1}\lambda_i(x) + \lambda_{\alpha-d}(x) + \sum_{i=\alpha-d+1}^{\alpha}\lambda_i(x) + \lambda_{\alpha+1}(x) + \sum_{i=\alpha+2}^{n-d}\lambda_i(x)\Big)$$

$$= K_1(x) + K_2(x) + K_3(x) + K_4(x) + K_5(x).$$

Since K_1 and K_2 are analogous to K_5 and K_4, it is sufficient to study the first three terms K_1, K_2 and K_3. We first consider the first term K_1:

$$K_1'(x) = \psi'(x) \sum_{i=0}^{\alpha-d-1} \lambda_i(x) + \psi(x) \sum_{i=0}^{\alpha-d-1} \lambda_i'(x).$$

Since the terms in both sums are increasing in absolute value and oscillating in sign, we derive that

$$|K_1'(x)| \leq 3d_\alpha^2 d_{\alpha+1}^2 (d_\alpha + d_{\alpha+1}) \prod_{k=\alpha-d-1}^{\alpha-1} d_k^{-3} + 3d_\alpha^3 d_{\alpha+1}^3 \prod_{m=\alpha-d-1}^{\alpha-1} d_m^{-1} \prod_{k=\alpha-d-1}^{\alpha-1} d_k^{-3}. \tag{3.15}$$

Next, we turn to K_2, after simplification which reads

$$K_2(x) = (x - x_{\alpha+1})^3 (-1)^{3\alpha-3d} \prod_{k=\alpha-d}^{\alpha-1} (x - x_k)^{-3}.$$

It follows that

$$|K_2'(x)| \leq 3d_{\alpha+1}^2 \prod_{k=\alpha-d}^{\alpha-1} d_k^{-3} + 3d_{\alpha+1}^3 \prod_{m=\alpha-d}^{\alpha-1} d_m^{-1} \prod_{k=\alpha-d}^{\alpha-1} d_k^{-3}.$$

Finally, we rewrite K_3 as

$$K_3(x) = \sum_{i=\alpha-d+1}^{\alpha} (-1)^{3i} \prod_{\substack{k=i \\ k\neq\alpha,\alpha+1}}^{i+d} (x - x_k)^{-3};$$

and its derivative has the following bound:

$$|K_3'(x)| \leq 3 \sum_{i=\alpha-d+1}^{\alpha} \sum_{\substack{m=i \\ m\neq\alpha,\alpha+1}}^{i+d} d_m^{-1} \prod_{\substack{k=i \\ k\neq\alpha,\alpha+1}}^{i+d} d_k^{-3}.$$

In order to derive a bound on N_1, we divide (3.15) with $|\tilde{B}^2(x)|$, choose $i = \alpha - d + 1$ in (3.13) and then obtain

$$\begin{aligned}
\frac{|K_1'(x)|}{|\tilde{B}^2(x)|} &\leq 6h^5 \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^6}{\prod_{k=\alpha-d-1}^{\alpha-1} d_k^3} + 3d_\alpha^3 d_{\alpha+1}^3 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^6}{d_m \prod_{k=\alpha-d-1}^{\alpha-1} d_k^3} \\
&\leq 6h^5 \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_{\alpha-d-1}^3 d_{\alpha-d}^3} + 3h^5 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_\alpha \prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_m d_{\alpha-d-1}^3 d_{\alpha-d}^3}.
\end{aligned}$$

Since $d_{\alpha-d+1}/d_{\alpha-d-1} \leq 1$, $d_{\alpha-d+2}/d_{\alpha-d} \leq 1$ and $d_\alpha/d_m \leq 1$ for $m \leq \alpha - 1$, we derive that

$$\frac{|K_1'(x)|}{|\tilde{B}^2(x)|} \leq 6h^5 \prod_{k=\alpha-d+3}^{\alpha-1} d_k^3 + 3h^5 \sum_{m=\alpha-d-1}^{\alpha-1} \prod_{k=\alpha-d+3}^{\alpha-1} d_k^3 \leq Kh^{3d-4}.$$

Using similar arguments, we may derive a bound of the same order for $|K_2'|/|\tilde{B}^2| \leq Kh^{3d-4}$:

$$\frac{|K_2'(x)|}{|\tilde{B}^2(x)|} \leq 3d_{\alpha+1}^2 \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^6}{\prod_{k=\alpha-d}^{\alpha-1} d_k^3} + 3d_{\alpha+1}^3 \sum_{m=\alpha-d}^{\alpha-1} \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^6}{\prod_{k=\alpha-d}^{\alpha-1} d_k^3}$$

$$\le 3d_{\alpha+1}^2 \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_{\alpha-d}^3} + 3d_{\alpha+1}^3 \sum_{m=\alpha-d}^{\alpha-1} \frac{\prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_m d_{\alpha-d}^3}.$$

Since $d_{\alpha-d+1}/d_{\alpha-d} \le 1$ and $d_{\alpha-1}/d_m \le 1$ for $m \le \alpha-1$, we derive the following bound:

$$\frac{|K_2'(x)|}{|\tilde{B}^2(x)|} \le 6h^5 \prod_{k=\alpha-d+2}^{\alpha-1} d_k^3 + 3h^3 \sum_{m=\alpha-d}^{\alpha-1} d_{\alpha-1}^2 \prod_{k=\alpha-d+2}^{\alpha-2} d_k^3 \le Kh^{3d-4}.$$

Using similar arguments, and for $|K_3'|/|\tilde{B}^2| \le Kh^{3d-4}$, the result is $|N_1(x)| \le Kh^{3d-4}$. To deal with N_2, we use the mesh ratio β in (3.6). Again we first consider the term involving K_1', choose $i = \alpha - d + 1$ in (3.13), and use the fact that $d_k/d_{k-1} \le 1$ for $k = \alpha - d + 1, \ldots, \alpha - 1$:

$$\begin{aligned}\frac{|\psi(x)K_1'(x)|}{|\tilde{B}(x)|} &\le 3d_\alpha^2 d_{\alpha+1}^2 (d_\alpha + d_{\alpha+1}) \frac{d_\alpha^3 d_{\alpha+1}^3 \prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{\prod_{k=\alpha-d-1}^{\alpha-1} d_k^3} + 3d_\alpha^3 d_{\alpha+1}^3 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_\alpha^3 d_{\alpha+1}^3 \prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_m \prod_{k=\alpha-d-1}^{\alpha-1} d_k^3}\\ &\le 3d_\alpha^2 d_{\alpha+1}^2 (d_\alpha + d_{\alpha+1}) \frac{d_\alpha^3 d_{\alpha+1}^3}{d_{\alpha-d-1}^3 d_{\alpha-d}^3} + 3d_{\alpha+1}^6 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_\alpha^6}{d_m d_{\alpha-d-1}^3 d_{\alpha-d}^3},\end{aligned}$$

since $d_\alpha/d_{\alpha-d-1} \le 1$ and $d_{\alpha-1}/d_m \le 1$ for $m \le \alpha - 1$, we obtain

$$|\psi(x)| \frac{|K_1'(x)|}{|\tilde{B}(x)|} \le 3h^5 \frac{d_{\alpha+1}}{d_{\alpha-1}} + 3h^5 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_{\alpha+1}}{d_{\alpha-1}} \le K\beta h^5.$$

Similar arguments result in a bound of the same order for $|\psi(x)||K_2'|/|\tilde{B}|$:

$$\begin{aligned}|\psi(x)| \frac{|K_2'(x)|}{|\tilde{B}(x)|} &\le 3d_{\alpha+1}^2 \frac{d_\alpha^3 d_{\alpha+1}^3 \prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{\prod_{k=\alpha-d}^{\alpha-1} d_k^3} + 3d_{\alpha+1}^3 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_\alpha^3 d_{\alpha+1}^3 \prod_{k=\alpha-d+1}^{\alpha-1} d_k^3}{d_m \prod_{k=\alpha-d}^{\alpha-1} d_k^3}\\ &\le 3d_\alpha^3 d_{\alpha+1}^5 \frac{1}{d_{\alpha-d}^3} + 3d_{\alpha+1}^6 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_\alpha^3}{d_m d_{\alpha-d}^3},\end{aligned}$$

since $d_\alpha/d_{\alpha-d} \le 1$ and $d_\alpha/d_m \le 1$ for $m \le \alpha - 1$, we obtain that

$$|\psi(x)| \frac{|K_2'(x)|}{|\tilde{B}(x)|} \le 3d_{\alpha+1}^5 + 3d_{\alpha+1}^5 \sum_{m=\alpha-d-1}^{\alpha-1} \frac{d_{\alpha+1}}{d_{\alpha-1}} \le Kh^5(1+\beta).$$

For $|\psi(x)||K_3'|/|\tilde{B}|$, the whole product in every term of the inner sum is eliminated without making use of the mesh ratio β:

$$|\psi(x)| \frac{|K_3'(x)|}{|\tilde{B}(x)|} \le d_{\alpha+1}^3 d_\alpha^3 \sum_{i=\alpha-d+1}^{\alpha} \sum_{\substack{m=i\\ m \ne \alpha, \alpha+1}}^{i+d} d_m^{-1} \le Kh^5.$$

Thus we have

$$|N_2(x)| \le K(1+\beta)h^5.$$

We may derive that N_3 is bounded by using similar arguments as for N_1 and the following observation: the differentiation of $\tilde{B}'$ results in a factor $(x - x_i)^{-3}$ in some of the terms of $\tilde{B}''$. Since $i \ne \alpha, \alpha + 1$, we can eliminate the absolute value of this factor through multiplication with $|\psi|$:

$$\frac{|\psi(x)|}{|x - x_i|} = \frac{d_\alpha^3 d_{\alpha+1}^3}{d_i} \le Kh^5.$$

Consequently,

$$|N_3(x)| \le Kh^{3d+1}.$$

Thus, we derive the result from bringing together all the bounds on the terms of the expansion (3.14). □

4. The barycentric form

We expect from [11] that r can be put in the barycentric form. In order to establish it, we write the interpolation polynomial p_i in the Hermite interpolation basis form

$$p_i(x) = \sum_{k=i}^{i+d} [\alpha_{ik}(x) f(x_k) + \beta_{ik}(x) f'(x_k) + \gamma_{ik}(x) f''(x_k)],$$

where $\alpha_{ik}(x)$, $\beta_{ik}(x)$ and $\gamma_{ik}(x)$ denote the Hermite polynomial interpolation basis functions corresponding to the $d+1$ points $x_i, \ldots, x_{i+d}$ respectively and satisfy the following conditions:

$$\begin{cases} \alpha_{ik}(x_j) = \delta_{kj} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad \alpha'_{ik}(x_j) = \alpha''_{ik}(x_j) = 0, \\ \beta_{ik}(x_j) = \beta''_{ik}(x_j) = 0, \quad \beta'_{ik}(x_j) = \delta_{kj}, & k, j = i, i+1, \ldots, i+d. \\ \gamma_{ik}(x_j) = \gamma'_{ik}(x_j) = 0, \quad \gamma''_{ik}(x_j) = \delta_{kj}, \end{cases}$$

According to the Hermite interpolation formula [20], we can derive that

$$\begin{cases} \alpha_{ik}(x) = \{[6(l_{ik}(x_k))^2 - 1.5 l''_{ik}(x_k)](x - x_k)^2 - 3 l'_{ik}(x_k)(x - x_k) + 1\} l^3_{ik}(x), \\ \beta_{ik}(x) = [-3 l'_{ik}(x_k)(x - x_k) + 1](x - x_k) l^3_{ik}(x), \\ \gamma_{ik}(x) = 0.5(x - x_k)^2 l^3_{ik}(x), \end{cases}$$

where $l_{ik}(x)$ $(k = i, \ldots, i+d)$ denote the Lagrange interpolation basis functions corresponding to the $d+1$ points $x_i, \ldots, x_{i+d}$. Substituting this into the numerator of (1.2), we can derive that

$$\begin{aligned} \sum_{i=0}^{n-d} \lambda_i(x) p_i(x) = &\sum_{i=0}^{n-d} \lambda_i(x) \sum_{k=i}^{i+d} \big\{[a_{ik}(x - x_k)^2 + b_{ik}(x - x_k) + 1] f(x_k) + \\ &[b_{ik}(x - x_k) + 1](x - x_k) f'(x_k) + 0.5(x - x_k)^2 f''(x_k)\big\} l^3_{ik}(x), \end{aligned}$$

where a_{ik} and b_{ik} denote the $6(l_{ik}(x_k))^2 - 1.5 l''_{ik}(x_k)$ and $-3 l'_{ik}(x_k)$, respectively. So we derive that

$$\begin{aligned} \sum_{i=0}^{n-d} \lambda_i(x) p_i(x) = &\sum_{i=0}^{n-d} (-1)^i \sum_{k=i}^{i+d} \big[\frac{a_{ik}}{(x - x_k)} + \frac{b_{ik}}{(x - x_k)^2} + \frac{1}{(x - x_k)^3}\big] \prod_{j=i, j\neq k}^{i+d} \frac{1}{(x_k - x_j)^3} f(x_k) \\ &+ \sum_{i=0}^{n-d} (-1)^i \sum_{k=i}^{i+d} \big[\frac{b_{ik}}{(x - x_k)} + \frac{1}{(x - x_k)^2}\big] \prod_{j=i, j\neq k}^{i+d} \frac{1}{(x_k - x_j)^3} f'(x_k) + \\ &\sum_{i=0}^{n-d} (-1)^i \sum_{k=i}^{i+d} \frac{0.5}{(x - x_k)} \prod_{j=i, j\neq k}^{i+d} \frac{1}{(x_k - x_j)^3} f''(x_k) \\ = &\sum_{k=0}^{n} \sum_{i \in J_\alpha} (-1)^i \big[\frac{a_{ik}}{(x - x_k)} + \frac{b_{ik}}{(x - x_k)^2} + \frac{1}{(x - x_k)^3}\big] \prod_{j=i, j\neq k}^{i+d} \frac{1}{(x_k - x_j)^3} f(x_k) + \end{aligned}$$

$$\sum_{k=0}^{n}\sum_{i\in J_\alpha}(-1)^i[\frac{b_{ik}}{(x-x_k)}+\frac{1}{(x-x_k)^2}]\prod_{j=i,j\neq k}^{i+d}\frac{1}{(x_k-x_j)^3}f'(x_k)+$$
$$\sum_{k=0}^{n}\sum_{i\in J_\alpha}\frac{(-1)^i0.5}{(x-x_k)}\prod_{j=i,j\neq k}^{i+d}\frac{1}{(x_k-x_j)^3}f''(x_k)$$
$$=\sum_{k=0}^{n}\frac{w_{1k}}{(x-x_k)}f(x_k)+\sum_{k=0}^{n}\frac{w_{2k}}{(x-x_k)^2}f(x_k)+\sum_{k=0}^{n}\frac{w_{3k}}{(x-x_k)^3}f(x_k)+$$
$$\sum_{k=0}^{n}\frac{w_{2k}}{(x-x_k)}f'(x_k)\sum_{k=0}^{n}\frac{w_{3k}}{(x-x_k)^2}f'(x_k)+\sum_{k=0}^{n}\frac{w_{3k}}{(x-x_k)}f''(x_k),$$

where

$$w_{1,k}=\sum_{i\in J_k}(-1)^i a_{ik}\prod_{j=i,j\neq k}^{i+d}\frac{1}{(x_k-x_j)^3},\quad w_{2,k}=\sum_{i\in J_k}(-1)^i b_{ik}\prod_{j=i,j\neq k}^{i+d}\frac{1}{(x_k-x_j)^3},$$
$$w_{3,k}=\sum_{i\in J_k}(-1)^i 0.5\prod_{j=i,j\neq k}^{i+d}\frac{1}{(x_k-x_j)^3},$$

Similarly, for the denominator, we use the fact that the function 1 is interpolated exactly, so that

$$1=\sum_{k=i}^{i+d}[a_{ik}(x-x_k)^2+b_{ik}(x-x_k)+1]l_{ik}^3(x),$$

which leads to

$$\sum_{i=0}^{n-d}\lambda_i(x)=\sum_{k=0}^{n}\frac{w_{1,k}}{(x-x_k)}+\sum_{k=0}^{n}\frac{w_{2,k}}{(x-x_k)^2}+\sum_{k=0}^{n}\frac{w_{3,k}}{(x-x_k)^3}.$$

This is the barycentric form we want for r, which provides an extremely fast and simple method for evaluating it. Note that the barycentric weights do not depend on the data values $f^{(k)}(x_i)$ ($k=0,1,2; i=0,1,\ldots,n$) and the Hermite rational interpolant r only depends linearly on $f^{(k)}(x_i)$ ($k=0,1,2; i=0,1,\ldots,n$), thus we call it the linear Hermite barycentric rational interpolant [21].

5. Numerical examples

In view of the good convergence properties of the linear Hermite barycentric rational interpolant r, it seems that it has good performance, but which has yet to be verified. Next, we will test it by using the Matlab software, and apply the approach to Runge's function $f(x)=1/(1+x^2)$ for $x\in[-5,5]$, which we sample at the equidistant points $x_i=-5+10i/n$, for various choices of n. Figure 1 shows plots of the linear Hermite barycentric rational interpolation function $r(x)$ with $d=3$ for $n=5,7$, respectively. Figure 2 shows plots of the first order derivative of Runge's function with $d=3$ for $n=5,7$, respectively. Figure 3 shows plots of the second order derivative of Runge's function with $d=3$ for $n=7,9$, respectively.

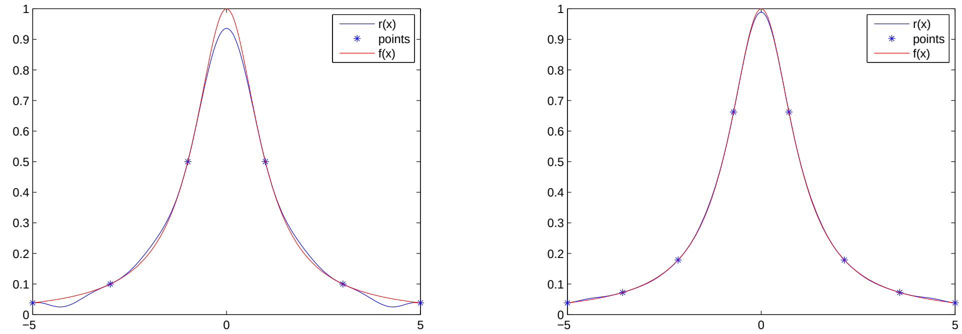

Figure 1 Interpolating Runge function with $d = 3$ for $n = 5$ and $n = 7$

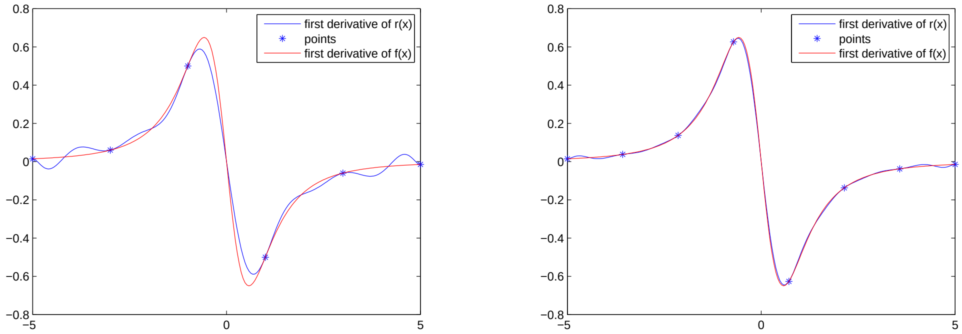

Figure 2 Interpolating first derivative of Sin function with $d = 3$ for $n = 5$ and $n = 7$

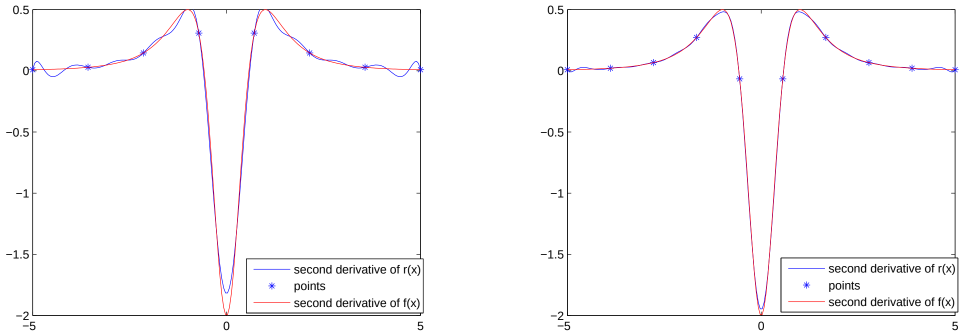

Figure 3 Interpolating second derivative of Runge function with $d = 3$ for $n = 7$ and $n = 9$

The second and third columns of Table 1 show the computed errors and convergence orders of $r(x)$, the fourth and fifth columns of Table 1 show the computed errors and orders of $r'(x)$, and the sixth and seventh columns of Table 1 show the computed errors and convergence orders of $r''(x)$. They support the convergence order predicted by Theorem 3.1, 3.4, 3.7, respectively. Comparing these results with the convergence orders in [13], we get that the convergence order is three units larger than the others at every differentiation.

n	$r(x)$	order	$r'(x)$	order	$r''(x)$	order
10	1.8e-03		6.1e-03		4.7e-02	
20	7.7e-07	11.2	4.8e-06	10.3	3.3e-05	9.4
40	1.7e-10	12.1	2.1e-09	11.2	1.4e-08	10.2
80	6.0e-14	11.5	1.6e-12	10.3	1.1e-11	9.7
160	1.5e-17	12.0	8.0e-16	11.0	5.4e-15	10.1
320	4.2e-21	11.8	4.2e-19	10.9	2.8e-18	9.8

Table 1 Error with Runge's function and $d = 3$

Finally, we compare it with the barycentric rational interpolation in [13] and C^2 cubic spline interpolation. The error of $f^{(k)}(x)$ is $O(h^{4-k})$ for $f \in C^4[a,b]$ (see [22, Chap. V]), the same order as for the barycentric rational interpolation in [13] with $d = 3$ (provided $f \in C^7[a,b]$), and the error of the linear Hermite rational interpolation is $O(h^{12-k})$ with $d = 3$ (provided $f \in C^{15}[a,b]$). Table 2 shows the errors in Runge's function of the three methods. The error in the linear Hermite rational interpolation is much smaller than that of the barycentric rational and C^2 cubic spline interpolation. Table 3 compares between the linear rational Hermite interpolation of Runge's function with n and barycentric rational and C^2 cubic spline interpolation with $3n$ points. The error in the linear Hermite rational interpolation is three units smaller than that of the barycentric rational and C^2 cubic spline interpolation.

n	$r(x)$	Paper [13]	cubic spline
10	1.8e-03	6.9e-02	2.2e-02
20	7.7e-07	2.8e-03	3.2e-03
40	1.7e-10	4.3e-06	2.8e-04
80	6.0e-14	5.1e-08	1.6e-05
160	1.5e-17	3.0e-09	9.5e-07
320	4.2e-21	1.8e-10	5.9e-08

Table 2 Error in linear Hermite barycentric rational, barycentric rational and spline interpolation of Runge's function

n	$r(x)$	n	Paper [13]	n	cubic spline
10	1.8e-03	30	6.2e-04	30	1.8e-03
20	7.7e-07	60	1.3e-07	60	6.4e-05
40	1.7e-10	120	2.7e-09	120	3.3e-06
80	6.0e-14	240	7.3e-10	240	1.9e-07
160	1.5e-17	480	4.3e-11	480	1.2e-08
320	4.2e-21	960	3.0e-12	960	7.4e-10

Table 3 Comparison between linear rational Hermite interpolation of Runge's function with n and barycentric rational and spline interpolation with $3n$ points

We also test the approach on the Sine function $f(x) = \sin(x)$ for $(x, y) \in [-5, 5]$ at the equidistant points $x_i = -5 + 10i/n$, but this time with $d = 1$ for various choices of n. Figure 4 shows plots of the linear Hermite barycentric rational interpolation function $r(x)$ with $d = 1$ for $n = 3, 5$, respectively. Figure 5 shows plots of the first order derivative of Sine function with $d = 1$ for $n = 3, 5$, respectively. Figure 6 shows plots of the second order derivative of Sine function with $d = 1$ for $n = 3, 5$, respectively.

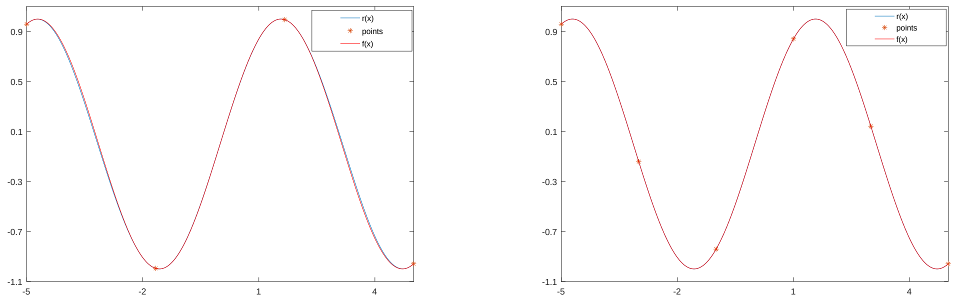

Figure 4 Interpolating Sine function with $d = 1$ for $n = 3$ and $n = 5$

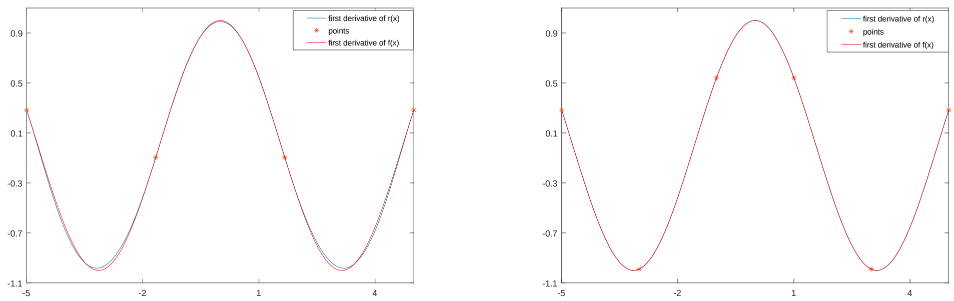

Figure 5 Interpolating first derivative of Sine function with $d = 1$ for $n = 3$ and $n = 5$

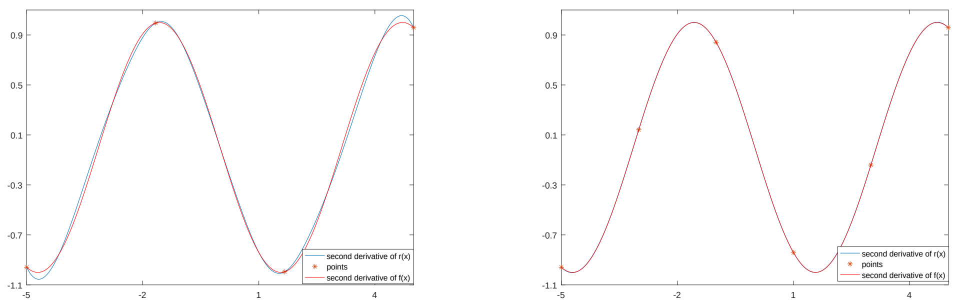

Figure 6 Interpolating second derivative of Sine function with $d = 1$ for $n = 3$ and $n = 5$

The other advantage of the linear Hermite barycentric rational interpolants is the ease with which we can change the degree d of the Hermite interpolation polynomials. In this way, we can find the best value of d which can minimize the numerically computed error for a fixed set of interpolation points, while for meromorphic functions we can give theoretically best value of d as a function of the location of the poles of f (see [23]).

6. Conclusions

In this paper, we present a family of linear Hermite barycentric rational interpolants which satisfies $r^{(k)}(x_i) = f^{(k)}(x_i)$ $(k = 0, 1, 2; i = 0, 1, 2, \ldots, n)$ and has the convergence rate $O(h^{3d+3-k})$ of the k-th derivative for $k = 0, 1, 2$, i.e., the case $s_i = 3$. It is natural to consider the case when larger k arises. This may be quite delicate, however, since the approximation error formula is put in the rational function form, the calculation of higher derivative of which is very complex. This would be an interesting subject for future research.

Acknowledgements We thank the referees for their time and comments.

References

[1] K. E. ATKINSON. *An Introduction to Numerical Analysis, 1st ed.* John Wiley, New York, 1989.
[2] J. P. BERRUT, L. N. TREFETHEN. *Barycentric Lagrange interpolation.* SIAM Rev., 2004, **46**(3): 501–517.
[3] H. E. SALZER. *Note on osculatory rational interpolation.* Math. Comput., 1962, **80**(16): 486–491.
[4] L. WUYTACK. *On the osculatory rational interpolation problem.* Math. Comput., 1975, **131**(29): 837–843.
[5] L. WUYTACK. *An algorithm for rational interpolation similar to the qd-algorithm.* Numer. Math., 1973, **20**: 418–424.
[6] L. WUYTACK. *Eigenschaften Eines Algorithmus Zur Rationalen Interpolation.* Birkhäuser, Basel, 1975.
[7] G. CLAESSENS. *A new algorithm for osculatory rational interpolation.* Numer. Math., 1976, **27**: 77–83.
[8] Jinbo WANG, Chuanqing GU. *Vector valued Thiele-Werner-type osculatory rational interpolants.* J. Comput. Appl. Math., 2004, **163**(1): 241–252.
[9] A. SIDI. *A new approach to vector-valued rational interpolation.* J. Approx. Theory, 2004, **130**(2): 177–187.
[10] A. SIDI. *Algebraic properties of some new vector-valued rational interpolants.* J. Approx. Theory, 2006, **141**(2): 142–161.
[11] C. SCHNEIDER, W. WERNER. *Hermite interpolation: The barycentric approach.* Computing, 1991, **46**(1): 35–51.
[12] J. P. BERRUT, H. D. MITTELMANN. *Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval.* Comput. Math. Appl., 1997, **33**(6): 77–86.
[13] M. S. FLOATER, K. HORMANN. *Barycentric rational interpolation with no poles and high rates of approximation.* Numer. Math., 2007, **107**(2): 315–331.
[14] J. P. BERRUT, M. S. FLOATER, G. KLEIN. *Convergence rates of derivatives of a family of barycentric rational interpolants.* Appl. Numer. Math., 2011, **61**(9): 989–1000.
[15] Ke JING, Ning KANG, Gongqin ZHU. *Convergence rates of a family of barycentric osculatory rational interpolation.* J. Appl. Math. Comput., 2017, **53**(1): 169–181.
[16] C. SCHULZ. *Topics in curve intersection and Barycentric interpolation.* Ph. D Thesis, University of Oslo, 2009.
[17] E. CIRILLO, K. HORMANN. *An iterative approach to barycentric rational Hermite interpolation.* Numer. Math., 2018, **140**(4): 939–962.
[18] E. CIRILLO, K. HORMANN. *On the Lebesgue constant of barycentric rational Hermite interpolants at equidistant nodes.* J. Comput. Appl. Math., 2019, **349**: 292–301.
[19] E. ISAACSON, H. N. KELLER. *Analysis of Numerical Methods.* Dover, New York, 1994.
[20] Renhong WANG. *Numerical Approximation.* Higher Education Press, Beijing, 2012. (in Chinese)
[21] C. DE BOOR. *A Practical Guide to Splines.* Springer-Verlag, New York, 2001.
[22] J. P. BERRUT, R. BALTENSPERGER, H. D. MITTELMANN. *Recent Developments in Barycentric Rational Interpolation.* Birkhäuser, Basel, 2005.
[23] S. GÜTTEL, G. KLEIN. *Convergence of linear barycentric rational interpolation for analytic functions.* SIAM J. Numer. Anal., 2012, **50**(5): 2560–2580.