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Finite Two-Arc-Regular Graphs Admitting an Almost Simple Group

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Abstract This paper completes the classification of $(G,2)$-arc-regular graphs of square-free order where G is an almost simple group.

Keywords coset graph; two-arc-regular; automorphism group; square-free

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1. Introduction

Denote by Γ a finite connected undirected graph with vertex set $V\Gamma$ and edge set $E\Gamma$. For a positive integer s, an s-arc of Γ is an $(s+1)$-tuple $(v_0,v_1,\ldots,v_s)$ of vertices such that $(v_{i-1},v_i)\in E\Gamma$ for $1\le i\le s$ and $v_{i-1}\ne v_{i+1}$ for $1\le i\le s-1$. Let $\mathsf{Aut}\Gamma$ denote the full automorphism group of Γ. If $G\le\mathsf{Aut}\Gamma$ is transitive on $V\Gamma$ and on the set of s-arcs of Γ, then Γ is called a (G,s)-arc-transitive graph. Further, if $G\le\mathsf{Aut}\Gamma$ is transitive on $V\Gamma$ and regular on the set of s-arcs of Γ, then Γ is called a (G,s)-arc-regular graph. In particular, if G itself is the full automorphism group, then a (G,s)-arc-regular is simply called an s-arc-regular graph.

The class of s-arc-regular graphs is closely connected to some important classes of combinatorial constructions, such as regular Mobius maps, near-polygonal graphs, and half-transitive graphs. There is a remarkable observation that, if a graph acts s-arc transitively on a graph for $s\ge2$, the vertex stabilizer is 2-transitive on the neighbors of that vertex. Thus the problems of classifying all finite 2-arc-transitive graphs, in particular, the graphs with square-free order, are highly attractive, and they have received considerable attentions [1–7]. In particular, the cases of 2-arc-transitive graphs admitting a Suzuki simple group and a Ree simple group are classified in 1999 (see [3,4]). And we have got some symmetric results on graphs with square-free order [8–12].

This paper aims to get a classification of $(G,2)$-arc-regular graphs with square-free order where G is an almost simple group. The following is our main result.

Theorem 1.1 *Let Γ be a $(G,2)$-arc-regular graph of square-free order, and G is an almost*

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simple group. Then the graph is isomorphic to one of the Coset graphs $\mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ *where* G *and* G_α *are listed as follows*

G	G_α	$\lvert\Gamma(\alpha)\rvert$	$\lvert V\Gamma\rvert$	Remarks
J_1	$2^3:7$	8	$3\times 5\times 11\times 19$	
M_{11}	$3^2:Q_8$	9	$2\times 5\times 11$	
$\mathrm{PSL}(2,q)$	A_4	4	$\frac{q(q^2-1)}{24}$	$q=\pm 3 \pmod 8$ and $q\geq 5$
$\mathrm{PSL}(2,q)$	$(Z_q:Z_{q-1})\times Z_{q-1}$	q	$\frac{q+1}{2}$	$q=p^d$ where p is odd prime
$\mathrm{PSL}(3,q)$	$Z_{q^2}:Z_{q^2-1}$	q^2	q^2+q+1	$q=p^d$ where p is odd prime

Table 1 Two-arc-regular graphs admitting almost simple group

This paper is organized as follows. Section 2 collects several preliminary results relating to this paper. In Section 3, we prove the main theorem by working out the corresponding $\mathsf{soc}(G), G_\alpha$ and constructing the corresponding graphs $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ which are 2-arc-regular.

2. Preliminaries

In this section, we collect some notations and results which will be used later. For an abstract group G, a subgroup $H \leq G$ is said to be core free if no non-trivial normal subgroups of G is contained in H. For a subset $S \subseteq G$ and a core free subgroup H of G, the coset graph $\Gamma = \mathsf{Cos}(G, H, HSH)$ is defined as the digraph with vertex set $V\Gamma := [G : H] = \{Hx | x \in G\}$ such that Hx is adjacent to Hy if and only if $yx^{-1} \in HSH$. It easily follows that each element $g \in G$ induces an automorphism of Γ acting by right multiplication, this is for all $x \in G$,

$$g : Hx \mapsto Hxg.$$

In the coset action, G is faithful on $V\Gamma$, and so we may assume that $G \leq \mathsf{Aut}\Gamma$. The following two lemmas collect some properties about coset graphs.

Lemma 2.1 ([13, P303, Theorem 11.1]) *Let* G *be a finite group with a core-free subgroup* H *and a* 2*-element* g*. Then the graph* $\Gamma = \mathsf{Cos}(G, H, HgH)$ *is a finite, connected,* $(G, 2)$*-arc-transitive graph with* G *transitive on vertices (acting by right multiplication) if and only if*

$$g \notin N_G(H), g^2 \in H, \langle H, g\rangle = G,$$

and the action of H *on* $[H : H \cap H^g]$ *by right multiplication is* 2*-transitive.*

Given a vertex $\alpha \in V\Gamma$, the stabilizer G_α induces an action on the neighborhood $\Gamma(\alpha)$. Let $G_\alpha^{\Gamma(\alpha)}$ be the group induced by G_α.

Lemma 2.2 ([13, P297, Lemma 9.4]) *Suppose that the graph* Γ *is* G*-vertex-transitive and let* $\alpha \in V\Gamma$*. Then* Γ *is* $(G, 2)$*-arc-transitive if and only if* $G_\alpha^{\Gamma(\alpha)}$ *is* 2*-transitive.*

To classify the $(G, 2)$-arc-regular graphs of square-free order, where G is an almost simple group, we also need two lemmas regarding finite non-abelian simple groups.

Lemma 2.3 ([14, P485, Theorem]) *If G is a nonabelian simple group with abelian Sylow 2-subgroup, then one of the following holds.*

(1) G is isomorphic to $\mathrm{PSL}(2,q), q \geq 3, q \equiv 3,5 \,(\mathrm{mod}\, 8)$*, or* $q = 2^m$*;*

(2) G is isomorphic to J_1;

(3) G is of Ree type.

The 2-rank $m_2(X)$ of X is the maximum rank of an abelian 2-subgroup A of X (the rank of A is by definition the number of factors in a direct product decomposition of A into cyclic subgroups).

Lemma 2.4 ([15, P72, Theorem 1.86]) *If G is a simple group of 2-rank at most 2, then $G \cong PSL(2,q)$ with $q \geq 5$ is odd,* $\mathrm{PSL}(3,q)$ *with q is odd, $U_3(q)$ with q is odd, $U_3(4)$, A_7 or M_{11}.*

3. The proof of the main Theorem

Let Γ be a $(G,2)$-arc-regular graph of square-free order, where G is an almost simple group and Let $L = \mathsf{soc}(G)$. From Lemma 2.2, it follows that G_α is a sharply 2-transitive permutation group on $\Gamma(\alpha)$. Then $G_\alpha < \mathrm{AGL}(d,p)$ where p is a prime and $d \geq 1$. Further $G_\alpha = N : H$, where $N \cong Z_p^d$, and $H = G_{\alpha\beta}$ for some $\beta \in \Gamma(\alpha)$ and $|H| = p^d - 1$. We split the proof in two cases.

Case 1. $p = 2$.

Let P be a Sylow 2-subgroup of G. Then P is isomorphic to Z_2^d or $Z_2^d.Z_2$, since $|G : G_\alpha|$ is square-free. There are two subcases. At first, we assume $|P : N| = 1$. That is, P is elementary abelian. It follows that by Lemma 2.3, L is isomorphic to $\mathrm{PSL}(2,q)$ with $q \geq 3$ and $q \equiv \pm 3 \pmod 8$, $\mathrm{PSL}(2,2^m)$, J_1 or $\mathrm{Ree}(3^e)$. We shall analyze these candidates in the following.

Suppose that $L \cong \mathrm{PSL}(2,q)$ where $q \geq 3$ and $q \equiv \pm 3 \pmod 8$. Let $q = {p'}^d$ for a prime p' and $p' \neq 2$, d is odd. Since $L \leq G \leq \mathsf{Aut}(L)$ and a Sylow 2-subgroup of L is isomorphic to Z_2^2 while a Sylow 2-subgroup of $\mathsf{Aut}(L)$ is isomorphic to a dihedral group of order 8, it follows that $P \cong Z_2^2$ and $G_\alpha \cong A_4$. Let $G = \mathrm{PSL}(2,q).Z_f$ where $f|d$. Since $|G : G_\alpha| = \frac{q(q+1)(q-1)f}{24}$ is square-free, we have $f = 1$ and $q = p'$ is prime. That is $G = \mathrm{PSL}(2,p')$ where $p' \geq 3$ and $p' \equiv \pm 3 \pmod 8$ is prime and $G_\alpha = A_4$. We can construct an infinite family of $(G,2)$-arc-regular graphs of square-free order. Let $G_i = \mathrm{PSL}(2,q_i)$ where $q_i \geq 3$ and $q_i \equiv 3,5 \pmod 8$ are primes, and let H_i be a subgroup of G_i with $H_i \cong A_4$. Then there exists a 2-element $g_i \in G_i$ such that $G_i = \langle H_i, g_i \rangle$ and $H_i \cap {H_i}^{g_i} \cong Z_3$. Note that, for each i, the triple G_i, H_i, g_i satisfies the conditions of Lemma 2.1, and hence the graph $\Gamma(i) = \mathsf{Cos}(G_i, H_i, H_i g_i H_i)$ is connected, G_i-vertex-transitive, and $(G_i,2)$-arc-transitive of valency 4. Moreover, $G_{i\alpha}$ is 2-transitive on $\Gamma(\alpha)$, and the order of $G_{i\alpha}$ is 12, thus the graph we constructed is $(G_i,2)$-arc-regular. Moreover, when $n = \frac{q(q+1)(q-1)}{24}$ is square-free, we constructed a class of $(G_i,2)$-arc-regular graphs of square-free order.

Suppose that $L \cong \mathrm{PSL}(2,2^m)$. Since $L \leq G \leq \mathsf{Aut}(L)$, it follows that $P \cong Z_2^m$ and $G_\alpha \cong Z_2^m : H$ where $|H| = 2^m - 1$. We shall prove there is no $(G,2)$-arc-regular graph corresponding to this kind of group.

We assume $G = L.Z_f$, then $f|(m, 2^m - 1)$. Further we can assume f is prime, otherwise there is a prime f' such that $f'|f$ and $f'|(m, 2^m-1)$.

Suppose $f^{i+1} \parallel 2^m-1$ (where $l^k \parallel n$ means the power of l dividing n is at most k), and $2^m-1 = kf^{i+1}$ where $(k,f)=1$. Then

$$H = (Z_k \times Z_{f^i}).Z_f \cong (\langle a\rangle \times \langle b\rangle).\langle c\rangle \cong \langle a\rangle \times \langle bc\rangle$$

where $o(a)=k$, $o(b)=f^i$ and $o(c)=f$. Since $o(bc)=f^{i+1}$ and H centralises no elements of $Z_2^m \setminus \{1\}$, $G_\alpha \cong Z_2^m : H \cong Z_2^m : (\langle a\rangle \times \langle bc\rangle)$ and

$$N_L(H) \le N_L(\langle a\rangle) \cong N_L(\langle Z_k\rangle) = D_{2(2^m-1)}.$$

Let $g \in N_L(H)$ be an involution,

$$g: a \longrightarrow a^{-1}, b \longrightarrow b^{-1}.$$

Then consider the induced action of g on $N_L(\langle a\rangle)/\langle a\rangle$ and $N_L(\langle a\rangle)/\langle a\rangle \times \langle b\rangle$, denoted by $\bar{g}$ and g', respectively. Then

$$\bar{g}: N_L(\langle a\rangle)/\langle a\rangle \longrightarrow N_L(\langle a\rangle)/\langle a\rangle,$$
$$\bar{b}\bar{c} \longrightarrow (\bar{b}\bar{c})^t$$

and

$$g': N_L(\langle a\rangle)/\langle a\rangle \times \langle b\rangle \longrightarrow N_L(\langle a\rangle)/\langle a\rangle \times \langle b\rangle,$$
$$c' \longrightarrow (c')^t.$$

Since $o(c)=f$, we have $t \equiv 1 \pmod f$. If $t=1$, then $\bar{b}\bar{c}^{\bar{g}} = \bar{b}^{\bar{g}}\bar{c}^{\bar{g}} = \bar{b}^{-1}\bar{c} = \bar{b}\bar{c}$. Thus $\bar{b}^{-1}=\bar{b}$, $b=b^{-1}$, which is impossible. So let $t=lf+1$, then

$$(\bar{b}\bar{c})^t = (\bar{b}\bar{c})^{lf+1} = (\bar{b}\bar{c})^{lf} \cdot (\bar{b}\bar{c}).$$

Since $o(b)=f^i$,

$$\bar{b}^{\bar{c}^{-1}} = \bar{b}^{1+jf^{i-1}}, \ \ 1 \le j \le f-1.$$

To simplify, we omit the symbol '-'. That is

$$(bc)^t = (bc)^{lf+1} = (bc)^{lf} \cdot (bc)$$

and

$$b^{c^{-1}} = b^{1+jf^{i-1}}, \ \ 1 \le j \le f-1.$$

Then it follows that

$$(bc)^2 = bcbc^{-1}c^2 = b \cdot b^{1+jf^{i-1}}c^2,$$
$$(bc)^3 = bcb^{2+jf^{i-1}}c^{-1}c^3 = b \cdot b^{(1+jf^{i-1})+(1+jf^{i-1})^2}c^3,$$
$$\cdots$$
$$(bc)^f = b \cdot b^{(1+jf^{i-1})+(1+jf^{i-1})^2+\cdots(1+jf^{i-1})^{f-1}}.$$

If we let $z := 1+jf^{i-1}$, then

$$(bc)^f = b^{1+z+z^2+\cdots z^{f-1}} = b^{\frac{z^f-1}{z-1}}.$$

Thus $(bc)^{lf+1} = b^{\frac{zf-1}{z-1}\cdot l+1}c = b^{-1}c$, that is $b^{\frac{zf-1}{z-1}\cdot l+1} = b^{-1}$. Then

$$[(1+jf^{i-1})^f - 1]l \equiv -2jf^{i-1} \pmod{f^i},$$

which is impossible, since $f^i|[(1+jf^{i-1})^f-1]l$ but $f^i \mid jf^{i-1}$. Thus there is no $(G,2)$-arc-regular graph corresponding to this kind of group.

Suppose $L \cong J_1$. Then by Atlas [16], we have $G = J_1$ and $G_\alpha \leq M$ where M is a maximal subgroup of J_1. Since $|G_\alpha| = 2^d(2^d-1)$ for some $d \geq 1$, it follows that $G_\alpha \cong Z_2^2.Z_3$ or $G_\alpha \cong Z_2^3 : Z_7$. For the former case, $G_\alpha \leq \mathrm{PSL}(2,11)$ and $G_{\alpha\beta} \cong Z_3$. Then there exists no element $g \in N_G(G_{\alpha\beta}) \setminus N_G(G_\alpha)$ and $g^2 \in Z_3$. For the latter case, there is an involution $g \in N_G(G_{\alpha\beta}) \cong Z_7 : Z_6$ such that $< G_\alpha, g >= G$ and G_α acts 2-regularly on $|\Gamma(\alpha)|$. So the corresponding graph $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is $(G,2)$-arc-regular graph of square-free order. Further, the graph is given in line 2 of our main Theorem 1.1.

Suppose that $L \cong \mathrm{Ree}(3^{2m+1})$ for some integer m. Then $|L| = q^3(q^3+1)(q-1)$, so $3^9||L|$. Therefore, $|G : G_\alpha|$ is not square-free. Thus there is no $(G,2)$-arc-regular graph of square-free order for this case.

Now we assume that $P \cong Z_2^d.Z_2 \cong Q.Z_2$ where $Q \cong Z_2^d$, we shall analyze two conditions based on $d = 2$ and $d \geq 3$. For these cases, there is no corresponding $(G,2)$-arc-regular graphs of square-free order.

If $d = 2$, then $L = \mathsf{soc}(G)$ has 2-rank 2, then by Lemma 2.4, $L \cong \mathrm{PSL}(2,q)$ (q is odd, and $q \geq 5$), $\mathrm{PSL}(3,q)$ or $U_3(q)$ (q is odd), $U_3(4)$, A_7 , or M_{11}. As an example, we prove the case $L \cong U_3(q)$ only, the others can be proved by similar arguments and checking by Atlas [16]. If $L \cong U_3(q)$ where $q = r^e$ for an odd prime r, it is obvious that $2^3||G : G_\alpha|$ if $q = 3$, and $q^3||G : G_\alpha|$ if $q > 3$, that is, there is no corresponding $(G,2)$-arc-regular graph.

If $d \geq 3$, let $X = N_G(Q)$, then $G_\alpha, P \leq X$, and $|G : X|$ is odd square-free. Let Y_2 and Y_1 be the subgroups of G such that $X \leq Y_2 < Y_1 \leq G$, and $\mathsf{soc}(Y_2) \neq \mathsf{soc}(Y_1) = \mathsf{soc}(G) = L$. Furthermore, Y_2 is maximal in Y_1, that is Y_1 acts faithfully and primitively on $[Y_1 : Y_2]$. Thus we can read out some information about Y_1 from [17]. Suppose there exists a $(G,2)$-arc-regular graph for corresponding group G, we can get that the following three conditions with respect to the four tables in [17] must be satisfied.

(1) n is odd square-free.

(2) If $G = Y_1 = L$, then $|G_{\bar{\alpha}} : G_\alpha|$ is even square free, where $G_{\bar{\alpha}}$ is the stabilizers in the four tables.

(3) If $G = L.O_1, Y_1 = L.O_2$, and $G_\alpha \leq L_{\bar{\alpha}}$ where $1 < O_2 \leq O_1$, then $|L_{\bar{\alpha}} : G_\alpha|$ is even square free.

Using the above three conditions and carefully computing the orders of the groups occurring in Tables [17] one by one, we can get that for all these groups, there exists no corresponding $(G,2)$-arc-transitive graph.

Case 2. $p \neq 2$.

Let P be a Sylow 2-subgroup of G. Then $P = Q.Z_2$, where Q is a Sylow 2-subgroup of H. Since H has only one involution by the structure of sharply 2-transitive graphs in [18], it follows that P has at most 2-rank-2. By Lemma 2.4, $L = \mathsf{soc}(G)$ is isomorphic to $\mathrm{PSL}(2, q)$, where q is odd and $q \geq 5$, $\mathrm{PSL}(3, q)$ or $U_3(q)$ with q is odd, $U_3(4)$, A_7, or M_{11}. We shall analyse these candidates one by one in the following.

Suppose that $L = \mathrm{PSL}(2, q)$ with $q = r^e$ and r is an odd prime number. Now $G_\alpha \cong Z_p^d : H$. Suppose $e \geq 1$. Then $r||G_\alpha|$, hence $G_\alpha \cong (Z_p^d : Z_{p^d-1}) \times Z_{p^d-1}$. Suppose $e = 1$. Then $|G||q(q+1)(q-1)$, and $p^d(p^d-1)|q(q+1)(q-1)$, since p and q are two odd prime numbers, $d = 1$, $q = p$, and $G_\alpha \cong (Z_p : Z_{p-1}) \times Z_{p-1}$. In both cases, we get $G_\alpha \cong (Z_q : Z_{q-1}) \times Z_{q-1}$. There is an involution g in Z_{q-1} satisfying the conditions in Lemma 2.1, so the corresponding graph $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is $(G, 2)$-arc-regular graph, which occurs in line 5 of of our main Theorem 1.1.

In order to analyze the following two cases, we need to use Zsigmondy Theorem [19]. If $a > b > 0$ are coprime numbers, then for any natural number $n > 1$, there is a prime number p that divides $a^n - b^n$ and does not divide $a^k - b^k$ for any $k < n$, with two exceptions: (1) $a = 2, b = 1$ and $n = 6$; or (2) $a + b$ is a power of two, and $n = 2$.

Suppose that $L = \mathrm{PSL}(3, q)$ where $q = r^e$ and r is an odd prime number. Then $L_\alpha \leq M$ for a maximal subgroup of G. If $L_\alpha \leq P_1 \cong [q^2].\mathrm{GL}(2, q)/Z_{(3,q-1)}$, then

$$G_\alpha \leq ([q^2].\mathrm{GL}(2, q)/Z_{(3,q-1)}).O, O \leq Z_2.Z_e.Z_{(3,q-1)}.$$

Since $G_\alpha \cong Z_p^d : H$ where $|H| = p^d - 1$, it follows that

$$p^d(p^d-1)|q^3(q+1)(q-1)^2 \times 2 \times e$$

and $\frac{q^3(q+1)(q-1)^2 \times 2 \times e}{p^d(p^d-1)}$ is square-free, then $p^d|q^3$, that is $p|q = r^e$, $d \geq 3e-1$. If $d = 3e$, then $p^d = p^{3e} = q^3$, so $p^d - 1 = q^3 - 1$, which does not divide $|G_\alpha|$, this is not possible. Thus $d = 3e-1$. If $e \geq 2$, then $p^d = p^{3e-1}$, $q = p^{d+1}$. By Zsigmondy Theorem, there exists a prime l such that $l \mid p^{d+1} - 1$ but $l \nmid p^d - 1$, it follows that $l^2 \mid |G : G_\alpha|$, that is $|G : G_\alpha|$ is not square-free. Therefore, $e = 1$. That is $L \cong L_3(q)$ with q prime. Then

$$G_\alpha \leq ([q^2].\mathrm{GL}(2, q)/Z_{(3,q-1)}).O, O \leq Z_2.Z_{(3,q-1)}.$$

By straightforward computation, it follows that $|G_\alpha| = q^2(q^2-1)$. So $(G_\alpha)_{q'} \leq \mathrm{GL}(2, q)$ or $(G_\alpha)_{q'} \leq \mathrm{GL}(2, q)/Z_{(3,q-1)}.Z_{(3,q-1)}$. For the former case,

$$(G_\alpha)_{q'} \cong Z_{q^2-1}, G_\alpha \cong Z_{q^2} : Z_{q^2-1}.$$

In this case, there is an involution g in Z_{q^2-1} satisfying the conditions in Lemma 2.1, so the corresponding graph $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is $(G, 2)$-arc-regular graph, which occurs in line 6 of our main Theorem 1.1. For the latter case, $(3, q-1) = 3$, then we can get $3^2||G : G_\alpha|$, contrary to that $|G : G_\alpha|$ is square-free.

As examples, we prove the cases $L \cong A_7$ and $L \cong M_{11}$ only, the others can be proved by similar arguments and checking by Atlas or GAP software.

Suppose $L = A_7$. Then $G = A_7$ or S_7. And $G_\alpha \leq M$ for a maximal subgroup of G. Since $|G_\alpha| = p^d(p^d-1)$ for some prime $p \neq 2$ and $|M : G_\alpha|$ is square-free. However, by Atlas [16] there is no maximal subgroup of G which contains such a subgroup. It follows that there is no $(G,2)$-arc-regular graph of square-free order for this group.

Suppose $L=M_{11}$. Then by Atlas [16], we have $G=M_{11}$ and $G_\alpha \le M$ for a maximal subgroup of M_{11}. Since $|G_\alpha| = 2^d(2^d-1)$ for some $d\ge 1$, it follows that $G_\alpha \cong Z_3^2 : Q_8, G_{\alpha\beta}\cong Q_8$. There is an involution $g\in N_G(G_{\alpha\beta})\cong Q_8.2$ such that $\langle G_\alpha, g\rangle = G$ and G_α acts 2-regularly on $|\Gamma(\alpha)|$. So the corresponding graph $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is $(G,2)$-arc-regular graph of square-free order which is given in line 3 of our main Theorem 1.1. □

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