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The Singular Integrals on the Intersection of Two Balls in $\mathbb{C}^n$

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Abstract In this paper the author studies the singular integral with the circularly deleted neighborhood on the boundary of the intersection of two balls, and obtain the principal value of the singular integral with holomorphic kernel and the Plemelj formula.

Keywords intersection of two balls; circularly deleted neighborhood; principal value; Plemelj formula

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1. Introduction

As well-known in several complex variables there is no uniform holomorphic kernel for different domains, and the principal value of the singular integral depends on the shape of the deleted neighborhood. Gong [1], Wolfgang Alt [2], Kerzman [3, 4] and [5, 6] et al. studied the singular integral with holomorphic kernel on the bounded domain with smooth strictly pseudoconvex boundary. In particular, Gong [1] studied the singular integral on the complex sphere with different deleted neighborhoods, i.e., circular, elliptic and rectangular neighborhoods, and obtained different principal values, further, they studied the singular integral on the bounded domain with smooth strictly pseudoconvex boundary with circular, elliptic and rectangular deleted neighborhoods, respectively, and obtained different principal values. It characterized the difference between one complex variable and several complex variables.

Range and Siu [7] studied $\bar{\partial}$-equation on the bounded strictly pseudoconvex domain with piecewise smooth boundary, and obtained the integral representation of holomorphic functions.

For the study of the singular integral on the bounded strictly pseudoconvex domain with piecewise smooth boundary, even for the basic case of the intersection of two balls, the calculation of the principal value, a key and first step of the the study of the singular integrals, is complicated. The difficulty is mainly due to the asymmetry of the deleted neighborhood on the boundary, and the integration being taken on the whole boundary including the lower dimensional intersection parts.

In this paper the author studies the singular integrals on the intersection of two balls D, for the deleted neighborhood on the boundary is asymmetric, the estimates of the singular

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integrals are more complicated than the smooth case. Compared with the singular integrals with Bochner-Martinelli kernel, the integral is on the whole boundary including the lower dimensional part [8–10].

We always suppose $n > 1$, and all the results may not hold for $n = 1$.

2. Notations and main results

The Cauchy kernel on the ball is invariant under translations, rotations or dilations [5]. So we define the intersection of two balls as the following special case for simplicity.

Definition 2.1 *$D = B_1 \cap B_2$ is said to be an intersection of two balls, if*

$$B_1 = \{z \in \mathbb{C}^n : |z| < 1\}, \quad B_2 = \{z \in \mathbb{C}^n : |z_1 - a|^2 + |z'|^2 < R^2\},$$

such that ∂B_1 and ∂B_2 intersect real transversally, $a = a_1 + ia_2 \in \mathbb{C}$, $a_2 > 0$, $R = |a - 1| \geq 1$, $0 < |a| < 1 + R$, and $z' = \{z_2, \ldots, z_n\} \in \mathbb{C}^{n-1}$.

For the singular integrals on the boundary ∂D, we focus on the standard position $z_0 = \{1, 0, \ldots, 0\} \in \partial D$. We first discuss the local geometry on the boundary ∂D at the standard position z_0. The real tangent hyperplanes of ∂B_1 and ∂B_2 at z_0 are

$$\pi_1 : \mathrm{Re} z_1 = x_1 = 1, \quad \pi_2 : \mathrm{Im} z_1 = \frac{1 - a_1}{a_2}(x_1 - 1),$$

respectively. The hyperplane that the manifold $\partial B_1 \cap \partial B_2$ lies in is

$$\pi : \mathrm{Im} z_1 = -\frac{a_1}{a_2}(x_1 - 1).$$

We denote by φ_j the included angle between the outer normal vectors of ∂B_j and the normal vector of the hyperplane π, at $z_0 = (1, 0, \ldots, 0)$, $j = 1, 2$. Then $0 < \varphi_j < \pi$, and

$$\varphi_1 = \cos^{-1} \frac{k}{\sqrt{1 + k^2}}, \quad \varphi_2 = \cos^{-1} \frac{R^2 + a_1 - 1}{R|a|}. \tag{2.1}$$

Range and Siu [7] constructed the integral representation for holomorphic functions on domains with piecewise smooth strictly pseudoconvex boundaries. We borrow their method to write out the concrete form of the holomorphic kernel on the intersection domain of two balls.

Suppose that the boundary definition functions for B_1 and B_2 are

$$\rho_1(\zeta) = |\zeta|^2 - 1, \quad \rho_2(\zeta) = |\zeta_1 - a|^2 + |\zeta'|^2 - R^2,$$

respectively, then we have

$$\nabla \rho_1 = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_n), \quad \nabla \rho_2 = (\bar{\zeta}_1 - \bar{a}, \bar{\zeta}_2, \ldots, \bar{\zeta}_n),$$

$$u(\zeta) = (u_1, u_2, \ldots, u_n) = \lambda \nabla \rho_1 + (1 - \lambda) \nabla \rho_2 = (\bar{\zeta}_1 - (1 - \lambda)\bar{a}, \bar{\zeta}_2, \ldots, \bar{\zeta}_n), \quad 0 \leq \lambda \leq 1.$$

Obviously, one has $\langle \frac{u(\zeta)}{< u(\zeta), \zeta - z >}, \zeta - z \rangle = 1$ for $\zeta \neq z$. So $\{u(\zeta)(< u(\zeta), \zeta - z >)^{-1}\}$ is a partition of unity. Using this partition of unity we can construct the Range-Siu's type integral representation for holomorphic functions [7]. We denote the barrier function by

$$\Phi(\zeta, z, \lambda) = \langle u(\zeta), \zeta - z \rangle = \lambda \Phi_1(\zeta, z) + (1 - \lambda)\Phi_2(\zeta, z),$$

where $\Phi_1(\zeta,z)=1-z\bar{\zeta}^t$, $\Phi_2(\zeta,z)=1-z\bar{\zeta}^t-\bar{a}(1-z_1)-a(1-\bar{\zeta}_1)$. Set

$$\mathrm{d}u_{[k]}=\mathrm{d}u_1\wedge\cdots\wedge\mathrm{d}u_{k-1}\wedge\mathrm{d}u_{k+1}\wedge\cdots\wedge\mathrm{d}u_n,$$

then the holomorphic Cauchy-Fantappiè kernel is

$$\Omega(\zeta,z,\lambda)=C_n(\Phi(\zeta,z,\lambda))^{-n}\sum_{k=1}^{n}(-1)^{n-1}u_k\mathrm{d}u_{[k]}\wedge\mathrm{d}\zeta, \tag{2.2}$$

where $C_n=\frac{(n-1)!}{(2\pi i)^n}$, $\mathrm{d}\zeta=\mathrm{d}\zeta_1\wedge\cdots\wedge\mathrm{d}\zeta_n$.

Since the domain D in Definition 2.1 is special case of strictly pseudoconvex domain with piecewise smooth boundary, and the kernel (2.2) is a special form of Range-Siu's kernel, we have the following Range-Siu's type formula of integral representation [7, P. 333, (2.5)].

Theorem 2.2 ([7]) *Let D be an intersection of two balls. Suppose $f(z)$ is holomorphic in $\overline{D}$, then for $z\in D$,*

$$\begin{aligned} f(z)&=\sum_I\int_{(\zeta,\lambda)\in S_I\times\Delta_I}f(\zeta)\Omega(\zeta,z,\lambda)\\ &=\int_{\zeta\in S_1}f(\zeta)\Omega(\zeta,z,1)+\int_{\zeta\in S_2}f(\zeta)\Omega(\zeta,z,0)+\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{12}}f(\zeta)\Omega(\zeta,z,\lambda), \end{aligned}\tag{2.3}$$

where $I=\{1,2\}$, $S_1=\partial B_1\cap D$, $S_2=\partial B_2\cap D$, $S_{12}=\partial B_1\cap\partial B_2$, and $\Delta_{12}=[0,1]$.

We first calculate the last term in (2.3).

$$\begin{aligned} &\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{12}}f(\zeta)\Omega(\zeta,z,\lambda)=C_n\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{12}}f(\zeta)\frac{\Sigma_{j=1}^n(-1)^{j-1}u_j\mathrm{d}u_{[j]}\wedge\mathrm{d}\zeta}{(\lambda\Phi_1+(1-\lambda)\Phi_2)^n}\\ &=C_n\int_{\zeta\in S_{12}}f(\zeta)\int_0^1\frac{\Sigma_{j=2}^n(-1)^{j-1}\overline{a\zeta_j}\mathrm{d}\lambda\wedge\mathrm{d}\bar{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta}{(\lambda\Phi_1+(1-\lambda)\Phi_2)^n}\\ &=C_n\int_{\zeta\in S_{12}}f(\zeta)\sum_{j=2}^n\frac{(-1)^{j-1}\overline{a\zeta_j}}{(n-1)(\Phi_1-\Phi_2)}\Big(\frac{1}{\Phi_2^{n-1}}-\frac{1}{\Phi_1^{n-1}}\Big)\mathrm{d}\bar{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta, \end{aligned}\tag{2.4}$$

where $\mathrm{d}\bar{\zeta}_{[1,j]}=\mathrm{d}\bar{\zeta}_2\wedge\cdots\wedge\mathrm{d}\bar{\zeta}_{j-1}\wedge\mathrm{d}\bar{\zeta}_{j+1}\wedge\cdots\wedge\mathrm{d}\bar{\zeta}_n$. So the kernel (2.2) is

$$\Omega(\zeta,z,\lambda)=C_n\sum_{j=2}^n\frac{(-1)^{j-1}\overline{a\zeta_j}}{(n-1)(\Phi_1-\Phi_2)}\Big(\frac{1}{\Phi_2^{n-1}}-\frac{1}{\Phi_1^{n-1}}\Big)\mathrm{d}\bar{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta. \tag{2.5}$$

The main results in this note are

Theorem 2.3 *Suppose $z\in\partial D$, $f(z)\in\mathcal{H}(\alpha,\partial D)$. Then the Cauchy principal value (3.1) exists, and*

$$\text{p.v.}\int_{\zeta\in\partial D}f(\zeta)\Omega(\zeta,z,\lambda)=\int_{\zeta\in\partial D}(f(\zeta)-f(z))\Omega(\zeta,z,\lambda)+\tau(z)f(z).$$

Theorem 2.4 (Plemelj Formula) *If $f\in\mathcal{H}(\alpha,\partial D)$, $z\in D$, and z approaches $z_0\in\partial D$, and satisfies $|z-z_0|/d(z,\partial D)<M$, M is a positive constant. Then*

$$\lim_{z\to z_0}\int_{\zeta\in\partial D}f(\zeta)\Omega(\zeta,z,\lambda)=\text{p.v.}\int_{\zeta\in\partial D}f(\zeta)\Omega(\zeta,z_0,\lambda)+(1-\tau(z_0))f(z_0).$$

3. Cauchy principal value

In several complex variables the principal values of singular integrals with holomorphic kernels depend on the shapes of the deleted neighborhood [1–5]. We define the Cauchy principal value at $z_0 \in \partial D$ by the circularly deleted neighborhood.

Definition 3.1 *For $f(z) \in C(\bar{D})$, $z \in \partial D$, the Cauchy principal value is defined as*

$$\text{p.v.} \sum_I \int_{(\zeta,\lambda)\in S_I \times \Delta_I} f(\zeta)\Omega(\zeta, z, \lambda) = \lim_{\varepsilon\to 0} \sum_I \int_{\substack{(\zeta,\lambda)\in S_I\times\Delta_I \\ |1-z\bar{\zeta}^t|\geq\varepsilon}} f(\zeta)\Omega(\zeta, z, \lambda). \tag{3.1}$$

Definition 3.2 *If $f(z)$ satisfies, for $z_1, z_2 \in \partial D$,*

$$|f(z_1) - f(z_2)| \lesssim |z_1 - z_2|^\alpha,$$

$0 < \alpha \leq 1$. *Then we say $f(z) \in \mathcal{H}(\alpha, \partial D)$. $A \lesssim B$ means that there exists a positive constant M, such that $A \lesssim MB$.*

We first give some lemmas.

Lemma 3.3 *Suppose $\alpha, \beta \in \mathbb{R}$. Then*

$$\int_\alpha^\beta \frac{\mathrm{d}\theta}{(1-\rho r e^{-i\theta})^n} = \sum_{m=1}^{n-1} \frac{1}{mi} \frac{1}{(1-\rho r e^{-i\alpha})^m} - \sum_{m=1}^{n-1} \frac{1}{mi} \frac{1}{(1-\rho r e^{-i\beta})^m} + \\ i\log(1-\rho r e^{-i\alpha}) - i\log(1-\rho r e^{-i\beta}) + \beta - \alpha. \tag{3.2}$$

The proof can be found in [1, proof of Lemma 1.2.1]. In this note we always assume that the logarithmic function is the principal branch.

The existence of the Cauchy principal value (3.1), for $f(z) = 1$, is the first and important step to study the boundary behavior of the Cauchy integral.

Lemma 3.4 *Suppose $z \in \partial D$, $f(z) = 1$. Then the Cauchy principal value (3.1) (denoted by $\tau(z)$) exists, and when z is a smooth point, $\tau(z) = \frac{1}{2}$, when z is a non-smooth point,*

$$\tau(z) = \frac{1}{2} - \frac{2^{n-2}}{\pi i} \int_{\varphi_2-\pi/2}^{\pi/2-\varphi_1} e^{-i(n-1)t} \cos^{n-2} t \sin t \mathrm{d}t + \\ \frac{1}{2\pi i} \int_{-e^{2\varphi_1 i}}^1 \frac{(1-e^{2\varphi_1 i}u)^{n-2}}{u} \mathrm{d}u - \frac{1}{2\pi i} \int_{-e^{-2\varphi_2 i}}^1 \frac{(1-e^{-2\varphi_2 i}u)^{n-2}}{u} \mathrm{d}u,$$

where φ_1 and φ_2 are defined in (2.1).

Proof For $z_0 \in \partial B_1 \cap B_2$, or $B_1 \cap \partial B_2$, i.e., z_0 is on the smooth parts of ∂D, $\tau(z) = \frac{1}{2}$ (see [1], Lemma 1.2.1 or [2]). Therefore, we only need consider $z_0 \in \partial B_1 \cap \partial B_2$, i.e., z_0 is on the non-smooth part. For convenience, we take the standard position $z_0 = (1, 0')$. The other points on $\partial B_1 \cap \partial B_2$ can be turned to the standard position. Let $\varepsilon > 0$ be sufficiently small,

$$\sigma = \{\zeta \in \partial D : |1 - z\bar{\zeta}^t| \leq \varepsilon\}, \quad \Sigma : \partial D \backslash \sigma.$$

Clearly, σ and Σ can be split into three disjoint parts, respectively,

$$\sigma_1 = \sigma \cap \partial B_1, \ \sigma_2 = \sigma \cap \partial B_2, \quad \sigma_{12} = \sigma \cap \partial B_1 \cap \partial B_2,$$

$$\Sigma_1=\Sigma\cap\partial B_1,\quad \Sigma_2=\Sigma\cap\partial B_2,\quad \Sigma_{12}=\Sigma\cap\partial B_1\cap\partial B_2.$$

Set $\widetilde{z}_0=(\rho+ik(1-\rho),0')\in D$, $0<\rho<1$, by Theorem 2.2, we have

$$1=\sum_I\Big(\int_{(\zeta,\lambda)\in\Sigma_I\times\Delta_I}+\int_{(\zeta,\lambda)\in\sigma_I\times\Delta_I}\Big)\Omega(\zeta,\widetilde{z}_0,\lambda). \tag{3.3}$$

Then one has

$$\begin{aligned}J&=\lim_{\varepsilon\to0}\lim_{\rho\to1}\sum_I\int_{(\zeta,\lambda)\in\sigma_I\times\Delta_I}\Omega(\zeta,\tilde{z}_0,\lambda)\\&=\lim_{\varepsilon\to0}\lim_{\rho\to1}\Big(\int_{\zeta\in\sigma_1}\Omega(\zeta,\widetilde{z}_0,1)+\int_{\zeta\in\sigma_2}\Omega(\zeta,\widetilde{z}_0,0)+\int_{(\zeta,\lambda)\in\sigma_{12}\times\Delta_{12}}\Omega(\zeta,\widetilde{z}_0,\lambda)\Big)\\&=\lim_{\varepsilon\to0}\lim_{\rho\to1}(J_1+J_2+J_3).\end{aligned} \tag{3.4}$$

Let us compute the three terms in (3.4). It is well-known that on the unit sphere the Cauchy-Fantappiè kernel is just the same as the Cauchy-Szegö kernel. So the term J_1 in (3.4) can be written as

$$J_1=\frac{1}{\omega_{2n-1}}\int_{\zeta\in\sigma_1}\frac{\sigma(\zeta)}{(1-\tilde{z}_0\bar{\zeta}^t)^n},$$

where $\omega_{2n-1}=2\pi^n/(n-1)!$, and $\sigma(\zeta)$ is the Lebesgue volume element of the sphere $\zeta\bar{\zeta}^t=1$. By Painleve's Theorem, the limit value of J_1 does not depend on the path of $\widetilde{z}_0$ approaching to the standard position point $z_0=(1,0')$. So we can choose a special path, i.e., taking $\tilde{z}_0=(\rho,0')$ $(0<\rho<1)$. Therefore, we have

$$\lim_{\varepsilon\to0}\lim_{\rho\to1}J_1=\lim_{\varepsilon\to0}\lim_{\rho\to1}\omega_{2n-1}^{-1}\int_{\zeta\in\sigma_1}\frac{\sigma(\zeta)}{(1-\rho\bar{\zeta}_1)^n},$$

where σ_1 is given as follows,

$$\sigma_1:|\zeta|=1,\ |1-\bar{\zeta}_1|\le\varepsilon,\quad k\mathrm{Re}\,\zeta_1+\mathrm{Im}\,\zeta_1\ge k,$$

where $k=a_1/a_2$, without loss of generality, suppose $a_1\ge0$, for $a_1<0$, the proof is similar. Suppose $\zeta_1=re^{i\theta}$, $v=(\zeta_2,\dots,\zeta_n)$. Then we have

$$\sigma_1:\begin{cases}v\bar{v}'=1-r^2\le\frac{2\varepsilon}{\sqrt{1+k^2}}-\varepsilon^2,\\ \alpha_1=\cos^{-1}\frac{k^2+\sqrt{(1+k^2)r^2-k^2}}{(1+k^2)r}<\theta<\cos^{-1}\frac{1+r^2-\varepsilon^2}{2r}=\alpha_2.\end{cases} \tag{3.5}$$

By Lemma 3.3,

$$\begin{aligned}J_1=&\omega_{2n-1}^{-1}\int_{v\bar{v}'\le\frac{2\varepsilon}{\sqrt{1+k^2}}-\varepsilon^2}\sigma(v)\int_{\alpha_1}^{\alpha_2}\frac{\mathrm{d}\theta}{(1-\rho re^{-i\theta})^n}\\=&\omega_{2n-1}^{-1}\int_{v\bar{v}'\le\frac{2\varepsilon}{\sqrt{1+k^2}}-\varepsilon^2}\Big(\sum_{m=1}^{n-1}\frac{1}{im}[(1-\rho re^{-i\alpha_1})^{-m}-(1-\rho re^{-i\alpha_2})^{-m}]+\\&i\log(1-\rho re^{-i\alpha_1})-i\log(1-\rho re^{-i\alpha_2})+(\alpha_2-\alpha_1)\Big)\sigma(v)\\=&\sum_{m=1}^{n-1}(\widetilde{P}_m-P_m)+\widetilde{P}_0-P_0+P^*.\end{aligned} \tag{3.6}$$

For the second term J_2 in (3.4), we take the coordinate transformation

$$\zeta = (\zeta_1, \zeta') \to \xi = ((\zeta_1 - a)/(1-a), \zeta'/R), \tag{3.7}$$

then, as $\tilde{z}_0 \to \tilde{u}_0$,

$$\sigma_2 : \begin{cases} |\zeta_1 - a|^2 + |\zeta'|^2 = R^2, \\ |1 - \bar{\zeta}_1| \le \varepsilon, \\ \mathrm{Im}\zeta_1 \le -k(\mathrm{Re}\zeta_1 - 1), \end{cases} \longrightarrow \sigma_2' : \begin{cases} |\xi| = 1, \\ (\mathrm{Re}\xi_1 - 1)^2 + \mathrm{Im}\xi_1^2 \le \varepsilon^2/R^2, \\ \mathrm{Im}\xi_1 \le \widetilde{k}(\mathrm{Re}\xi_1 - 1), \end{cases}$$

where $\widetilde{k} = (|a|^2 - a_1)/a_2$. Let $\xi_1 = re^{i\theta}$, $v = (\xi_2, \ldots, \xi_n)$. We have

$$\sigma_2' : \begin{cases} v\bar{v}' = 1 - r^2 \le \frac{2\varepsilon}{R\sqrt{1+\widetilde{k}^2}} - \frac{\varepsilon^2}{R^2}, \\ -\beta_1 = -\cos^{-1}\frac{1+r^2-\varepsilon^2/R^2}{2r} < \theta < -\cos^{-1}\frac{\tilde{k}^2+\sqrt{(1+\tilde{k}^2)r^2-\tilde{k}^2}}{r(1+\tilde{k}^2)} = -\beta_2. \end{cases}$$

Taking $\widetilde{u}_0 = (\rho, 0')$ $(0 < \rho < 1)$, we have by Lemma 3.3

$$\begin{aligned} J_2 =& \omega_{2n-1}^{-1} \int_{v\bar{v}' \le \frac{2\varepsilon}{R\sqrt{1+\widetilde{k}^2}} - \frac{\varepsilon^2}{R^2}} \sigma(v) \int_{-\beta_1}^{-\beta_2} \frac{\mathrm{d}\theta}{(1-\rho re^{-i\theta})^n} \\ =& \omega_{2n-1}^{-1} \int_{v\bar{v}' \le \frac{2\varepsilon}{R\sqrt{1+\widetilde{k}^2}} - \frac{\varepsilon^2}{R^2}} \Big(\sum_{m=1}^{n-1} \frac{1}{im} [(1-\rho re^{i\beta_1})^{-m} - (1-\rho re^{i\beta_2})^{-m}] + \\ & i\log(1-\rho re^{i\beta_1}) - i\log(1-\rho re^{i\beta_2}) + (\beta_1 - \beta_2) \Big) \sigma(v) \\ =& \sum_{m=1}^{n-1} (Q_m - \widetilde{Q}_m) + Q_0 - \widetilde{Q}_0 + Q^*. \end{aligned} \tag{3.8}$$

For $\alpha_1 = O(1)$, $\alpha_2 = O(1)$, $\beta_1 = O(1)$, $\beta_2 = O(1)$, we have

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1} P^* = 0, \quad \lim_{\varepsilon\to 0}\lim_{\rho\to 1} Q^* = 0. \tag{3.9}$$

We compute $\widetilde{P}_0$ in (3.6) as follows. First we have

$$1 - \rho re^{-i\alpha_1} = 1 - \rho + \rho\frac{1+ki}{1+k^2}(1 - \sqrt{(1+k^2)r^2 - k^2}). \tag{3.10}$$

Set $c = 1-\rho$, $b = \rho(1+ki)$, $g = [2b + c(1+k^2)]/(1+k^2)$, one has

$$\begin{aligned} \widetilde{P}_0 =& \omega_{2n-1}^{-1} \int_{v\bar{v}' \le 2\varepsilon/\sqrt{1+k^2} - \varepsilon^2} \log[1-\rho+\rho\frac{1+ki}{1+k^2}(1-\sqrt{1-(1+k^2)v\bar{v}'})]\sigma(v) \\ =& \frac{2\pi^{n-1} i\omega_{2n-1}^{-1}}{\Gamma(n-1)} \int_0^{\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}} \log[1-\rho+\rho\frac{1+ki}{1+k^2}(1-\sqrt{1-(1+k^2)s^2})] s^{2n-3}\mathrm{d}s \\ =& \frac{(n-1)i}{2\pi} \int_0^{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2} \log[1-\rho+\rho\frac{1+ki}{1+k^2}(1-\sqrt{1-(1+k^2)t})] t^{n-2}\mathrm{d}t \\ =& \frac{(n-1)i}{\pi} \int_0^{\varepsilon/\sqrt{1+k^2}} y^{n-2}[2-(1+k^2)y]^{n-2}[1-(1+k^2)y]\log(c+by)\mathrm{d}y. \end{aligned}$$

For an integer $m \ge 0$,

$$\lim_{\varepsilon\to 0}\lim_{c\to 0} \int_0^{b\varepsilon/\sqrt{1+k^2}} y^m \log(c+by)\mathrm{d}y = 0,$$

it yields

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\widetilde{P}_0 = 0. \tag{3.11}$$

For the same reason, we have

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\widetilde{Q}_0 = 0. \tag{3.12}$$

For the terms $\Sigma_{m=1}^{n-1}P_m$ and $\Sigma_{m=1}^{n-1}Q_m$, we have

$$\begin{aligned}
P_m =&\frac{1}{im\omega_{2n-1}}\int_{v\bar{v}'\le 2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}\frac{\sigma(\zeta)}{(1-\rho re^{-i\alpha_2})^m}\\
=&\frac{1}{im\omega_{2n-1}}\int_0^{\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}}\frac{s^{2n-3}\mathrm{d}s}{[1-\rho+\rho\frac{\varepsilon^2+s^2}{2}+\frac{i\rho}{2}\sqrt{2\varepsilon^2(2-s^2)-s^2-\varepsilon^2}]^m}\cdot\\
&\int_0^\pi \sin^{2n-4}\varphi_1\mathrm{d}\varphi_1\ldots\int_0^\pi\sin\varphi_{2n-4}\mathrm{d}\varphi_{2n-4}\int_0^{2\pi}\mathrm{d}\varphi_{2n-3}\\
=&\frac{2\pi^{n-1}}{im(n-2)!\omega_{2n-1}}\int_0^{\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}}\frac{s^{2n-3}\mathrm{d}s}{[1-\rho+\rho\frac{\varepsilon^2+s^2}{2}+\frac{i\rho}{2}\sqrt{4\varepsilon^2-(s^2+\varepsilon^2)^2}]^m}.
\end{aligned}$$

When $1\le m\le n-2$, if $1/2\le\rho\le 1$,

$$\begin{aligned}
&|1-\rho+\rho\frac{\varepsilon^2+s^2}{2}+\frac{i\rho}{2}\sqrt{4\varepsilon^2-(s^2+\varepsilon^2)^2}|^{-m}\le\big((\rho\frac{\varepsilon^2+s^2}{2})^2+\frac{\rho^2}{4}(4\varepsilon^2-(s^2+\varepsilon^2)^2)\big)^{-m/2}\\
&=\frac{1}{(\rho\varepsilon)^m}\le\frac{2^m}{\varepsilon^m}.
\end{aligned}$$

For $1\le m\le n-2$, by Lebesgue's Theorem,

$$|P_m|\lesssim\varepsilon^{-m}\int_0^{\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}}s^{2n-3}\mathrm{d}s\to 0,\quad \varepsilon\to 0. \tag{3.13}$$

When $m=n-1$, setting $s=\eta y$, $\eta=\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}$, then by Lebesgue's Theorem

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{\rho\to 1}P_{n-1}&=\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_0^1\frac{y^{2n-3}\mathrm{d}y}{[\frac{y^2+\varepsilon^2\eta^{-2}}{2}+\frac{i}{2}\sqrt{4\eta^{-4}\varepsilon^2-(y^2+\varepsilon^2\eta^{-2})^2}]^{n-1}}\\
&=\frac{2^{n-1}}{\pi i}\int_0^1\frac{y^{2n-3}\mathrm{d}y}{(y^2+i\sqrt{1+k^2-y^4})^{n-1}}\\
&=\frac{2^{n-2}}{\pi i}\Big(\int_0^{\frac{\pi}{2}}-\int_0^{\psi_1}\Big)e^{-i(n-1)t}\cos^{n-2}t\sin t\mathrm{d}t.
\end{aligned} \tag{3.14}$$

$\psi_1=\cos^{-1}\frac{1}{\sqrt{1+k^2}}=\pi/2-\varphi_1$. While for P_0,

$$\begin{aligned}
&\log|1-\rho+\rho\frac{\varepsilon^2+s^2}{2}+\frac{i\rho}{2}\sqrt{4\varepsilon^2-(s^2+\varepsilon^2)^2}|\lesssim\log\big((1-\rho+\rho\frac{\varepsilon^2+s^2}{2})^2+\frac{\rho^2}{4}(4\varepsilon^2-(s^2+\varepsilon^2)^2\big)\\
&\lesssim\log\frac{1}{4}((1+\varepsilon^2+s^2)^2+4\varepsilon^2-(s^2+\varepsilon^2)^2)\lesssim\log\frac{1}{2}(1+2\varepsilon),
\end{aligned}$$

so we have

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}P_0 = 0. \tag{3.15}$$

Similarly, when $0\le m\le n-2$,

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}Q_m = 0 \tag{3.16}$$

and

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1} Q_{n-1} = \frac{2^{n-2}}{\pi i}\Big(\int_0^{\frac{\pi}{2}} - \int_0^{\psi_2}\Big) e^{i(n-1)t}\cos^{n-2} t \sin t \mathrm{d}t. \tag{3.17}$$

$\psi_2 = \cos^{-1}\frac{1}{\sqrt{1+\widetilde{k}^2}} = \pi/2 - \varphi_2$. Next we consider $\widetilde{P}_m$. We have

$$\widetilde{P}_m = \frac{1}{mi\omega_{2n-1}}\int_{v\bar{v}'\le 2\varepsilon/\sqrt{1+k^2}-\varepsilon^2} [1-\rho+\rho\frac{1+ki}{1+k^2}(1-\sqrt{1-(1+k^2)v\bar{v}'})]^{-m}\sigma(v).$$

By spheroidal coordinates, one has

$$\widetilde{P}_m = \frac{\pi^{n-1}}{m(n-2)!i\omega_{2n-1}}\int_0^{\sqrt{2\varepsilon/\sqrt{1+k^2}-\varepsilon^2}} \frac{t^{n-2}\mathrm{d}t}{[1-\rho+\rho\frac{1+ki}{1+k^2}(1-\sqrt{1-(1+k^2)t})]^m}. \tag{3.18}$$

Set $1-\sqrt{1-(1+k^2)t} = (1+k^2)y$, then, for $0 < m < n-1$,

$$\widetilde{P}_m = \frac{\pi^{n-1}}{m(n-2)!i\omega_{2n-1}}\int_0^{\frac{\varepsilon}{\sqrt{1+k^2}}} \frac{2y^{n-2}[2-(1+k^2)y]^{n-2}[1-(1+k^2)y]\mathrm{d}y}{[1-\rho+\rho(1+ki)y]^m} \to 0, \quad \varepsilon\to 0, \ \rho\to 1.$$

For $m = n-1$,

$$\widetilde{P}_{n-1} = \frac{1}{2\pi i}\int_0^{\varepsilon/\sqrt{1+k^2}} \frac{2y^{n-2}[2-(1+k^2)y]^{n-2}[1-(1+k^2)y]\mathrm{d}y}{[1-\rho+\rho(1+ki)y]^{n-1}}.$$

Set $c = 1-\rho$, $b = \rho(1+ki)$,

$$F_{11}(y) = (2y^{n-2}[2-(1+k^2)y]^{n-2}[1-(1+k^2)y])/[c+by]^{n-1},$$

then we have

$$\widetilde{P}_{n-1} = \frac{1}{2\pi i}\int_0^{\varepsilon/\sqrt{1+k^2}} F_{11}(y)\mathrm{d}y. \tag{3.19}$$

For $\widetilde{Q}_m$, it can be treated similarly to $\widetilde{P}_m$, when $m < n-1$, $\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\widetilde{Q}_m = 0$, and

$$\widetilde{Q}_{n-1} = \frac{1}{2\pi i}\int_0^{\frac{2\varepsilon}{R\sqrt{1+\widetilde{k}^2}}-\frac{\varepsilon^2}{R^2}} [1-\rho+\rho\frac{1-\tilde{k}i}{1+\tilde{k}^2}(1-\sqrt{1-(1+\tilde{k}^2)t})]^{1-n} t^{n-2}\mathrm{d}t.$$

Set $1-\sqrt{1-(1+\widetilde{k}^2)t} = (1+\widetilde{k}^2)y$,

$$F_{21}(y) = (2y^{n-2}[2-(1+\widetilde{k}^2)y]^{n-2}[1-(1+\widetilde{k}^2)y])/[c+\rho(1-\widetilde{k}i)y]^{n-1},$$

then

$$\widetilde{Q}_{n-1} = \frac{1}{2\pi i}\int_0^{\varepsilon/(R\sqrt{1+\widetilde{k}^2})} F_{21}(y)\mathrm{d}y. \tag{3.20}$$

Now let us compute J_3. On σ_{12}, $x^2 + k^2(x-1)^2 + |\zeta'|^2 = 1$, $x = \mathrm{Re}\zeta_1$, then

$$\mathrm{d}\overline{\zeta}_2 = \frac{-1}{\zeta_2}\sum_{j=3}^{n}\zeta_j\mathrm{d}\overline{\zeta}_j \ \mathrm{mod}(\mathrm{d}x, \mathrm{d}\zeta_2, \mathrm{d}\zeta_3, \ldots, \mathrm{d}\zeta_n), \quad \zeta_2\neq 0. \tag{3.21}$$

So we have, by (2.3),

$$J_3 = \frac{C_n\bar{a}}{n-1}\int_{\sigma_{12}}\frac{|\zeta'|^2\mathrm{d}\overline{\zeta}_{[12]}\wedge\mathrm{d}\zeta}{(\Phi_1-\Phi_2)\zeta_2\Phi_1^{n-1}} - \frac{C_n\bar{a}}{n-1}\int_{\sigma_{12}}\frac{|\zeta'|^2\mathrm{d}\overline{\zeta}_{[12]}\wedge\mathrm{d}\zeta}{(\Phi_1-\Phi_2)\zeta_2\Phi_2^{n-1}} = J_{31} - J_{32}. \tag{3.22}$$

Set $v_1 = (\zeta_3, \ldots, \zeta_n) \in \mathbb{C}^{n-2}$, $z_1 = \rho, 0 < \rho < 1$, by (3.21)

$$\begin{aligned} J_{31} =& \frac{2\pi i C_n (1-ki)\bar{a}}{n-1} \int_{1-\varepsilon/\sqrt{1+k^2}}^{1} \mathrm{d}x \int_{v_1\overline{v}_1' \le 1-x^2-k^2(1-x)^2} \frac{(1-x^2-k^2(1-x)^2)\mathrm{d}\overline{\zeta}_{[12]} \wedge \mathrm{d}\zeta_{[12]}}{(\Phi_1-\Phi_2)(1-z_1\bar{\zeta}_1)^{n-1}} \\ =& \frac{1-ki}{2\pi i} \int_{1-\frac{\varepsilon}{\sqrt{1+k^2}}}^{1} \frac{(1-x^2-k^2(1-x)^2)^{n-1}\mathrm{d}x}{(1-\rho-(1-ki)(1-x))(1-\rho(x+ki(x-1)))^{n-1}} \\ =& \frac{1}{2\pi i} \int_0^{\frac{\varepsilon}{\sqrt{1+k^2}}} \frac{(1-ki)t^{n-1}(2-(1+k^2)t)^{n-1}\mathrm{d}t}{(1-\rho-(1-ki)t)(1-\rho+\rho(1+ki)t)^{n-1}}. \end{aligned}$$

Set

$$F_{12}(t) = (1-ki)t^{n-1}(2-(1+k^2)t)^{n-1}[(1-\rho-(1-ki)t)(1-\rho+\rho(1+ki)t)^{n-1}]^{-1},$$

then

$$J_{31} = \frac{1}{2\pi i} \int_0^{\varepsilon/\sqrt{1+k^2}} F_{12}(t)\mathrm{d}t. \tag{3.23}$$

For the second term J_{32} in (3.22), by the transformation (3.7), we have

$$\begin{aligned} J_{32} =& \frac{C_n}{n-1} \int_{\sigma'_{12}} \frac{|\xi'|^2 \mathrm{d}\overline{\xi}_{[12]} \wedge \mathrm{d}\xi}{(1-u_1-(1+\widetilde{k}i)/(1-\widetilde{k}i)(1-\bar{\xi}_1))\xi_2(1-u_1\bar{\xi}_1)^{n-1}} \\ =& \frac{2\pi i C_n(1+\tilde{k}i)}{n-1} \int_{1-\varepsilon/(R\sqrt{1+\tilde{k}^2})}^{1} \mathrm{d}x \\ &\times \int_{v_1\overline{v}_1' \le 1-x^2-\tilde{k}^2(1-x)^2} \frac{(1-x^2-\tilde{k}^2(1-x)^2)\mathrm{d}\overline{\xi}_{[12]} \wedge \mathrm{d}\xi_{[12]}}{(1-u_1-(1+\widetilde{k}i)/(1-\widetilde{k}i)(1-\bar{\xi}_1))\xi_2(1-u_1\bar{\xi}_1)^{n-1}} \\ =& \frac{1+\widetilde{k}i}{2\pi i} \int_{1-\varepsilon/(R\sqrt{1+\tilde{k}^2})}^{1} \frac{(1-x^2-\widetilde{k}^2(1-x)^2)^{n-1}\mathrm{d}x}{(1-u_1-(1+\widetilde{k}i)(1-x))(1-u_1+(1+\widetilde{k}i)(1-x))^{n-1}} \\ =& \frac{1}{2\pi i} \int_0^{\varepsilon/(R\sqrt{1+\tilde{k}^2})} \frac{(1+\tilde{k}i)t^{n-1}(2-(1+\tilde{k}^2)t)^{n-1}\mathrm{d}t}{(1-u_1-(1+\tilde{k}i)t)(1-u_1+u_1(1-\tilde{k}i)t)^{n-1}}. \end{aligned}$$

Set $u_1 = (\rho, 0')$, $0 < \rho < 1$,

$$F_{22}(t) = \frac{(1+\widetilde{k}i)t^{n-1}(2-(1+\widetilde{k}^2)t)^{n-1}}{(1-\rho-(1+\widetilde{k}i)t)(1-\rho+\rho(1-\widetilde{k}i)t)^{n-1}},$$

then

$$J_{32} = \frac{1}{2\pi i} \int_0^{\varepsilon/(R\sqrt{1+\tilde{k}^2})} F_{22}(t)\mathrm{d}t. \tag{3.24}$$

By (3.22) and (3.23), we have

$$J_3 = \frac{1}{2\pi i} \int_0^{\frac{\varepsilon}{\sqrt{1+k^2}}} F_{12}(t)\mathrm{d}t - \frac{1}{2\pi i} \int_0^{\frac{\varepsilon}{R\sqrt{1+\tilde{k}^2}}} F_{22}(t)\mathrm{d}t. \tag{3.25}$$

Let us consider (3.19) and (3.23). Set $c = 1-\rho$, $b = \rho(1+ki)$, $d = -(1-ki)$, we have

$$\begin{aligned} F_{11}(t) =& \frac{2t^{n-2}[2-(1+k^2)t]^{n-1}}{(c+bt)^{n-1}} - \frac{2t^{n-2}[2-(1+k^2)t]^{n-2}}{(c+bt)^{n-1}}, \\ F_{12}(t) =& \frac{ct^{n-2}[2-(1+k^2)t]^{n-1}}{(c+dt)(c+bt)^{n-1}} - \frac{t^{n-2}[2-(1+k^2)t]^{n-1}}{(c+bt)^{n-1}}, \end{aligned}$$

$$
\begin{aligned}
Z =&\frac{1}{2\pi i}\int_0^{\frac{\varepsilon}{\sqrt{1+k^2}}}(F_{11}(t)+F_{12}(t))\mathrm{d}t\\
=&\frac{1}{2\pi i}\int_0^{\frac{\varepsilon}{\sqrt{1+k^2}}}\Big(\frac{ct^{n-2}[2-(1+k^2)t]^{n-1}}{(c+bt)^{n-1}(c+dt)}-\frac{(1+k^2)t^{n-1}[2-(1+k^2)t]^{n-2}}{(c+bt)^{n-1}}\Big)\mathrm{d}t\\
=&Z_1-Z_2. \qquad (3.26)
\end{aligned}
$$

Put $c+bt=x$, and $[2b+(1+k^2)c]/(1+k^2)=g$, then

$$
\lim_{\varepsilon\to0}\lim_{c\to0}Z_2=\frac{(-1)^{n-2}}{2\pi i}\frac{(1+k^2)^{n-1}}{(1+ki)^{2n-2}}\lim_{\varepsilon\to0}\lim_{c\to0}\int_c^{c+\frac{b\varepsilon}{\sqrt{1+k^2}}}\frac{(x-c)^{n-1}(x-g)^{n-2}}{x^{n-1}}\mathrm{d}x=0.
$$

While

$$
\begin{aligned}
\lim_{\varepsilon\to0}\lim_{c\to0}Z_1 =&\frac{(-1)^{n-1}(1+k^2)^{n-1}}{2\pi i(1+ki)^{2n-3}}\lim_{\varepsilon\to0}\lim_{c\to0}\int_c^{c+\frac{b\varepsilon}{\sqrt{1+k^2}}}\frac{c(x-c)^{n-2}(x-g)^{n-1}}{x^{n-1}(cb-cd+dx)}\mathrm{d}x\\
=&\frac{(-1)^{n-1}(1+k^2)^{n-1}}{2\pi i(1+ki)^{2n-3}}\lim_{\varepsilon\to0}\lim_{c\to0}(-g)^{n-1}c\sum_{j=1}^{n-1}\int_c^{c+\frac{b\varepsilon}{\sqrt{1+k^2}}}\frac{C_{n-2}^{n-1-j}(-c)^{j-1}}{x^j(cb-cd+dx)}\mathrm{d}x\\
=&-\frac{2^{n-2}}{2\pi i(1+ki)^{n-2}}\sum_{s=0}^{n-2}C_{n-2}^s(\frac{d}{2})^s\int_{-e^{2\varphi_1 i}}^1\frac{(1-u)^s}{u}\mathrm{d}u\\
=&\frac{-1}{2\pi i}\int_{-e^{2\varphi_1 i}}^1\frac{(1-e^{2\varphi_1 i}u)^{n-2}}{u}\mathrm{d}u.
\end{aligned}
$$

Let us consider (3.20) and (3.24). Set $c=1-\rho$, $p=\rho(1-\tilde{k}i)$, $q=-(1+\tilde{k}i)$, we have

$$
F_{21}(t)=\frac{2t^{n-2}[2-(1+\tilde{k}^2)t]^{n-1}}{(c+pt)^{n-1}}-\frac{2t^{n-2}[2-(1+\tilde{k}^2)t]^{n-2}}{(c+pt)^{n-1}},
$$

$$
F_{22}(t)=\frac{ct^{n-2}[2-(1+\tilde{k}^2)t]^{n-1}}{(c+qt)(c+pt)^{n-1}}-\frac{t^{n-2}[2-(1+\tilde{k}^2)t]^{n-1}}{(c+pt)^{n-1}},
$$

$$
\begin{aligned}
\tilde{Z} =&\frac{1}{2\pi i}\int_0^{\frac{\varepsilon}{R\sqrt{1+\tilde{k}^2}}}(F_{21}(t)+F_{22}(t))\mathrm{d}t\\
=&\frac{1}{2\pi i}\int_0^{\frac{\varepsilon}{R\sqrt{1+\tilde{k}^2}}}\Big(\frac{ct^{n-2}[2-(1+\tilde{k}^2)t]^{n-1}}{(c+pt)^{n-1}(c+qt)}-\frac{(1+\tilde{k}^2)t^{n-1}[2-(1+\tilde{k}^2)t]^{n-2}}{(c+pt)^{n-1}}\Big)\mathrm{d}t\\
=&\tilde{Z}_1-\tilde{Z}_2. \qquad (3.27)
\end{aligned}
$$

Put $c+pt=x$, and $[2p+(1+\tilde{k}^2)c]/(1+\tilde{k}^2)=\tilde{g}$, then

$$
\lim_{\varepsilon\to0}\lim_{c\to0}\tilde{Z}_2=\frac{(-1)^{n-2}}{2\pi i}\frac{(1+\tilde{k}^2)^{n-1}}{(1+\tilde{k}i)^{2n-2}}\lim_{\varepsilon\to0}\lim_{c\to0}\int_c^{c+\frac{p\varepsilon}{R\sqrt{1+\tilde{k}^2}}}\frac{(x-c)^{n-1}(x-\tilde{g})^{n-2}}{x^{n-1}}\mathrm{d}x=0.
$$

While

$$
\begin{aligned}
\lim_{\varepsilon\to0}\lim_{c\to0}\tilde{Z}_1 =&\frac{(-1)^{n-1}(1+\tilde{k}^2)^{n-1}}{2\pi i(1-\tilde{k}i)^{2n-3}}\lim_{\varepsilon\to0}\lim_{c\to0}\int_c^{c+\frac{p\varepsilon}{R\sqrt{1+\tilde{k}^2}}}\frac{c(x-c)^{n-2}(x-\tilde{g})^{n-1}}{x^{n-1}(cp-cq+qx)}\mathrm{d}x\\
=&-\frac{2^{n-2}}{2\pi i(1-\tilde{k}i)^{n-2}}\sum_{s=0}^{n-2}C_{n-2}^s(\frac{q}{2})^s\int_{-e^{-2\varphi_2 i}}^1\frac{(1-u)^s}{u}\mathrm{d}u\\
=&\frac{-1}{2\pi i}\int_{-e^{-2\varphi_2 i}}^1\frac{(1-e^{-2\varphi_2 i}u)^{n-2}}{u}\mathrm{d}u.
\end{aligned}
$$

By (3.3), (3.4), (3.6), (3.8), (3.15)–(3.27),

$$\begin{aligned}\tau(z) =&1-\lim_{\varepsilon\to 0}\lim_{\rho\to 1}(J_1+J_2+J_3)=1-\lim_{\varepsilon\to 0}\lim_{\rho\to 1}(Q_{n-1}-P_{n-1}+Z_1-\tilde{Z}_1)\\ =&1+\frac{2^{n-2}}{\pi i}\Big(\int_0^{\frac{\pi}{2}}-\int_0^{\psi_1}\Big)e^{-i(n-1)t}\cos^{n-2}t\sin t\mathrm{d}t-\\ &\frac{2^{n-2}}{\pi i}\Big(\int_0^{\frac{-\pi}{2}}+\int_{-\psi_2}^{0}\Big)e^{-i(n-1)t}\cos^{n-2}t\sin t\mathrm{d}t+\\ &\frac{1}{2\pi i}\int_{-e^{2\varphi_1 i}}^{1}\frac{(1-e^{2\varphi_1 i}u)^{n-2}}{u}\mathrm{d}u-\frac{1}{2\pi i}\int_{-e^{-2\varphi_2 i}}^{1}\frac{(1-e^{-2\varphi_2 i}u)^{n-2}}{u}\mathrm{d}u\\ =&\frac{1}{2}-\frac{2^{n-2}}{\pi i}\int_{\varphi_2-\pi/2}^{\pi/2-\varphi_1}e^{-i(n-1)t}\cos^{n-2}t\sin t\mathrm{d}t+\\ &\frac{1}{2\pi i}\int_{-e^{2\varphi_1 i}}^{1}\frac{(1-e^{2\varphi_1 i}u)^{n-2}}{u}\mathrm{d}u-\frac{1}{2\pi i}\int_{-e^{-2\varphi_2 i}}^{1}\frac{(1-e^{-2\varphi_2 i}u)^{n-2}}{u}\mathrm{d}u.\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.3 For $z\in\partial D$ a smooth point, the case is the same as in the complex sphere [1], so we only need to consider $z\in\partial D$ a non-smooth point. Without loss of generality, let $z=(1,0')$. By Lemma 3.4, we have

$$\text{p.v.}\sum_I\int_{(\zeta,\lambda)\in S_I\times\triangle_I}f(\zeta)\Omega(\zeta,z,\lambda)=\lim_{\varepsilon\to 0}\sum_I\int_{(\zeta,\lambda)\in\Sigma_\varepsilon\times\triangle_I}(f(\zeta)-f(z))\Omega(\zeta,z,\lambda)+\tau(z)f(z),$$

while

$$\begin{aligned}&\sum_I\int_{(\zeta,\lambda)\in\Sigma_\varepsilon\times\triangle_I}(f(\zeta)-f(z))\Omega(\zeta,z,\lambda)\\ &=\Big(\int_{\zeta\in\Sigma_{\varepsilon(1)},\lambda=1}+\int_{\zeta\in\Sigma_{\varepsilon(2)},\lambda=0}+\int_{(\zeta,\lambda)\in\Sigma_{\varepsilon(12)}\times\triangle_I}\Big)(f(\zeta)-f(z))\Omega(\zeta,z,\lambda)\\ &=J_1+J_2+J_3.\end{aligned}$$

For J_1, $|J_1|\lesssim\int_{\zeta\in\Sigma_{\varepsilon(1)}}\frac{\sigma(\zeta)}{|1-\overline{\zeta}_1|^{n-\alpha/2}}=O(1)$. Similarly, we have $|J_2|\lesssim O(1)$. For J_3,

$$\begin{aligned}|J_3|\lesssim&\int_{\zeta\in\Sigma_{\varepsilon(12)}}\Big(\frac{1}{|\Phi_1|^{n-1}}+\frac{1}{|\Phi_2|^{n-1}}\Big)\frac{|1-\overline{\zeta}_1|^\alpha|\Sigma_{j=2}^n(-1)^{j-1}\overline{a}\overline{\zeta_j}\mathrm{d}\overline{\zeta}_{[1j]}\wedge\mathrm{d}\zeta|}{|\Phi_1-\Phi_2|}\\ =&J_{31}+J_{32}.\\ J_{32}\lesssim&\int_{\zeta\in\Sigma_{\varepsilon(12)}}\frac{|1-\overline{\zeta}_1|^\alpha|1-x|^{n-1}|\mathrm{d}\overline{\zeta}_{[12]}\wedge\mathrm{d}\zeta_{[12]}\wedge\mathrm{d}x|}{|1-\overline{\zeta}_1|^n}\\ \lesssim&\int_{(a_1^2-a_2^2)/|a|^2}^{1-\frac{2\varepsilon}{\sqrt{1+k^2}}+\varepsilon^2}\frac{\mathrm{d}x}{(1-x)^{n-\alpha/2}}=O(1).\end{aligned}$$

Similarly, we have $J_{31}\lesssim O(1)$. Hence, $J_1+J_2+J_3$ is a convergent generalized integral. The proof is completed. □

4. The limit value of Cauchy type integral and Plemelj formula

We introduce a symbol $d(z,\partial D)=\min_{\zeta\in\partial D}|1-z\bar{\zeta}^t|$.

Theorem 4.1 *Suppose $f(\zeta)\in\mathcal{H}(\alpha,\partial D)$, $0<\alpha\le 1$, $\zeta\in\partial D$. Let $z\in D$ approach $z_0\in\partial D$, when*

$$|\zeta-z|/d(z,\partial D)<M,$$

M is a positive constant, then

$$\lim_{z\to z_0}\sum_I\int_{(\zeta,\lambda)\in\partial D}(f(\zeta)-f(z_0))\Omega(\zeta,z,\lambda)=\sum_I\int_{(\zeta,\lambda)\in\partial S_I\times\triangle_I}(f(\zeta)-f(z_0))\Omega(\zeta,z_0,\lambda). \quad (4.1)$$

Proof We only consider the case of z_0 the non-smooth point. Without loss of generality we can take $z_0=(1,0')$. Then the integral on the left hand side of (4.1) equals

$$\begin{aligned}T=&\Big(\int_{\zeta\in\sigma_1,\lambda=1}+\int_{\zeta\in\sigma_2,\lambda=0}+\int_{(\zeta,\lambda)\in\sigma_{12}\times\triangle_{12}}+\int_{\zeta\in\Sigma_1,\lambda=1}+\int_{\zeta\in\Sigma_2,\lambda=0}+\int_{(\zeta,\lambda)\in\Sigma_{12}\times\triangle_{12}}\Big)\\ &(f(\zeta)-f(z_0))(\Omega(\zeta,z,\lambda)-\Omega(\zeta,z_0,\lambda))\\ =&T_1+T_2+T_3+T_4+T_5+T_6.\end{aligned}$$

Taking similar operation in [1, Theorem 1.4.1], we have

$$|T_j|=O(\varepsilon),\quad j=1,2,4,5.$$

While

$$\begin{aligned}|T_3|\lesssim&\int_{\zeta\in\sigma_{12}}\Big(\frac{1}{|\Phi_1|^{n-1}}-\frac{1}{|\Phi_2|^{n-1}}\Big)\frac{|1-\overline{\zeta}_1|^\alpha|\Sigma_{j=2}^n(-1)^{j-1}\overline{a}\overline{\zeta_j}\mathrm{d}\overline{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta|}{|\Phi_1-\Phi_2|}\\ \lesssim&\int_{\zeta\in\sigma_{12}}\frac{\left(\begin{array}{c}|1-\overline{\zeta}_1|^{n-2}+|1-\overline{\zeta}_1|^{n-3}|1-\overline{\zeta}_1+\bar{a}(1-\zeta_1)|\\ +\cdots+|1-\overline{\zeta}_1+\bar{a}(1-\zeta_1)|^{n-3}\end{array}\right)|\Sigma_{j=2}^n(-1)^{j-1}\overline{a}\overline{\zeta_j}\mathrm{d}\overline{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta|}{|1-\overline{\zeta}_1|^{n-\alpha}|1-\overline{\zeta}_1+\bar{a}(1-\zeta_1)|^{n-1}}\\ \lesssim&\int_{\zeta\in\sigma_{12}}\frac{(K_1(1-x)^{n-2}+\cdots+K_{n-2}(1-x)^{n-2})|\Sigma_{j=2}^n(-1)^{j-1}\overline{a}\overline{\zeta_j}\mathrm{d}\overline{\zeta}_{[1,j]}\wedge\mathrm{d}\zeta|}{(1-x)^{2n-2-\alpha}}\\ \lesssim&\int_{(a_1^2-a_2^2)/|a|^2}^{1-\frac{2\varepsilon}{\sqrt{1+k^2}}+\varepsilon^2}\frac{\mathrm{d}x}{(1-x)^{1-\alpha}}=O(1),\end{aligned}$$

where K_j is a positive constant ($j=1,\dots,n-2$). While $|T_6|=O(1)$. The proof is completed. □

Proof of Theorem 2.4 (Plemelj Formula) For

$$\begin{aligned}&\sum_I\int_{(\zeta,\lambda)\in S_I\times\Delta_I}f(\zeta)\Omega(\zeta,z,\lambda)\\ &=\sum_I\int_{(\zeta,\lambda)\in S_I\times\Delta_I}(f(\zeta)-f(z_0))\Omega(\zeta,z,\lambda)+f(z_0)\sum_I\int_{(\zeta,\lambda)\in S_I\times\Delta_{01}}\Omega(\zeta,z,\lambda)\\ &=J_1+J_2,\end{aligned}$$

by Theorem 2.3, for $z\in D$, $z_0\in\partial D$, then

$$\lim_{z\to z_0}J_1=\sum_I\int_{(\zeta,\lambda)\in S_I\times\Delta_I}(f(\zeta)-f(z_0))\Omega(\zeta,z_0,\lambda)$$

$$=\text{p.v.} \int_{(\zeta,\lambda)\in\triangle_I} f(\zeta)\Omega(\zeta, z_0, \lambda) - \tau(z_0)f(z_0).$$

For $\sum_I \int_{(\zeta,\lambda)\in S_I\times\triangle_I} \Omega(\zeta, z, \lambda) = 1$, we have $\lim_{z\to z_0} J_2 = f(z_0)$. The proof is completed. □

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