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Vertex Decomposable Property of Graphs Whose Complements Are r-Partite

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Abstract Let G be a non-complete graph such that its complement $\overline{G}$ is r-partite. In this paper, properties of the graph G are studied, including the Cohen-Macaulay property and the sequential Cohen-Macaulay property. For $r = 2, 3$, some constructions are established for G to be vertex decomposable and some sufficient conditions are provided for $r \geq 4$.

Keywords vertex decomposable graph; Cohen-Macaulay; graph complement; r-partite

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1. Introduction

Simplicial complexes are classical objects in combinatorial commutative algebra. Every simplicial complex Δ corresponds to monomial ideals, e.g., the facet ideal $I(\Delta)$ and the Stanley-Reisner ideal I_Δ. If Δ is pure and vertex decomposable, then Δ is pure and shellable, and I_Δ is Cohen-Macaulay. The motivation of this research comes from the following established results:

Theorem 1.1 ([1]) *Let G be a bipartite graph with a vertex partition $V(G) = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\}$. Then the following statements are equivalent:*

(1) G is well-covered and vertex decomposable;

(2) G is well-covered and shellable;

(3) G is well-covered and constructible;

(4) G is Cohen-Macaulay;

(5) $n = m$, and there is a labeling such that

(a) $\{x_i, y_i\} \in E(G)$ for each i;

(b) $\{x_i, y_j\} \in E(G)$ implies $i \leq j$; and

(c) for $i < j < k$, $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\} \in E(G)$ imply $\{x_i, y_k\} \in E(G)$.

Note that the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are well-known; the equivalence of (4) with (5) is the classical [1, Theorem 3.4]. In order to derive (5) $\Longrightarrow$ (1), note that condition (b) implies that y_n is a weak shedding vertex, and one can use mathematical induction to conclude that both $G \setminus y_n$ and $G \setminus N_G[y_n]$ are vertex decomposable. Thus G is well-covered and vertex

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decomposable.

Proposition 1.2 *Let Δ be a simplicial complex of dimension 1. Then the following statements are equivalent:*

(1) Δ is pure and vertex decomposable;

(2) Δ is pure and shellable;

(3) Δ is pure constructible;

(4) Δ is Cohen-Macaulay;

(5) Δ is connected.

Again, the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are well-known. For the equivalence of (4) and (5), one can refer to [2, Corollary 6.3.14]. For proving the implication (5) $\Longrightarrow$ (1), one can use mathematical induction on the number of edges in Δ.

As a corollary of Proposition 1.2, it is known that for a non-complete graph G whose complement $\overline{G}$ is bipartite, G is well-covered and vertex decomposable if and only if $\overline{G}$ is connected as a graph.

In this paper, we study properties of the graph G whose complement $\overline{G}$ is r-partite for $r \geq 3$, and it is organized as follows. In Section 2, we recall some preliminaries on both graph theory and combinatorics and in Section 3, some constructions of vertex decomposable graphs are given for the graphs whose complements are r-partite, where $r = 2, 3$. In Section 4, the vertex decomposable property of the graphs is discussed for $r \geq 4$, whose complements are r-partite.

2. Preliminaries

In this section, we recall some relevant definitions and results on graphs and simplicial complexes, which are commonly used in combinatorial commutative algebra.

Recall that a simplicial complex Δ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $E \subseteq F$, then $E \in \Delta$. Each F in Δ is called a face, and a facet F is a maximal face with respect to inclusion. A simplicial complex Δ is called pure if all facets have the same cardinality. The set of all facets of Δ is denoted by $\mathcal{F}(\Delta)$ and, if $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_t\}$, then Δ can be written as $\Delta = \langle F_1, F_2, \ldots, F_t\rangle$. The dimension of a face F is $\dim F = |F| - 1$ and the dimension of Δ is defined by $\dim \Delta = \max\{\dim F | F \in \Delta\}$. Recall that the Alexander dual $\Delta^\vee$ of a simplicial complex Δ is defined by $\Delta^\vee = \{[n] \setminus F | F \notin \Delta\}$.

For a face H of a simplicial complex Δ, recall the following notations, namely deletion and link:

$$\Delta \setminus H =: \{F \in \Delta \mid H \cap F = \emptyset\},$$

$$\mathrm{lk}_\Delta(H) =: \{F \in \Delta \mid H \cap F = \emptyset,\, F \cup H \in \Delta\}.$$

Note that whenever $H = \{x\}$ is a vertex, the notations are usually written as $\Delta \setminus x$ and $\mathrm{lk}_\Delta(x)$, respectively. Recall the following concept of a vertex decomposable simplicial complex, which is introduced by Provan and Billera [3] in the pure case and extended to the nonpure case by Björner and Wachs [4, 5]:

Definition 2.1 *A simplicial complex Δ over $[\,n\,]$ is called vertex decomposable, if either Δ is a simplex, or Δ contains a vertex x such that the following requirements are satisfied:*

(1) Both $\Delta \setminus x$ and $\mathrm{lk}_\Delta(x)$ are vertex decomposable, and

(2) No facet of $\mathrm{lk}_\Delta(x)$ is a facet of $\Delta \setminus x$, or equivalently, $\Delta \setminus x = \langle\, \{F \mid x \notin F \in \mathcal{F}(\Delta)\}\,\rangle$.

A vertex x satisfying the conditions (1) and (2) is called a shedding vertex of Δ. If x only satisfies the condition (2), then it is called a weak shedding vertex.

It is well known that a vertex cover of a graph G is a subset C of the vertex set $V(G)$ such that $C \cap \{i,\, j\} \neq \emptyset$ holds for all $\{i,\, j\} \in E(G)$. A vertex cover is called minimal if no proper subset of it is a vertex cover. An independent vertex set I of graph G is a subset of $V(G)$ such that there is no edge between any pair of vertices in I. An independent vertex set is called maximal if there is no other independent vertex set of G containing it. Clearly, a subset C of $V(G)$ is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent vertex subset of G. Recall also that a graph G is said to be well-covered, if all the minimal vertex covers of G have the same cardinality. For a graph G, the cluster of all maximal independent vertex sets is a simplicial complex, which is called the independent simplicial complex of graph G and is denoted by $\mathrm{Ind}(G)$.

A translation in the language of graphs is restated in the following.

Definition 2.2 *A graph G is called vertex decomposable if either it has no edges, or else has some vertex x such that the following conditions hold:*

(1) Both graphs $G \setminus N_G[x]$ and $G \setminus x$ are vertex decomposable, where $N_G[x]$ is the union of the neighbourhood $N_G(x)$ together with $\{x\}$;

(2) For every independent vertex set S in $G \setminus N_G[x]$, there exists some $y \in N_G(x)$ such that $S \cup \{y\}$ is independent in $G \setminus x$.

A vertex x with above properties is called a shedding vertex of G. A vertex with the second property is called a weak shedding vertex.

Following [6], a vertex v of a graph G is called codominated if there exists a vertex $u \neq v$ such that $N_G[u] \subseteq N_G[v]$. In particular, a vertex which has an adjacent end vertex is codominated.

Lemma 2.3 ([5, Lemma 6] or [6]) *A codominated vertex of a graph G is a weak shedding vertex of G.*

By Definition 2.2, we easily get the following general construction:

Theorem 2.4 *Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. Let H_i be an induced subgraph of a graph G_i $(i = 1, 2)$; and let $u, w_1, \ldots, w_r$ $(r \geq 1)$ be some additional vertices such that $\{u, w_1, \ldots, w_r\} \cap (V(G_1) \cup V(G_2)) = \emptyset$. Let L be an enlarged graph such that $V(L) = V(G_1) \cup \{u, w_1, \ldots, w_r\} \cup V(G_2)$, in which each G_i is an induced subgraph of L, and*

$$\{\{u, w_i\}, \{u, h\} \mid 1 \leq i \leq r, h \in V(G_i \setminus H_i)\ (i = 1, 2)\} \cup E(G_1) \cup E(G_2) = E(L).$$

If each H_i and G_i are vertex decomposable, then L is also vertex decomposable.

Proof Note that $L \setminus N_L[u] = H_1 \cup H_2$ and $L \setminus u$ is identical with $G_1 \cup G_2$ together with some isolated vertices w_i. Any independent subset S of $V(H_1 \cup H_2)$ can be enlarged to an independent vertex set $S \cup \{w_1\}$, thus u is a shedding vertex of L, so L is vertex decomposable. $\Box$

Vertex decomposable simplicial complexes and graphs are important in combinatorial commutative algebra and combinatorial topology because they provide examples of shellable simplicial complexes, while well-covered vertex decomposable graphs are further Cohen-Macaulay; refer to [7–13] for related studies. It is known that each matroid is well-covered and vertex decomposable [14]. Furthermore, each vertex is a shedding vertex. Note that all chordal graphs are also vertex decomposable. For each graph G and a clique partition π of $V(G)$, an extension graph G^π of G is constructed (called a clique-whiskered graph of G), which is well-covered and vertex decomposable [15]. As a generalization, a construction Δ_χ is provided for each simplicial complex Δ and any s-coloring χ on Δ, and it is proved that all Δ_χ are well-covered and vertex decomposable simplicial complexes [16].

3. Vertex decomposable graphs whose complements are 3-partite

Let us begin with the following observation on graphs whose complement is bipartite:

Lemma 3.1 *Let G be a non-complete graph whose complement is bipartite. Then G is not well-covered if and only if there is a clone vertex in G, i.e., a vertex which is adjacent to all other vertices.*

Proof Note that $\dim \mathrm{Ind}(\mathrm{G}) = 1$ and that $\overline{G}$ is a bipartite graph, thus G is not well-covered if and only if $\mathrm{Ind}(G)$ has isolated vertices. The latter holds true if and only if there exists a clone vertex in G. $\Box$

Lemma 3.2 *Let G be any vertex decomposable graph, and v a clone vertex in G. Then v is a shedding vertex of G.*

Proof Since v is a clone vertex, we have $G \setminus N_G[v] = \emptyset$, thus v is a weak shedding vertex of G by Lemma 2.3. Therefore, it is only necessary to show that $G \setminus v$ is vertex decomposable. For this, we use induction on $|V(G)|$.

Let w be any shedding vertex of G with $w \neq v$. Then both $G_1 =: G \setminus N_G[w]$ and $G_2 =: G \setminus w$ are vertex decomposable, and for any independent subset S of $V(G_1)$, there exists a vertex $y \in N_G(w)$ such that $S \cup \{y\}$ is independent in G_2. Clearly, $y \neq v$ whenever S is nonempty. Hence it follows from

$$(G \setminus v) \setminus w = (G \setminus w) \setminus v,\ G \setminus N_G[w] = (G \setminus v) \setminus N_{G \setminus v}[w]$$

and induction on the vertex decomposable graph $G \setminus w$ that w is a shedding vertex of the graph $G \setminus v$, therefore, $G \setminus v$ is vertex decomposable. $\Box$

Now the following result is an immediate corollary of Proposition 1.2 and the previous lemmas:

Proposition 3.3 *Let G be any graph such that $\overline{G}$ is bipartite. Let $v_1, \ldots, v_t$ be the clone vertexes of G. Then G is vertex decomposable if and only if the complement of the graph $G \setminus \{v_1, \ldots, v_r\}$*

is connected.

In the following we record further two immediate corollaries of Proposition 1.2.

Corollary 3.4 *Let G be a non-complete graph consisting of two cliques A and B, with some additional edges in between. If there exists a vertex w in B such that w is not adjacent to any vertex of A, then G is vertex decomposable.*

Proof We use induction on the number of clone vertices of G. If G contains no clone vertex, then $\overline{G}$ is connected since $A \subseteq N_{\overline{G}}(w)$. Thus, G is well-covered and by vertex decomposable by Proposition 1.2. If G contains a clone vertex v_1, then $v_1 \in B$ holds. Thus, $G \setminus v_1$ satisfies the assumed condition and therefore, it is vertex decomposable by induction assumption. Consequently, G is vertex decomposable. □

Corollary 3.5 *Let G be a non-complete graph consisting of two cliques A and B, with some additional edges in between (called proper edges), such that $|B| \geq 3$. Assume further that there exists a proper edge e such that $V(e) \cap V(h) = \emptyset$ holds for other proper edge h. If further, there exists a vertex $w \in B \setminus V(e)$ such that w is a vertex of exactly one proper edge, then G is well-covered and vertex decomposable.*

Proof Clearly, there is no clone vertex in G, thus G is well-covered. Assuming further that $e = \{x, y\}, h = \{w, u\}$, where $\{x, w\} \subseteq B, \{y, u\} \subseteq A$. We claim that $\overline{G}$ is connected since in $\overline{G}$ we have the following edges:

$$\{x, u\}, \{w, y\}, \{w, a\}, \{y, b\}, \forall\, a \in A \setminus \{u\}, \forall\, b \in B \setminus \{x\}.$$

Finally, it follows from Lemma 1.2 that G is well-covered and vertex decomposable. □

In the remaining part of this section, a graph G is always assumed such that $\overline{G}$ is tripartite. We introduce the following:

Definition 3.6 *Let G be a graph whose complement is 3-partite. Then G is called quasi-dual-connected if it satisfies the following inductive conditions: either G is a discrete graph or there is a weak shedding vertex v such that the following conditions are fulfilled:*

(1) The graph $H =: G \setminus N_G[v]$ either satisfies $|V(H)| \leq 1$, or is such that $\overline{H}$ is connected;

(2) The graph $L =: G \setminus v$ satisfies one of the following three conditions: (a) quasi-dual-connected; (b) bipartite and $\overline{L}$ is connected; (c) $|V(L)| \leq 1$.

By Proposition 1.2, we immediately have

Proposition 3.7 *Let G be a graph whose complement is 3-partite. Then G is vertex decomposable if and only if it is quasi-dual-connected.*

In the end of this section, Proposition 3.7 is applied to construct a series of vertex decomposable graphs whose complement is 3-partite.

Example 3.8 Let G_1 be any vertex decomposable graph whose complement is 2-partite, with parts A_1 and B_1.

(1) Take any additional u_{11}, w_{1i}, where $1 \leq i \leq r_1$. Let G_2 be a graph on the vertex set $V(G_1) \cup \{u_{11}, w_{11}, \ldots, w_{1,r_1}\}$, and let $E(G_1) \cup [\cup_{1\leq j\leq r_1}\{u_{11}, w_{1j}\}] \subseteq E(G_2)$. Add some edges between u_{11} and vertices of G_1 in such a way that it makes $G_2 \setminus N_{G_2}[u_{11}]$ a vertex decomposable graph. Then G_2 will be a vertex decomposable graph whose complement is 3-partite, with parts

$$A_2 = A_1 \cup \{w_{11}, \ldots, w_{1r_1}\},\ B_2 = B_1,\ C_2 = \{u_{11}\}.$$

(2) Take any additional u_{21}, w_{2i}, where $1 \leq i \leq r_2$. Let G_3 be a graph on the vertex set $V(G_2) \cup \{u_{21}, w_{21}, \ldots, w_{2,r_2}\}$, and let the edge set be $E(G_2) \cup [\cup_{1\leq j\leq r_2}\{u_{21}, w_{2j}\}] \cup \{u_{21}, u_{11}\}$. Then G_3 will be a vertex decomposable graph whose complement is 3-partite, with parts

$$A_3 = A_2 \cup \{u_{21}\}, B_3 = B_2 \cup \{w_{21}, \ldots, w_{2r_2}\}, C_3 = \{u_{11}\}.$$

(3) Take any additional u_{31}, w_{3i}, where $1 \leq i \leq r_3$. Let G_4 be a graph on the vertex set $V(G_3) \cup \{u_{31}, w_{31}, \ldots, w_{3,r_3}\}$, and let the edge set be $E(G_3) \cup [\bigcup_{1\leq j\leq r_3}\{u_{31}, w_{3j}\}] \cup \{u_{31}, u_{21}\}$. Then G_4 will be a vertex decomposable graph whose complement is 3-partite, with parts

$$A_4 = A_3, B_4 = B_3 \cup \{u_{31}\}, C_3 = \{u_{11}, w_{31}, \ldots, w_{3r_3}\}.$$

This process can be continued in a similar way until we have a large enough graph.

This construction is illustrated in the following simple example. Note that all the four graphs are vertex decomposable, and it also follows by Theorem 2.4.

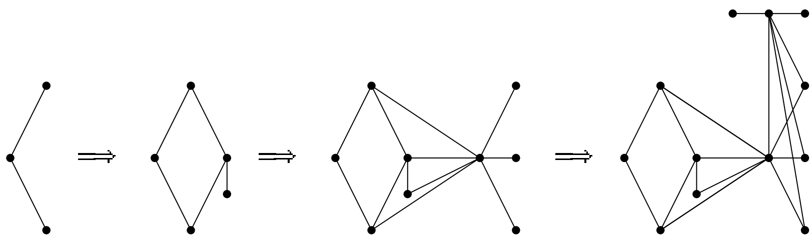

Figure 1 The enlargements of vertex decomposable graphs

4. Vertex decomposable graphs whose complements are r-partite ($r \geq 4$)

Throughout this section, all graphs are assumed to be connected and non-complete, consisting of several finite cliques with some additional edges in between. These additional edges are called proper edges, i.e., a proper edge is the one that its related two vertices belong to distinct cliques. If a vertex of a proper edge does not belong to another proper edge, then this vertex is called a rigid vertex. If a proper edge has both rigid vertices, then this edge is called a rigid edge. If a vertex is not used in any proper edge, it is called an improper vertex.

Proposition 4.1 *Let G be a non-complete graph consisting of r cliques A_i $(1 \leq i \leq r)$, together with some additional edges. Assume further that of the r cliques, $r-1$ cliques have an improper vertex each. Then the graph G is vertex decomposable.*

Proof We use mathematical induction on r. For $r = 2$, it is vertex decomposable by Proposition 1.2. Now assume $r \geq 3$, and suppose it is true for the $r - 1$ case. Assume further that each of $A_2, \ldots, A_r$ has an improper vertex. For r, take a vertex $v \in A_r$ such that v is not an improper vertex. Note that $G \setminus N_G[v]$ is a non-complete graph consisting of $r-1$ cliques, having $r-2$ improper vertices in distinct cliques together with some additional edges, so it is vertex decomposable. While, $G \setminus v$ consists of r cliques with $r-1$ improper vertices. Furthermore, for any independent vertex set M of the graph $G \setminus N_G[v]$, there is an improper vertex $w \in N(v)$ such that $M \cup \{w\}$ is an independent set of $G \setminus v$. Note that $G \setminus v$ shares the same property with G, thus mathematical induction can be applied to conclude that $G \setminus A_r$ is a non-complete graph consisting of $r-1$ cliques with some additional edges, of which $r-2$ cliques have improper vertex each, so it is vertex decomposable. □

Theorem 4.2 *Let G be a non-complete graph consisting of r cliques $A_1, A_2, \ldots, A_r$ with some additional edges. Assume further that there is one improper vertex $w \in A_1$, and from A_1 to each A_i $(2 \leq i \leq r)$ there exists a rigid edge. Then the graph G is vertex decomposable.*

Proof Take any vertex $v \in A_1$ such that v is a vertex of a rigid edge. Clearly, $G \setminus N_G[v]$ is a non-complete graph consisting of $r-1$ cliques of which $r-2$ cliques have an improper vertex each, so it is vertex decomposable by Proposition 4.1. While, $G \setminus v$ consists of r cliques with some additional edges, in which there are two cliques having an improper vertex each. Furthermore, for any independent vertex set M of $G \setminus N_G[v]$, there exists $w \in N_G(v)$ such that $M \cup \{w\}$ is an independent vertex set of $G \setminus v$. Then, we use mathematical induction on A_1 to conclude that $G \setminus A_1$ is a graph consisting of $r-1$ cliques and each clique has an improper vertex, so it is vertex decomposable by [15, Lemma 3.2, Theorem 3.3]. □

Proposition 4.3 *Let G be a non-complete graph consisting of r cliques A_i $(1 \leq i \leq r)$, with some additional edges in between. Set*

$$\mathcal{I} = \{A_i \mid 1 \leq i \leq r, \text{and } A_i \text{ contains improper vertices of } G\}, \ \mathcal{P} = \{A_1, \ldots, A_r\} \setminus \mathcal{I},$$

and assume $|\mathcal{I}| = m \geq 1$ and $|\mathcal{P}| = n \geq 1$. Then G is vertex decomposable if one of the following conditions is satisfied:

(1) There exists an element $A \in \mathcal{I}$ and a subset $\mathcal{S}$ of $\mathcal{P}$ with $|\mathcal{S}| = n-1$, such that for each $B \in \mathcal{S}$, there is a rigid edge of G from A to B, and there is a vertex v in A which is adjacent to a rigid vertex of B $(\forall B \in \mathcal{S})$.

(2) There exist $A \in \mathcal{I}$ and $B \in \mathcal{P}$, and take $v \in A$ and $u \in B$ such that v is adjacent to u, and every $A_i \neq B$ from $\mathcal{P}$ has two vertices rigid in G, one of which is adjacent to v and the other is adjacent to u.

Proof For $A \in \mathcal{I}$, there is an improper vertex $w \in A$. Take a vertex $v \in A$ and consider $G \setminus N_G[v]$. Clearly, $G \setminus N_G[v]$ is a non-complete graph consisting of $r-1$ cliques with $r-2$ improper vertices, so by Proposition 4.1 it is vertex decomposable. While $G \setminus v$ consists of r cliques with $r-1$ improper vertices, so it is again vertex decomposable by Proposition 4.1.

Furthermore, for any independent vertex set M of $G \setminus N_G[v]$, clearly, there exists $w \in N_G(v)$ such that $M \cup \{w\}$ is an independent vertex set of $G \setminus v$. This completes the proof. □

Proposition 4.4 *Let G be a non-complete graph consisting of r cliques A_i $(1 \leq i \leq r)$, with some additional edges in between. Assume that no A_i has an improper vertex in G. Then,*

(1) G is vertex decomposable if the following two conditions are fulfilled:

(a) There exist some clique A_1 and other $r-2$ cliques A_j $(2 \leq j \leq r-1)$, such that there is a vertex v in A_1 which is adjacent to a rigid vertex of A_j $(\forall\, 2 \leq j \leq r-1)$, and for each $2 \leq j \leq r-1$, there is an edge in $E(A_1, A_j)$ which is rigid in the graph G.

(b) There exists a number j $(2 \leq j \leq r-1)$, such that $E(A_1, A_j)$ contains one rigid edge in G, and there exists a number t $(t \neq j,\ 2 \leq t \leq r)$, such that $E(A_j, A_t)$ contains two edges rigid in G.

or

(2) G is vertex decomposable if the following two conditions are fulfilled:

(c) There exists a couple, say $\{A_1, A_2\}$, such that $v \in A_1$ and $u \in A_2$, and v is adjacent to u. And every A_j $(j \neq 1, 2)$ has two rigid vertices, one of which is adjacent to v and the other is adjacent to u.

(d) There exists a number k $(2 \leq k \leq r)$, such that $E(A_1, A_k)$ contains three edges rigid in G.

Proof (1) Let $A_j = A_2$ and $A_t = A_3$, then $e_1 = \{x_1, y_1\}$, $e_2 = \{x_2, y_2\}$ and $e_3 = \{x_3, y_3\}$ are rigid edges, where $x_1 \in A_1$, $\{y_1, x_2, x_3\} \subset A_2$ and $\{y_2, y_3\} \subset A_3$. For the vertex $y_1 \in A_2$, the graph $G \setminus N_G[y_1]$ consists of $r-1$ cliques with an improper vertex in A_1, which satisfies condition (1) of Proposition 4.3, so it is vertex decomposable. While, the graph $G \setminus y_1$ consists of r cliques with an improper vertex in A_1, which satisfies condition (1) of Proposition 4.3 also, so it is also vertex decomposable. For any independent vertex set M of $G \setminus N_G[y_1]$, we have $\{x_2, x_3\} \subset N_G(y_1)$, if $M \cap V(e_2) = \emptyset$ holds. In this case, $M \cup \{x_2\}$ is an independent set of $G \setminus y_1$; if $M \cap V(e_2) \neq \emptyset$ holds, then $M \cup \{x_3\}$ is an independent vertex set of $G \setminus y_1$. This proves that G is vertex decomposable.

(2) Take $A_k = A_2$. Then there are three rigid edges, $e_1 = \{x_1, y_1\}$, $e_2 = \{x_2, y_2\}$ and $e_3 = \{x_3, y_3\}$, from A_1 to A_2, where $x_i \in A_1$ and $y_i \in A_2$ for $i = \{1, 2, 3\}$. Take $y_1 \in A_2$, then $G \setminus N_G[y_1]$ consists of $r-1$ cliques and every clique has an improper vertex, so the graph $G \setminus N_G[y_1]$ is vertex decomposable by [15, Lemma 3.2, Theorem 3.3]. The graph $G \setminus y_1$ consists of r cliques with an improper vertex $x_1 \in A_1$. So condition (2) of Proposition 4.3 is fulfilled and hence, $G \setminus y_1$ is vertex decomposable. Further, for any independent vertex set M of $G \setminus N_G[y_1]$, we have $\{y_2, y_3\} \subset N_G(y_1)$, if $M \cap V(e_2) = \emptyset$ holds. Then $M \cup \{y_2\}$ is an independent vertex set of $G \setminus y_1$; if $M \cap V(e_2) \neq \emptyset$ holds, then $M \cup \{y_3\}$ is an independent vertex set of $G \setminus y_1$. Note that when $A_k = A_3$, $G \setminus N_G[y_1]$ consists of $r-1$ cliques satisfying conditions (2) of Proposition 4.3, so it is vertex decomposable. This proves that G is vertex decomposable. □

We have the following two remarks:

Remark 4.5 Condition (b) can be replaced by any one of the following:

(b$_2$) There exists a number j, where $2 \leq j \leq r-1$, such that $E(A_1, A_j)$ contains two edges rigid in G.

(b$_3$) There exist three edges in $E(A_1, A_r)$, which are rigid in graph G.

(b$_4$) There exist two rigid edges in $E(A_1, A_r)$ and there exists a number t, where $2 \leq t \leq r-1$, such that $E(A_r, A_t)$ contains two rigid edges in G.

Remark 4.6 Condition (d) can be replaced by any one of the following:

(d$_2$) There exists a number k, where $3 \leq k \leq r$, such that $E(A_1, A_k)$ contains two edges rigid in G, and there exists a number x where $x \neq k$, $3 \leq x \leq r$, such that $E(A_k, A_x)$ contains two edges rigid in G.

(d$_3$) There exists one rigid edge in $E(A_1, A_2)$ and there exists a number k, where $3 \leq k \leq r$, such that $E(A_2, A_k)$ contains two edges rigid in G.

In the following theorem, let $(A_1, A_2, \ldots, A_r)$ be cyclic, i.e., we regard A_1 as A_{r+1}.

Theorem 4.7 *Let G be a non-complete graph with r cliques $A_1, A_2, \ldots, A_r$ together with some additional edges in between. Then G is vertex decomposable if the following conditions are fulfilled:*

(1) There exist two disjoint cycles, each containing exactly one edge in $E(A_i, A_{i+1})$ ($\forall 1 \leq i \leq r$) which is rigid in G.

(2) For each $1 \leq i \leq r$ and each $j \neq i-1, i, i+1$, A_j has a rigid vertex adjacent to some vertex of A_i.

Proof Let G be a non-complete graph with the above construction. By condition (1), there exist two disjoint cycles C_1 and C_2, each containing exactly one edge in $E(A_i, A_{i+1})$ ($\forall 1 \leq i \leq r$), which is rigid in G.

Without loss of generality, let $e_i = \{x_i, y_i\} \in C_i$ ($i = 1, 2, x_i \in A_r, y_i \in A_1$) be two rigid edges in the cycles. Consider the induced subgraph $G \setminus N_G[x_1]$, which consists of $r-1$ cliques A_i ($1 \leq i \leq r-1$). Clearly, y_2 is an improper vertex in $G \setminus N_G[x_1]$ since e_2 is a rigid edge. Due to the same reason, A_{r-1} also contains at leat two improper vertices in $G \setminus N_G[x_1]$. Note that condition (2) ensures that every other A_i also contains at least one improper vertex in $G \setminus N_G[x_1]$, hence $G \setminus N_G[x_1]$ consists of $r-1$ cliques, each of which has an improper vertex. So, $G \setminus N_G[x_1]$ is vertex decomposable by [15, Lemma 3.2, Theorem 3.3].

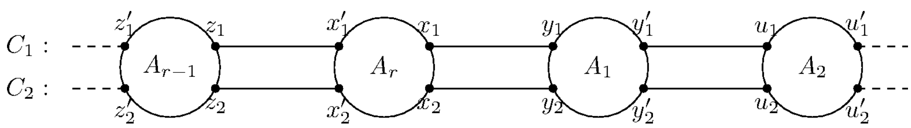

Figure 2 C_1 and C_2

Now let $e_3 = \{x_1', z_1\} \in C_1, e_4 = \{x_2', z_2\} \in C_2$, where $\{z_1, z_2\} \in A_{r-1}$ and $\{x_1', x_2'\} \subseteq A_r$. For any independent vertex set M of the graph $G \setminus N_G[x_1]$, if $M \cap V\{e_3\} = \emptyset$ holds, then

$M \cup \{x'_1\}$ is independent set in $G \setminus x_1$. If $M \cap V\{e_3\} \neq \emptyset$ holds, then $M \cup \{x'_2\}$ is independent in the graph $G \setminus x_1$. This shows that x_1 is a weak shedding vertex of G.

Next, we proceed to show that $H =: G \setminus x_1$ is also vertex decomposable. For this, consider the two rigid edges $e_5 = \{y'_1, u_1\} \in C_1$ and $e_6 = \{y'_2, u_2\} \in C_2$, where $\{u_1, u_2\} \subseteq A_2, \{y'_1, y'_2\} \subseteq A_1$. Consider $H \setminus N_H[y'_1]$ and note that x_2 is an improper vertex of A_r and u_2 is an improper vertex of A_2 in $H \setminus N_H[y'_1]$, thus condition (2) ensures that $H \setminus N_H[y'_1]$ consists of $r-2$ cliques, each of which has at least one improper vertex, so it is vertex decomposable by [15, Lemma 3.2, Theorem 3.3]; while $H \setminus y'_1$ consists of r cliques with two improper vertices $y_1 \in A_1$ and $u_1 \in A_2$. For any independent set M of $H \setminus N_H[y'_1]$, there exists $y_1 \in N_H(y'_1)$ such that $M \cup \{y_1\}$ is an independent set of $H \setminus y'_1$. This shows that y'_1 is a weak shedding vertex of H.

Now apply mathematical induction on pathes $A_2A_3, A_3A_4, \ldots, A_{r-1}A_r$, respectively. In the end, we get a graph consisting of r cliques, where each clique has an improper vertex, so by [15, Lemma 3.2, Theorem 3.3] it is vertex decomposable.

Finally, x_1 is a shedding vertex of G, thus G is vertex decomposable. □

Theorem 4.8 *Let G be a non-complete graph consisting of cliques A_i $(1 \leq i \leq r)$, with some additional edges in between. Assume further that each A_i contains no improper vertices in G. Then G is vertex decomposable if the following conditions are fulfilled:*

(1) For each t where $3 \leq t \leq r$, there exists one rigid edge e_t in $E(A_1, A_t)$ and two rigid edges e_1, e_2 in $E(A_1, A_2)$.

(2) There exist $a_i \in A_i$ $(i = 1, 2)$ such that neither is adjacent to a vertex of A_j $(3 \leq j \leq r)$, and $a_i \cap V(e_j) = \emptyset$ holds, where $1 \leq j \leq r$ and $1 \leq i \leq 2$.

Proof For $1 \leq j \leq r$, let $e_j = \{x_j, y_j\}$ be rigid edges in G, where $x_j \in A_1$ holds for all $1 \leq j \leq r$, $\{y_1, y_2\} \subset A_2$ and $y_t \in A_t$ holds for each $3 \leq t \leq r$. Consider the vertex $x_3 \in A_1$ from the rigid edge e_3, and note that $G \setminus N_G[x_3]$ consists of $r-1$ cliques with some additional edges, of which $r-2$ cliques have an improper vertex each. So it is vertex decomposable by Proposition 4.1. While $G \setminus x_3$ consists of r cliques with some additional edges and one improper vertex $y_3 \in A_3$. For any independent vertex set M of $G \setminus N_G[x_3]$, we have $\{x_1, x_2\} \subset N_G(x_3)$ and if $M \cap V(e_1) = \emptyset$ holds, then $M \cup \{x_1\}$ is an independent vertex set. But if $M \cap V(e_1) \neq \emptyset$ holds, then $M \cup \{x_2\}$ is an independent vertex set of $G \setminus x_3$. Next, apply the same procedure on the sequence of vertices $x_4, x_5, \ldots, x_r$. Let $H = G \setminus \{x_3, x_4, \ldots, x_r\}$ consist of r cliques with some additional edges, of which $r-2$ cliques have an improper vertex each. For $x_2 \in A_1$, the graph $H \setminus N_H[x_2]$ consists of $r-1$ cliques and each clique has an improper vertex, so by [15, Lemma 3.2, Theorem 3.3] it is vertex decomposable. $H \setminus x_2$ consists of r cliques with some additional edges, of which $r-1$ cliques have an improper vertex each. So by Proposition 4.1, it is vertex decomposable. For any independent vertex set M of $H \setminus N_H[x_2]$, $\{x_1, a_1\} \subset N_H(x_2)$ holds true if $M \cap V(e_1) = \emptyset$. In the case, $M \cup \{x_1\}$ is an independent vertex set of $H \setminus x_2$. But if $M \cap V(e_1) \neq \emptyset$ holds, then $M \cup \{a_1\}$ is an independent vertex set of $H \setminus x_2$. This shows that G is vertex decomposable. □

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References

[1] J. HERZOG, T. HIBI. *Distributive lattices, bipartite graphs and Alexandar duality*. J. Algebraic Combin., 2005, **22**(3): 289–302.

[2] R. H. VILLARREAL. *Monomial Algebras*. Marcel Dekker, Inc., New York, 2001.

[3] J. S. PROVAN, L. J. BILLERA. *Decompositions of simplicial complexes related to diameters of convex polyhedra*. Math. Oper. Res., 1980, **5**(4): 576–594.

[4] A. BJÖRNER, M. L. WACHS. *Shellable nonpure complexes and posets (I)*. Trans. Amer. Math. Soc., 1996, **348**(4): 1299–1327.

[5] A. BJÖRNER, M. L. WACHS. *Shellable nonpure complexes and posets (II)*. Trans. Amer. Math. Soc., 1997, **349**(10): 3945–3975.

[6] T. BIYIKǑGLU, Y. CIVAN. *Vertex-decomposable graphs, codismantlability, Cohen-Macaulayness, and Castelnuovo-Mumford regularity*. Electron. J. Combin., 2014, **21**(1): 1–17.

[7] R. FRÖBERG. *On Stanley-Reisner Rings*. PWN, Warsaw, 1990.

[8] R. WOODROOFE. *Vertex decomposable graphs and obstructions to shellability*. Proc. Amer. Math. Soc., 2009, **137**(10): 3235–3246.

[9] J. HERZOG, T. HIBI. *Monomial Ideals*. Springer-Verlag London, Ltd., London, 2011.

[10] D. ARMSTRONG, B. RHOADES, N. WILLIAMS. *Rational associahedra and noncrossing partitions*. Electron. J. Combin., 2013, **20**(3): Paper 54, 27 pp.

[11] A. DOCHTERMANN. *Face rings of cycles, associahedra and standard Young tableaux*. Electron. J. Combin., 2016, **23**(3): Paper 3.22, 17 pp.

[12] Jinping WU, Jin GUO. *On the vertex decomposable property of f-graphs and their complements*. J. Jiangxi Normal University, 2018, **42**(4): 354–356, 373. (in Chinese)

[13] B. LITJENS, S. POLAK, B. SEVENSTER, et al. *On the chromatic number of a subgraph of the Kneser graph*. Preprint available at arXiv:1803.04342.

[14] R. STANLEY. *Combinatorics and Commutative Algebra*. Birkhäuser Boston, Inc., Boston, MA, 1996.

[15] D. COOK II, U. NAGEL. *Cohen-Macaulay graphs and face vectors of flag complexes*. SIAM J. Discrete Math., 2012, **26**(1): 89–101.

[16] J. BIERMANN, A. VAN TUYL. *Balanced vertex decomposable simplicial complexes and their h-vectors*. Electron. J. Combin., 2013, **20**(3): Paper 15, 12 pp.