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Global Stability of the Deterministic and Stochastic SIS Epidemic Models with Vaccination

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Abstract We study the stability of endemic equilibriums of the deterministic and stochastic SIS epidemic models with vaccination. The deterministic SIS epidemic model with vaccination was proposed by Li and Ma (2004), for which some sufficient conditions for the global stability of the endemic equilibrium were given in some earlier works. In this paper, we first prove by Lyapunov function method that the endemic equilibrium of the deterministic model is globally asymptotically stable whenever the basic reproduction number is larger than one. For the stochastic version, we obtain some sufficient conditions for the global stability of the endemic equilibrium by constructing a class of different Lyapunov functions.

Keywords SIS epidemic model; vaccination; global stability

MR(2020) Subject Classification 34D05; 34D23; 60H10; 93E03; 93E15

1. Introduction

Studies of epidemic models with vaccination have become an important area in the mathematical theory of epidemiology, and they have largely been inspired by the works [1–5]. The vaccination enables the vaccinated to acquire a permanent or temporary immunity. When the immunity is temporary, the immunity can be lost after a period of time. In [6], Li and Ma proposed the following SIS model with vaccination:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = (1-q)A - \beta SI - (\mu+p)S + \gamma I + \varepsilon V, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - (\mu+\gamma+\alpha)I, \\ \frac{\mathrm{d}V}{\mathrm{d}t} = qA + pS - (\mu+\varepsilon)V. \end{cases} \tag{1.1}$$

Here $S(t)$ denotes the number of members who are susceptible to an infection at time t. $I(t)$ denotes the number of members who are infective at time t. $V(t)$ denotes the number of members who are immune to an infection at time t as a result of vaccination. A stands for a constant input of new members into the population per unit time, q denotes the fraction of vaccinated for new born; μ denotes the natural death rate of S, I, V compartments; β is the transmission coefficient between compartments S and I; p represents the proportional coefficient of vaccinated for the

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susceptible individuals; γ is the recovery rate of I; ε denotes the rate of losing their immunity for vaccinated individuals; α is the disease-caused mortality rate. The parameters $q, A, \beta, \mu, p, \gamma, \varepsilon$ are positive with $0 < q < 1$.

They obtained the threshold value $R_0 = \frac{A\beta[\varepsilon+(1-q)A]}{\mu(\mu+\gamma+\alpha)(\mu+\varepsilon+p)}$ of system (1.1), and proved that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable in the invariant set Ω, where $\Omega = \{(S, I, V) : S > 0, I \geq 0, V \geq 0, S + I + V \leq \frac{A}{\mu}\}$, while if $R_0 > 1$, there exists a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$, where

$$(S^*, I^*, V^*) = \big(\frac{\mu+\gamma+\alpha}{\beta}, \frac{\mu(\mu+\gamma+\alpha)(\mu+\varepsilon+p)}{\beta(\mu+\alpha)(\mu+\varepsilon)}(R_0-1), \frac{q\beta A + p(\mu+\gamma+\alpha)}{\beta(\mu+\varepsilon)}\big),$$

which is globally asymptotically stable in Ω under certain sufficient condition. Recently, a new sufficient condition for the global stability of E^* is given in [7]. Some related works can be referred to [8, 9]. In this paper, we prove the following

Theorem 1.1 *Let $R_0 > 1$. Then the endemic equilibrium E^* of system (1.1) is globally asymptotically stable.*

However, the evolving process of the epidemic disease over the time is naturally subject to random and environmental perturbations. To understand the impacts due to such randomness and fluctuations, it is convenient and effective to model the disease spreading through a stochastic differential equation (SDE) approach [11–18]. In [15], Zhao, Jiang and O'Regan take into account the effect of randomly fluctuating environment in model (1.1) by assuming that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter β: $\beta \to \beta + \sigma\dot{B}$, where B is standard Brownian motion and σ^2 represents its intensity, and obtained the following SDE model:

$$\begin{cases} \mathrm{d}S = [(1-q)A - \beta SI - (\mu+p)S + \gamma I + \varepsilon V]\mathrm{d}t - \sigma SI\mathrm{d}B, \\ \mathrm{d}I = [\beta SI - (\mu+\gamma+\alpha)I]\mathrm{d}t + \sigma SI\mathrm{d}B, \\ \mathrm{d}V = [qA + pS - (\mu+\varepsilon)V]\mathrm{d}t + \sigma_3\mathrm{d}B_3. \end{cases} \tag{1.2}$$

They proved that when the noise is large, the infective decays exponentially to zero regardless of the magnitude of R_0. When the noise is small, some sufficient conditions on extinction and persistence are established. Some further works on system (1.2) can be referred to [19–23]. In [16], Zhao and Jiang considered the following stochastic system:

$$\begin{cases} \mathrm{d}S = [(1-q)A - \beta SI - (\mu+p)S + \gamma I + \varepsilon V]\mathrm{d}t - \sigma_1 S\mathrm{d}B_1, \\ \mathrm{d}I = [\beta SI - (\mu+\gamma+\alpha)I]\mathrm{d}t + \sigma_2 I\mathrm{d}B_2, \\ \mathrm{d}V = [qA + pS - (\mu+\varepsilon)V]\mathrm{d}t + \sigma_3\mathrm{d}B_3, \end{cases} \tag{1.3}$$

where B_i $(i = 1, 2, 3)$ are independent Brownian motions and σ_i $(i = 1, 2, 3)$ are their intensities. When the perturbations and the disease-related death rate α are small, they showed that there is a stationary distribution and it is ergodic when $R_0 > 1$, whereas the asymptotic behavior of the solution around the disease-free equilibrium prevails when $R_0 \leq 1$. In [24], the sufficient conditions for extinction and persistence in mean are obtained, and a threshold of the stochastic model which determines the outcome of the disease was established when the white noises are small.

From the above we know that system (1.1) admits a unique endemic equilibrium E^* when $R_0 > 1$. Furthermore, we assume stochastic perturbations are of white noise type, which are directly proportional to distances S, I, V from values of S^*, I^*, V^*, influence on the S, I, V respectively. Thus system (1.1) results in

$$\begin{cases} \mathrm{d}S = [(1-q)A - \beta SI - (\mu+p)S + \gamma I + \varepsilon V]\mathrm{d}t + \sigma_1(S-S^*)\mathrm{d}B_1(t), \\ \mathrm{d}I = [\beta SI - (\mu+\gamma+\alpha)I]\mathrm{d}t + \sigma_2(I-I^*)\mathrm{d}B_2(t), \\ \mathrm{d}V = [qA + pS - (\mu+\varepsilon)V]\mathrm{d}t + \sigma_3(V-V^*)\mathrm{d}B_3(t), \end{cases} \tag{1.4}$$

where B_i $(i = 1, 2, 3)$ are independent standard Brownian motions and σ_i $(i = 1, 2, 3)$ represent their intensities. Obviously, stochastic system (1.4) has the same equilibrium points as system (1.1). In this paper, we will investigate asymptotic stability of the equilibrium E^* of stochastic system (1.4). We obtain

Theorem 1.2 Let $R_0 > 1$. Then the equilibrium E^* of system (1.4) is stochastically asymptotically stable in the large if the following conditions are satisfied:

$$\begin{cases} \sigma_1^2 < \frac{2\beta\eta^* I^*}{\eta^*+1}, \\ \sigma_2^2 < (\mu+\alpha)(1 + \frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{\beta I^*})^{-1}, \\ \sigma_3^2 < \frac{\varepsilon^2}{4\eta^*(\mu+\alpha)(\eta^*+1)M^*}, \end{cases}$$

where $M^* = \frac{1}{p^2}[2(\mu+\varepsilon)(\mu+p) - p\varepsilon]$ and $\eta^* = \frac{p^2\varepsilon^2}{4\mu(\mu+\alpha)(\mu+\varepsilon)(\mu+p+\varepsilon)}$.

This paper is organized as follows. In Section 2, we will prove Theorem 1.1 by using Lyapunov function method. In Section 3, the proof of Theorem 1.2 will be given by constructing a class of different Lyapunov functions.

2. Proof of Theorem 1.1

In the section, we will give the proof of Theorem 1.1 by using Lyapunov function method.

Proof Let us consider a nonnegative solution (S, I, V) of system (1.1). Denote

$$x = S - S^*,\ y = V - V^*,\ z = N - N^*,\ N = S + I + V,$$

where $N^* = S^* + I^* + V^*$. Adding the three equations in (1.1) yields

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - (\mu+\alpha)N + \alpha(S+V). \tag{2.1}$$

Thus, (x, I, y, z) satisfies

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -\beta Ix - (\mu+p)x - (\mu+\alpha)(I-I^*) + \varepsilon y, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta Ix, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = px - (\mu+\varepsilon)y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} = \alpha(x+y) - (\mu+\alpha)z. \end{cases} \tag{2.2}$$

Define the functions V_i $(i = 1, 2, 3)$ along the solution (x, I, y, z) of system (2.2) by

$$V_1 = \frac{1}{2}x^2 + \frac{\mu+\alpha}{\beta}(I - I^* - I^* \ln \frac{I}{I^*}),\ V_2 = \frac{1}{2}y^2,\ V_3 = \frac{1}{2}z^2.$$

Consider the Lyapunov function $V_{X,Y}(t) = V_1(t) + XV_2(t) + YV_3(t)$, where the positive constants X and Y will be determined later. Next, we calculate the derivatives $\frac{\mathrm{d}V_i}{\mathrm{d}t}$ along the solution of system (2.2). By the identities

$$\begin{cases} S^* = \frac{\mu+\gamma+\alpha}{\beta},\ qA + pS^* - (\mu+\varepsilon)V^* = 0, \\ (1-q)A - \beta S^* I^* - (\mu+p)S^* + \gamma I^* + \varepsilon V^* = 0, \end{cases} \tag{2.3}$$

we have

$$\begin{cases} \frac{\mathrm{d}V_1}{\mathrm{d}t} = -\beta I x^2 - (\mu+p)x^2 + \varepsilon xy, \\ \frac{\mathrm{d}V_2}{\mathrm{d}t} = pxy - (\mu+\varepsilon)y^2, \\ \frac{\mathrm{d}V_3}{\mathrm{d}t} = -(\mu+\alpha)z^2 + \alpha xz + \alpha yz. \end{cases}$$

Then for any positive constants X and Y, we have

$$\begin{aligned} \frac{\mathrm{d}V_{X,Y}}{\mathrm{d}t} \leq & -(\mu+p)x^2 + (pX+\varepsilon)xy - X(\mu+\varepsilon)y^2 - \\ & Y(\mu+\alpha)z^2 + Y\alpha xz + Y\alpha yz \\ =: & -\mathbf{x}\mathcal{M}(X,Y)\mathbf{x}^{\mathrm{T}}, \end{aligned} \tag{2.4}$$

where $\mathbf{x} = (x, y, z)$, T denotes the transpose, and the matrix $\mathcal{M}(X,Y)$ is defined by

$$\mathcal{M}(X,Y) = \begin{pmatrix} \mu+p & -\frac{pX+\varepsilon}{2} & -\frac{Y\alpha}{2} \\ -\frac{pX+\varepsilon}{2} & X(\mu+\varepsilon) & -\frac{Y\alpha}{2} \\ -\frac{Y\alpha}{2} & -\frac{Y\alpha}{2} & Y(\mu+\alpha) \end{pmatrix}.$$

It is easy to see that by some elementary transformations, $\mathcal{M}(X,Y)$ can be transferred into

$$\widetilde{\mathcal{M}}(X,Y) = \begin{pmatrix} \mu+p & -\frac{pX+\varepsilon}{2} & -\frac{Y\alpha}{2} \\ 0 & X(\mu+\varepsilon) - \frac{(pX+\varepsilon)^2}{4(\mu+p)} & -\frac{Y\alpha}{2} - \frac{Y\alpha(pX+\varepsilon)}{4(\mu+p)} \\ 0 & -\frac{Y\alpha}{2} - \frac{Y\alpha(pX+\varepsilon)}{4(\mu+p)} & Y(\mu+\alpha) - \frac{(Y\alpha)^2}{4(\mu+p)} \end{pmatrix}.$$

Clearly, $\mathcal{M}(X,Y)$ is positive definite if the following conditions are satisfied:

$$\begin{cases} \Delta_1(X) =: X(\mu+\varepsilon) - \frac{(pX+\varepsilon)^2}{4(\mu+p)} > 0, \\ \Delta_2(X,Y) =: \Delta_1(X)\cdot[Y(\mu+\alpha) - \frac{(Y\alpha)^2}{4(\mu+p)}] - [\frac{Y\alpha}{2} + \frac{Y\alpha(pX+\varepsilon)}{4(\mu+p)}]^2 > 0. \end{cases}$$

Note that

$$\begin{aligned} \Delta_2(X,Y) &= Y\big\{(\mu+\alpha)\Delta_1(X) - Y\big[\frac{X(\mu+\varepsilon)\alpha^2}{4(\mu+p)} + \frac{\alpha^2}{4} + \frac{\alpha^2(pX+\varepsilon)}{4(\mu+p)}\big]\big\} \\ &=: Y[(\mu+\alpha)\Delta_1(X) - Y\Delta_3(X)], \end{aligned}$$

and

$$\Delta_1(X) = -\frac{1}{4(\mu+p)}\big\{p^2X^2 - 2[2(\mu+\varepsilon)(\mu+p) - p\varepsilon]X + \varepsilon^2\big\}.$$

Since $2(\mu+\varepsilon)(\mu+p) - p\varepsilon > 0$ and $[2(\mu+\varepsilon)(\mu+p) - p\varepsilon]^2 - p^2\varepsilon^2 > 0$, the equation $\Delta_1(X) = 0$ has two positive roots X_1 and X_2 with $X_1 < X_2$. Taking $X^* = \frac{1}{2}(X_1 + X_2)$ yields $\Delta_1(X^*) > 0$. By choosing $Y = Y^* = \frac{(\mu+\alpha)\Delta_1(X^*)}{2\Delta_3(X^*)}$, one has $\Delta_2(X^*, Y^*) > 0$ and hence, the matrix $\mathcal{M}(X^*, Y^*)$ is positive definite. It follows from (2.4) that $\frac{\mathrm{d}V_{X^*,Y^*}}{\mathrm{d}t}$ is negative-definite. On the other hand,

it is clear that $\frac{\mathrm{d}V_{X^*,Y^*}}{\mathrm{d}t} = 0$ if and only if $(S, I, V) = (S^*, I^*, V^*)$. According to the LaSalle's invariant principle [10], E^* is globally asymptotically stable. The proof is completed. □

3. Proof of Theorem 1.2

In the section, we study the stochastically asymptotic stability of E^* of system (1.4). To this end, we will construct a class of different Lyapunov functions to finish the proof of Theorem 1.2.

Proof Let $x(t) = S(t) - S^*, y(t) = I(t) - I^*$ and $z(t) = V(t) - V^*$. Noticing (2.3), we have

$$\begin{cases} \mathrm{d}x = [-\beta xy - \beta I^* x - (\mu+\alpha)y - (\mu+p)x + \varepsilon z]\mathrm{d}t + \sigma_1 x \mathrm{d}B_1, \\ \mathrm{d}y = (\beta xy + \beta I^* x)\mathrm{d}t + \sigma_2 y \mathrm{d}B_2, \\ \mathrm{d}z = [px - (\mu+\varepsilon)z]\mathrm{d}t + \sigma_3 z \mathrm{d}B_3. \end{cases} \tag{3.1}$$

To prove the theorem, it suffices to show that the zero solution of system (3.1) is stochastically asymptotically stable in the large. Let $\mathbf{x} = (x, y, z)$. Define the Lyapunov function

$$V(\mathbf{x}) = \sum_{i=1}^{4} a_i V_i(\mathbf{x}),$$

where a_i are positive constants to be chosen later, and V_i $(i = 1, 2, 3, 4)$ are defined as follows:

$$V_1(x) = \frac{1}{2}x^2,\ V_2(y) = \frac{1}{2}y^2,\ V_3(z) = \frac{1}{2}z^2,\ V_4(x, y) = \frac{1}{2}(x+y)^2.$$

Let L be the differential operator associated with (3.1). By Itô formula [25, Theorem 6.2 of Chapter 1], we have

$$\begin{cases} LV_1 = -\beta I^* x^2 - (\mu+\alpha)xy - (\mu+p)x^2 + \varepsilon xz - \beta x^2 y + \frac{1}{2}\sigma_1^2 x^2, \\ LV_2 = \beta I^* xy + \beta xy^2 + \frac{1}{2}\sigma_2^2 y^2, \\ LV_3 = pxz - (\mu+\varepsilon)z^2 + \frac{1}{2}\sigma_3^2 z^2, \\ LV_4 = -(\mu+p)x^2 - (\mu+\alpha)y^2 - (p+\alpha+2\mu)xy + \varepsilon xz + \varepsilon yz + \frac{1}{2}\sigma_1^2 x^2 + \frac{1}{2}\sigma_2^2 y^2. \end{cases}$$

Due to $2(\mu+\varepsilon)(\mu+p) - p\varepsilon > 0$, taking $M^* = \frac{1}{p^2}[2(\mu+\varepsilon)(\mu+p) - p\varepsilon]$ yields

$$\begin{aligned} L(V_1 + \frac{\mu+\alpha}{\beta I^*}V_2 + M^* V_3) =& -\beta I^* x^2 - [(\mu+p)x^2 - (M^* p + \varepsilon)xz + M^*(\mu+\varepsilon)z^2] - \\ & \beta x^2 y + \frac{\mu+\alpha}{I^*}xy^2 + \frac{1}{2}\sigma_1^2 x^2 + \frac{\mu+\alpha}{2\beta I^*}\sigma_2^2 y^2 + \frac{M^*}{2}\sigma_3^2 z^2 \\ \le& -\beta I^* x^2 - \frac{f(M^*)}{4(\mu+p)}z^2 - \beta x^2 y + \frac{\mu+\alpha}{I^*}xy^2 + \\ & \frac{1}{2}\sigma_1^2 x^2 + \frac{\mu+\alpha}{2\beta I^*}\sigma_2^2 y^2 + \frac{M^*}{2}\sigma_3^2 z^2, \end{aligned} \tag{3.2}$$

where $f(M^*) = -\{p^2(M^*)^2 - 2[2(\mu+\varepsilon)(\mu+p) - p\varepsilon]M^* + \varepsilon^2\} > 0$. Similarly, we have

$$\begin{aligned} & L(V_4 + \frac{p+\alpha+2\mu}{\beta I^*}V_2 + M^* V_3) \\ &= -[(\mu+p)x^2 - (M^* p + \varepsilon)xz + M^*(\mu+\varepsilon)z^2] - (\mu+\alpha)y^2 + \varepsilon yz + \\ & \frac{p+\alpha+2\mu}{I^*}xy^2 + \frac{1}{2}\sigma_1^2 x^2 + \frac{1}{2}\big(1 + \frac{p+\alpha+2\mu}{\beta I^*}\big)\sigma_2^2 y^2 + \frac{M^*}{2}\sigma_3^2 z^2 \end{aligned}$$

$$\leq -\frac{1}{4(\mu+p)}f(M^*)z^2-(\mu+\alpha)y^2+\varepsilon yz+ \frac{p+\alpha+2\mu}{I^*}xy^2+\frac{1}{2}\sigma_1^2x^2+\frac{1}{2}\big(1+\frac{p+\alpha+2\mu}{\beta I^*}\big)\sigma_2^2y^2+\frac{M^*}{2}\sigma_3^2z^2. \tag{3.3}$$

Note that $\eta^*=\frac{2\varepsilon^2(\mu+p)}{(\mu+\alpha)f(M^*)}$. It follows from (3.2) and (3.3) that

$$\begin{aligned}
&L(V_4+M^*V_3+\frac{p+\alpha+2\mu}{\beta I^*}V_2+\eta^*(V_1+M^*V_3+\frac{\mu+\alpha}{\beta I^*}V_2))\\
&\leq -((\mu+\alpha)y^2-\varepsilon yz+\frac{\eta^*f(M^*)}{4(\mu+p)}z^2)-\frac{f(M^*)}{4(\mu+p)}z^2-\beta\eta^*I^*x^2+\\
&\quad\frac{p+\alpha+2\mu}{I^*}xy^2+\frac{1}{2}\sigma_1^2x^2+\frac{1}{2}(1+\frac{p+\alpha+2\mu}{\beta I^*})\sigma_2^2y^2+\frac{M^*}{2}\sigma_3^2z^2-\\
&\quad\beta\eta^*x^2y+\eta^*(\frac{\mu+\alpha}{I^*}xy^2+\frac{1}{2}\sigma_1^2x^2+\frac{M^*}{2}\sigma_3^2z^2+\frac{\mu+\alpha}{2\beta I^*}\sigma_2^2y^2)\\
&=-\frac{\mu+\alpha}{2}(y-\frac{\varepsilon}{\mu+\alpha}z)^2-\frac{\mu+\alpha}{2}y^2-\frac{f(M^*)}{4(\mu+p)}z^2-\beta\eta^*I^*x^2-\\
&\quad\beta\eta^*x^2y+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{I^*}xy^2+\\
&\quad\frac{\eta^*+1}{2}\sigma_1^2x^2+\frac{1}{2}(1+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{\beta I^*})\sigma_2^2y^2+\frac{\eta^*+1}{2}M^*\sigma_3^2z^2\\
&\leq -\beta\eta^*I^*x^2-\frac{\mu+\alpha}{2}y^2-\frac{f(M^*)}{4(\mu+p)}z^2-\beta\eta^*x^2y+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{I^*}xy^2+\\
&\quad\frac{\eta^*+1}{2}\sigma_1^2x^2+\frac{1}{2}(1+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{\beta I^*})\sigma_2^2y^2+\frac{\eta^*+1}{2}M^*\sigma_3^2z^2.
\end{aligned}$$

By choosing a_i $(i=1,2,3,4)$ as follows:

$$a_1=\eta^*,\ a_2=\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{\beta I^*},\ a_3=(\eta^*+1)M^*,\ a_4=1,$$

we have

$$LV\leq -(Ax^2+By^2+Cz^2)-\beta\eta^*x^2y+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{I^*}xy^2,$$

where

$$\begin{cases}
A=\beta\eta^*I^*-\frac{\eta^*+1}{2}\sigma_1^2>0,\\
B=\frac{\mu+\alpha}{2}-\frac{1}{2}(1+\frac{p+\alpha+2\mu+\eta^*(\mu+\alpha)}{\beta I^*})\sigma_2^2>0,\\
C=\frac{f(M^*)}{4(\mu+p)}-\frac{\eta^*+1}{2}M^*\sigma_3^2>0.
\end{cases}$$

Let $\lambda=\min\{A,B,C\}$. Then $LV\leq-\lambda\|\mathbf{x}(t)\|^2+o(\|\mathbf{x}(t)\|^2)$, where $\|\mathbf{x}\|=\sqrt{x^2+y^2+z^2}$, and $o(\|\mathbf{x}(t)\|^2)$ is an infinitesimal of higher order of $\|\mathbf{x}(t)\|^2$ for $t\geq0$. Hence LV is negative-definite in a sufficiently small neighborhood of $\mathbf{x}=0$ for $t\geq0$. Besides, it is clear that $V(\mathbf{x})$ is positive-definite decrescent. According to [25, Theorem 2.4 of Chapter 4], we therefore conclude that the zero solution of system (3.1) is stochastically asymptotically stable in the large. The proof is completed. □

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