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n-Gorenstein Projective Modules and Dimensions over Frobenius Extensions

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Abstract In this paper, we study n-Gorenstein projective modules over Frobenius extensions and n-Gorenstein projective dimensions over separable Frobenius extensions. Let $R \subset A$ be a Frobenius extension of rings and M any left A-module. It is proved that M is an n-Gorenstein projective left A-module if and only if $A \otimes_R M$ and $\mathrm{Hom}_R(A, M)$ are n-Gorenstein projective left A-modules if and only if M is an n-Gorenstein projective left R-module. Furthermore, when $R \subset A$ is a separable Frobenius extension, n-Gorenstein projective dimensions are considered.

Keywords Frobenius extensions; n-Gorenstein projective modules; n-Gorenstein projective dimensions

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1. Introduction

The study of Gorenstein homological algebra stems from finitely generated modules of G-dimensions zero over any noetherian rings, introduced by Auslander and Bridger [1] in 1969 as a generalization of finite generated projective modules. In order to complete the analogy, in 1995, Enochs and Jenda introduced the Gorenstein projective modules (not necessarily finitely generated) over any associative rings; and dually, Gorenstein injective modules were defined in [2]. In 2004, Holm further studied the properties of these modules in [3]. In 2015, n-Gorenstein projective modules and n-Gorenstein injective modules were introduced by Tang in [4] as a generalization of these modules. Tang used these two classes of modules to give a new characterization of Gorenstein rings in terms of top local cohomology modules of flat modules.

The theory of Frobenius extensions was developed by Kasch [5] in 1954 as a generalization of Frobenius algebras, and was further studied by Nakayama and Tsuzuku [6, 7] in 1960–1961, Morita [8] in 1965, and Kadison [9] in 1999. In 2018, Ren studied the Gorenstein projective properties of modules and Gorenstein homological dimensions along Frobenius extensions of rings in [10, 11].

Inspired by above conclusions, in this paper, we intend to study the n-Gorenstein projective properties of modules and n-Gorenstein homological dimensions along Frobenius extensions of rings.

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Throughout this paper, let R be an associative ring with identity. All modules will be unitary left R-modules. We write $P(R)$ and $GP(R)$ for the classes of projective and Gorenstein projective left R-modules, respectively. For each positive integer n and the class of modules $\mathcal{X}$, we denote by ${}^{\perp_n}\mathcal{X} := \{M|\mathrm{Ext}^i_R(M,X)=0 \text{ for any } X\in\mathcal{X}, \text{and } 1\le i\le n\}$.

2. n-Gorenstein projective modules over Frobenius extensions

As a generalization of Gorenstein projective modules, Tang defined n-Gorenstein projective modules in [4].

Definition 2.1 *Suppose that n is a positive integer. An R-module M is said to be n-Gorenstein projective, if there exists an acyclic complex of projective modules $\mathbf{P}=\cdots\to P_1\to P_0\to P_{-1}\to P_{-2}\to\cdots$ such that $M\cong \mathrm{Im}(P_0\to P_{-1})$ and such that for any projective module Q the complex $\mathrm{Hom}_R(\mathbf{P},Q)=\cdots\to P_1^*\to P_0^*\to P_{-1}^*\to P_{-2}^*\to\cdots$ is exact at P_i^* for all $i\ge -n$, where $P_i^*=\mathrm{Hom}_R(P_{-i},Q)$. The class of n-Gorenstein projective modules is denoted by n-$GP(R)$.*

Clearly, by the definitions we have $P(R)\subseteq GP(R)\subseteq n\text{-}GP(R)$. However, there are n-Gorenstein projective modules which are not Gorenstein projective by [4, Example 2.4].

Lemma 2.2 ([4, Proposition 2.2]) *Suppose that M is an R-module, and m, n are positive integers such that $m<n$, then the following statements hold.*

(1) M is n-Gorenstein projective if and only if M belongs to the class ${}^{\perp_n}P(R)$, and admits a co-proper right $P(R)$-resolution.

(2) n-Gorenstein projective modules are m-Gorenstein projective modules.

(3) $GP(R)=\cap_{n=1}^{\infty}$ n-$GP(R)$.

(4) If M is n-Gorenstein projective, then there is an exact sequence $0\to M\to P\to G\to 0$ such that P is projective and G is $(n+1)$-Gorenstein projective.

Lemma 2.3 ([4, Proposition 2.6 and Corollary 2.7]) *n-$GP(R)$ is closed under direct sums, direct summands and extensions.*

Lemma 2.4 ([4, Corollary 3.2]) *Let $0\to G_1\to G\to M\to 0$ be an exact sequence, where G and G_1 are n-Gorenstein projective and $\mathrm{Ext}^1(M,Q)=0$ for all projective modules Q. Then M is n-Gorenstein projective.*

Definition 2.5 ([9, Definition 1.1 and Theorem 1.2]) *A ring extension $R\subset A$ is a Frobenius extension, which provided that one of the following equivalent conditions holds:*

(1) The functors $\mathrm{T}=A\otimes_R -$ and $\mathrm{H}=\mathrm{Hom}_R(A,-)$ are naturally equivalent.

(2) ${}_RA$ is finite generated projective and ${}_AA_R\cong({}_RA_A)^*=\mathrm{Hom}_R({}_RA_A,R)$.

(3) A_R is finite generated projective and ${}_RA_A\cong({}_AA_R)^*=\mathrm{Hom}_{R^{\mathrm{op}}}({}_AA_R,R)$.

Example 2.6 ([9, Definition 1.1 and Theorem 1.2]) (1) For a finite group G, $\mathbb{Z}\subset\mathbb{Z}G$ is a Frobenius extension.

(2) ([11, Lemma 3.1]) Let R be an arbitrary ring, and $A=R[x]/(x^2)$ is the quotient of the

polynomial ring, where x is a variable which is supposed to commute with all the elements of R. Then the ring extension $R \subset A$ is a Frobenius extension.

In [11], Ren studied the Gorenstein projective properties of modules along Frobenius extensions of rings. Let M be any left A-module. It is proved that M is a Gorenstein projective left A-module if and only if M is a Gorenstein projective left R-module if and only if $A \otimes_R M$ and $\mathrm{Hom}_R(A, M)$ are Gorenstein projective left A-modules. Analogously, we have the following conclusions for n-Gorenstein projective modules.

Proposition 2.7 *Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. If ${}_AM$ is n-Gorenstein projective, then the underlying left R-module ${}_RM$ is also n-Gorenstein projective.*

Proof Let M be an n-Gorenstein projective left A-module. There exists an acyclic complex of projective left A-module $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and for any projective left A-module Q the complex $\mathrm{Hom}_A(\mathbf{P}, Q) = \cdots \to P_1^* \to P_0^* \to P_{-1}^* \to P_{-2}^* \to \cdots$ is exact at P_i^* for all $i \geq -n$, where $P_i^* = \mathrm{Hom}_A(P_{-i}, Q)$. Note that each P_i is a projective left R-module. Then by restricting $\mathbf{P}$ one gets an acyclic complex of projective R-modules.

Let F be a projective left R-module. It follows from isomorphisms $\mathrm{Hom}_R(A, F) \cong A \otimes_R F$ that $\mathrm{Hom}_R(A, F)$ is a projective left A-module. Then the complex

$$\mathrm{Hom}_A(\mathbf{P}, \mathrm{Hom}_R(A, F)) = \cdots \to T_1^* \to T_0^* \to T_{-1}^* \to T_{-2}^* \to \cdots$$

is exact at T_i^* for all $i \geq -n$, where $T_i^* = \mathrm{Hom}_A(P_{-i}, \mathrm{Hom}_R(A, F)$. Moreover, there are isomorphisms

$$\mathrm{Hom}_R(\mathbf{P}, F) \cong \mathrm{Hom}_R(A \otimes_A \mathbf{P}, F) \cong \mathrm{Hom}_A(\mathbf{P}, \mathrm{Hom}_R(A, F)).$$

This implies that the complex $\mathrm{Hom}_R(\mathbf{P}, F)$ is exact at H_i^* for all $i \geq -n$, where $H_i^* = \mathrm{Hom}_R(P_{-i}, F)$, and hence the underlying R-module M is n-Gorenstein projective. □

Proposition 2.8 *Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. Then $A \otimes_R M(\mathrm{Hom}_R(A, M))$ is an n-Gorenstein projective left A-module if and only if the underlying left R-module ${}_RM$ is also n-Gorenstein projective.*

Proof $\Leftarrow$. Let M be an n-Gorenstein projective left R-module. There exists an acyclic complex of projective left R-module $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and for any projective left R-module Q the complex $\mathrm{Hom}_R(\mathbf{P}, Q) = \cdots \to P_1^* \to P_0^* \to P_{-1}^* \to P_{-2}^* \to \cdots$ is exact at P_i^* for all $i \geq -n$, where $P_i^* = \mathrm{Hom}_R(P_{-i}, Q)$. It is easy to see that $A \otimes_R \mathbf{P}$ is an acyclic complex of projective A-modules, and

$$A \otimes_R M \cong \mathrm{Im}(A \otimes_R P_0 \to A \otimes_R P_{-1}).$$

Moreover, for any projective left A-module P, there are isomorphisms

$$\mathrm{Hom}_A(A \otimes_R \mathbf{P}, P) \cong \mathrm{Hom}_R(\mathbf{P}, P).$$

This implies that the complex $\mathrm{Hom}_A(A \otimes_R \mathbf{P}, P)$ is exact at U_i^* for all $i \geq -n$, where $U_i^* =$

$\mathrm{Hom}_A(A\otimes_R P_{-i},P)$. Hence $A\otimes_R M$ is an n-Gorenstein projective left A-module. There is an isomorphism

$$\mathrm{Hom}_R(A,M)\cong A\otimes_R M.$$

This implies that the module $\mathrm{Hom}_R(A,M)$ is an n-Gorenstein projective left A-module.

$\Rightarrow$. Note that for the ring extension $R\subset A$ and any A-module M, the module M is a left R-module. By Proposition 2.7, it suffices to prove that when the left A-module $A\otimes_R M$ is n-Gorenstein projective, $A\otimes_R M$ is an n-Gorenstein projective left R-module. It is easy to see that the module ${}_RM$ is a direct summand of the left R-module $A\otimes_R M$. According to Lemma 2.3, ${}_RM$ is an n-Gorenstein projective left R-module. $\square$

Theorem 2.9 *Suppose M is any left A-module. Then $A\otimes_R M$ ($\mathrm{Hom}_R(A,M)$) is an n-Gorenstein projective left A-module if and only if M is an n-Gorenstein projective left A-module.*

Proof By Propositions 2.7 and 2.8, it suffices to prove that n-Gorenstein projective left R-module M is also an n-Gorenstein projective left A-module.

Let Q be any projective left A-module. Then Q is a projective left R-module. Note that for the ring extension $R\subset A$ and any A-module M, the module M is a left R-module. Therefore, by the isomorphisms

$$\mathrm{Hom}_A(M,A\otimes_R Q)\cong \mathrm{Hom}_A(M,\mathrm{Hom}_R(A,Q))\cong \mathrm{Hom}_R(A\otimes_A M,Q)\cong \mathrm{Hom}_R(M,Q),$$

we get the cohomology isomorphisms

$$\mathrm{Ext}^i_A(M,A\otimes_R Q)\cong \mathrm{Ext}^i_A(M,\mathrm{Hom}_R(A,Q))\cong \mathrm{Ext}^i_R(A\otimes_A M,Q)\cong \mathrm{Ext}^i_R(M,Q).$$

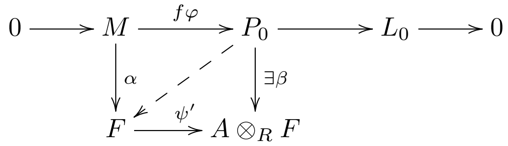
Since M is an n-Gorenstein projective left R-module, it follows from Lemma 2.2(1) that $M\in {}^{\perp_n}P(R)$, i.e., $\mathrm{Ext}^i_R(M,Q)=0$ for all $1\le i\le n$. Then we have $\mathrm{Ext}^i_A(M,A\otimes_R Q)\cong \mathrm{Ext}^i_R(M,Q)=0$. Moreover, since ${}_AQ$ is a direct summand of $A\otimes_R Q$, and then $\mathrm{Ext}^i_A(M,Q)=0$.

Since $\mathrm{Hom}_R(A,M)$ is an n-Gorenstein projective left A-module, by Lemma 2.2(2), there is an exact sequence $0\to \mathrm{Hom}_R(A,M)\xrightarrow{f} P_0\to L\to 0$ of left A-modules, where P_0 is projective and L is $(n+1)$-Gorenstein projective. By Lemma 2.2 (4), L is n-Gorenstein projective. There is a map $\varphi: M\to \mathrm{Hom}_R(A,M)$ given by $\varphi(m)(a)=am$, which is an A-monomorphism, and split when we restrict it as an R-homomorphism. Hence we have an R-homomorphism $\varphi':\mathrm{Hom}_R(A,M)\to M$ such that $\varphi'\varphi=\mathrm{id}_M$. Let P be any projective R-module, and $g:M\to P$ be any R-homomorphism. Since L is also n-Gorenstein projective as an R-module, for the R-homomorphism $g\varphi':\mathrm{Hom}_R(A,M)\to P$, there is an R-homomorphism $h:P_0\to P$, such that $g\varphi'=hf$. That is, we have the following commutative diagram:

$$\begin{array}{ccccc} 0\longrightarrow & \mathrm{Hom}_R(A,M) & \xrightarrow{f} & P_0 & \longrightarrow L\longrightarrow 0\\ & \downarrow{\scriptstyle g\varphi'} & \swarrow{\scriptstyle \exists h} & & \\ & P & & & \end{array}$$

Now we have an A-monomorphism $f\varphi: M\to P_0$. Consider the exact sequence $0\to M\xrightarrow{f\varphi} P_0\to L_0\to 0$ of A-modules, where P_0 is projective, $L_0=\mathrm{Coker}(f\varphi)$. Restricting the sequence, we note that it is $\mathrm{Hom}_R(-,P)$-exact for any projective R-modules P, since for any R-homomorphism $g : M \to P$, there is an R-homomorphism $h : P_0 \to P$, such that $g = (g\varphi')\varphi = h(f\varphi)$. Then, it follows from the exact sequence $\mathrm{Hom}_R(P_0, P) \to \mathrm{Hom}_R(M, P) \to \mathrm{Ext}^1_R(L_0, P) \to 0$ that $\mathrm{Ext}^1_R(L_0, P) = 0$. Moreover, M and P_0 are n-Gorenstein projective left R-modules, it follows from Lemma 2.4 that L_0 is an n-Gorenstein projective left R-module.

Let F be any projective left A-module. There is a split epimorphism $\psi : A \otimes_R F \to F$ of A-modules given by $\psi(a \otimes_R x) = ax$ for any $a \in A$ and $x \in F$, and then there exists an A-homomorphism $\psi' : F \to A \otimes_R F$ such that $\psi\psi' = \mathrm{id}_F$. Note that F is also projective as an R-module. Then, it follows from $\mathrm{Ext}^1_A(L_0, A \otimes_R F) \cong \mathrm{Ext}^1_R(L_0, F) = 0$ that the exact sequence $0 \to M \xrightarrow{f\varphi} P_0 \to L_0 \to 0$ remains exact after applying $\mathrm{Hom}_A(-, A \otimes_R F)$. For any A-homomorphism $\alpha : M \to F$, we consider the following diagram



For $\psi'\alpha : M \to A \otimes_R F$, there exists an A-map $\beta : P_0 \to A \otimes_R F$ such that $\psi'\alpha = \beta(f\varphi)$. And then, we have $\psi\beta : P_0 \to F$, such that $\alpha = (\psi\psi')\alpha = (\psi\beta)(f\varphi)$. This implies that the sequence $0 \to M \xrightarrow{f\varphi} P_0 \to L_0 \to 0$ is $\mathrm{Hom}_A(-, F)$-exact.

Note that L_0 is an n-Gorenstein projective left R-module, and then $\mathrm{Hom}_R(A, L_0)$ is an n-Gorenstein projective left A-module. Repeating the process we followed with M, we inductively construct an exact sequence $0 \to M \to P_0 \to P_1 \to P_2 \to \cdots$ of A-modules, with each P_i projective and which is also exact after applying $\mathrm{Hom}_A(-, F)$ for any projective left A-module F. It follows from Lemma 2.2 (1) that M is an n-Gorenstein projective left A-module. □

3. n-Gorenstein projective dimensions over Frobenius extensions

In [10], Ren studied the Gorenstein projective dimensions along Frobenius extensions of rings. In this section, we consider similar conclusions for n-Gorenstein projective dimensions.

Definition 3.1 *Let R be a ring. The n-Gorenstein projective dimension of a left R-module M, denote by $n\text{-}\mathrm{Gpd}_R M$, is defined as* $\inf\{m|$ *there exists an exact sequence* $0 \to G_m \to \cdots \to G_1 \to G_0 \to M \to 0$ *of R-modules, where G_i is an n-Gorenstein projective left R-module*$\}$. *If such m does not exist, then $n\text{-}\mathrm{Gpd}_R M = \infty$. Obviously, M is an n-Gorenstein projective left R-module if and only if $n\text{-}\mathrm{Gpd}_R M = 0$.*

Lemma 3.2 ([4, Proposition 3.1]) *Let M be an R-module with finite n-Gorenstein projective dimension m. Then there exists an exact sequence $0 \to K \to G \to M \to 0$, where G is n-Gorenstein projective and $\mathrm{pd}_R K = m - 1$.*

Proposition 3.3 *Let $0 \to K \to G \to M \to 0$ be an exact sequence of left R-module, where G is n-Gorenstein projective. If $1 \le n\text{-}\mathrm{Gpd}_R M < \infty$, then $n\text{-}\mathrm{Gpd}_R K = n\text{-}\mathrm{Gpd}_R M - 1$.*

Proof Let $1 \leq n\text{-Gpd}_R M < \infty$. On the one hand, by Lemma 3.2 and inclusion relation $P(R) \subseteq n\text{-}GP(R)$, we have an inequality $n\text{-Gpd}_R K \leq \text{pd}_R K = n\text{-Gpd}_R M - 1$.

On the other hand, let $n\text{-Gpd}_R K = s < \infty$. Then there exists an exact sequence $0 \to K_s \to K_{s-1} \to \cdots \to K_1 \to K_0 \to K \to 0$, where $K_j \in n\text{-}GP(R), j = 0, 1, \ldots, s-1, s$. There exists another exact sequence $0 \to K_s \to K_{s-1} \to \cdots \to K_1 \to K_0 \to G \to M \to 0$. So, we have an inequality $n\text{-Gpd}_R M \leq s + 1 = n\text{-Gpd}_R K + 1$, i.e., $n\text{-Gpd}_R K \geq n\text{-Gpd}_R M - 1$. □

Proposition 3.4 *Let R be a ring. If $(M_i)_{i \in I}$ is any family of left R-module, then we have an equality,*

$$n\text{-Gpd}_R(\oplus_{i \in I} M_i) = \sup\{n\text{-Gpd}_R M_i | i \in I\}.$$

Proof The inequality '$\leq$' is clear since $n\text{-}GP(R)$ is closed under direct sums by Lemma 2.3. For the converse inequality '$\geq$', it suffices to show that if M_1 is any direct summand of an R-module M, then $n\text{-Gpd}_R M_1 \leq n\text{-Gpd}_R M$. Naturally we may assume that $n\text{-Gpd}_R M = m$ is finite, and then proceed by induction on m.

The induction start is clear, because if M is n-Gorenstein projective, then so is M_1, by Lemma 2.3. If $m \geq 1$, we write $M = M_1 \oplus M_2$ for some module M_2. Suppose that when we have equality $n\text{-Gpd}_R M = m - 1$, there is an inequality $n\text{-Gpd}_R M \geq n\text{-Gpd}_R M_1$. Naturally we have $n\text{-Gpd}_R M_i < \infty$, where $i = 1, 2$. By Lemma 3.2, there are exact sequences $0 \to K_1 \to G_1 \to M_1 \to 0$ and $0 \to K_2 \to G_2 \to M_2 \to 0$ of left R-modules, where G_1 and G_2 are n-Gorenstein projective. We get commutative diagram with split-exact rows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_1 & \to & K_1 \oplus K_2 & \to & K_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_1 & \to & G_1 \oplus G_2 & \to & G_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & M & \to & M_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Diagram 1 By Horseshoe Lemma

In Diagram 1, $G_1 \oplus G_2$ is n-Gorenstein projective. Applying Proposition 3.3 to the middle column in Diagram 1, we get that

$$n\text{-Gpd}_R(K_1 \oplus K_2) = n\text{-Gpd}_R(M_1 \oplus M_2) - 1 = m - 1.$$

Hence the induction hypothesis yields that $n\text{-Gpd}_R K_1 \leq m - 1$, and thus the short exact sequence $0 \to K_1 \to G_1 \to M_1 \to 0$ shows that $n\text{-Gpd}_R M_1 \leq m$, as desired. □

Proposition 3.5 *Let $R \subset A$ be a Frobenius extension of rings. For any left R-module M, if $n\text{-Gpd}_R M < \infty$, then*

$$n\text{-Gpd}_R M = n\text{-Gpd}_A(A \otimes_R M) = n\text{-Gpd}_R(A \otimes_R M).$$

Proof It follows from Proposition 2.7 that $n\text{-Gpd}_R(A \otimes_R M) \leqslant n\text{-Gpd}_A(A \otimes_R M)$. For any n-Gorenstein projective left R-module M, it follows from Proposition 2.8 that $A \otimes_R M$ is an n-Gorenstein projective left A-module. Then $n\text{-Gpd}_A(A \otimes_R M) \leqslant n\text{-Gpd}_R M$. As R-modules, M is a direct summand of $A \otimes_R M$. It follows immediately from Proposition 3.4 that $n\text{-Gpd}_R M \leqslant n\text{-Gpd}_R(A \otimes_R M)$. Hence, we get the desired equality. □

Definition 3.6 ([11, Definition 2.8]) *A ring extension $R \subset A$ is separable provided that the multiplication map $\varphi : A \otimes_R A \to A (a \otimes_R b \mapsto ab)$ is a split epimorphism of A-bimodules. If $R \subset A$ is simultaneously a Frobenius and separable extension, then it is called a separable Frobenius extension.*

Example 3.7 (1) ([11, Example 2.10]) For a finite group G, the integral group ring extension $\mathbb{Z} \subset \mathbb{Z}G$ is a separable Frobenius extension.

(2) ([9, Example 2.7]) Let F be a field and set $A = M_4(F)$. Let R be the subalgebra of A with F-basis consisting of the idempotents and matrix units $e_1 = e_{11} + e_{44}, e_2 = e_{22} + e_{33}, e_{21}, e_{31}, e_{41}, e_{42}, e_{43}$. Then $R \subset A$ is a separable Frobenius extension.

Lemma 3.8 ([11, Lemma 2.9]) *The following are equivalent:*

(1) $R \subset A$ is a separable extension.

(2) For any A-bimodule M, $\theta : A \otimes_R M \to M$ is a split epimorphism of A-bimodules.

Proposition 3.9 *Let $R \subset A$ be a separable Frobenius extension of rings. For any left A-module M, if $n\text{-Gpd}_A M < \infty$, then $n\text{-Gpd}_A M = n\text{-Gpd}_R M$.*

Proof By Proposition 2.7, any n-Gorenstein projective left A-module is also n-Gorenstein projective left R-module. It is easy to see that $n\text{-Gpd}_R M \leq n\text{-Gpd}_A M < \infty$. For the converse, we can assume that $n\text{-Gpd}_R M = m < \infty$, then there exists an exact sequence $0 \to G_m \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ of R-modules, where G_i is n-Gorenstein projective. By Proposition 2.8, $A \otimes_R G_i$ is n-Gorenstein projective left A-modules, where $i = 0, 1, \ldots, m-1, m$. Then there exists an exact sequence $0 \to A \otimes_R G_m \to A \otimes_R G_{m-1} \to \cdots \to A \otimes_R G_1 \to A \otimes_R G_0 \to A \otimes_R M \to 0$ of left A-modules. Then $n\text{-Gpd}_A(A \otimes_R M) \leq m$. By Lemma 3.8, left A-module M is direct summand of $A \otimes_R M$. By Proposition 3.4, we have inequalities $n\text{-Gpd}_A M \leq n\text{-Gpd}_A(A \otimes_R M) \leq m$. □

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