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Enumeration of Protected Nodes in Motzkin Trees

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Abstract In this paper, we enumerate the set of Motzkin trees with n edges according to the number of leaves, the number of vertices adjacent to a leaf, the number of protected nodes, the number of (protected) branch nodes, and the number of (protected) lonely nodes. Explicit formulae as well as generating functions are obtained. We also find that, as n goes to infinity, the proportion of protected branch nodes and protected lonely nodes among all vertices of Motzkin trees with n edges approaches $4/27$ and $2/9$.

Keywords Motzkin trees; protected nodes; Motzkin number; Bivariate generating function; Lagrange inversion

MR(2020) Subject Classification 05A05; 05A10; 05A15; 05A16; 05C05

1. Introduction

A Motzkin tree is an ordered tree in which the degree of each vertex is 0, 1, or 2 with 0 indicating a leaf [1, 2]. It is also referred to as a $\{0, 1, 2\}$-tree, or unary-binary tree [3]. An ordered tree is defined recursively as having a root and an ordered set of subtrees [4–6]. For each vertex v of an ordered tree, the number of subtrees of v is called the degree of v. A vertex of degree zero is called a leaf, a vertex of degree 1 is called a lonely node, and a vertex of degree at least 2 is called a branch node.

Let $\mathcal{M}$ denote the family of Motzkin trees. Then every tree in $\mathcal{M}$ is one of the forms shown in Figure 1.

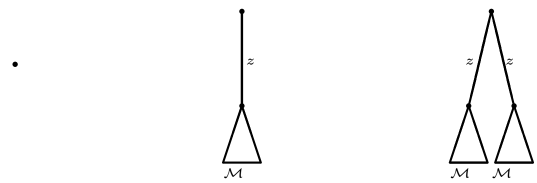

Figure 1 Decomposition of the Motzkin trees

Let m_n be the number of Motzkin trees in $\mathcal{M}$ with n edges, and let $M(z)$ be the corresponding generating function $M(z) = \sum_{n=0}^{\infty} m_n z^n$. The symbolic equation in Figure 1 can be translated

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into the following recurrence relation for the generating function

$$M(z) = 1 + zM(z) + z^2M(z)^2. \tag{1.1}$$

From this, an explicit form for $M(z)$ is easily obtained

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}. \tag{1.2}$$

The total number v_n of vertices in all Motzkin trees with n edges has the generating function

$$V(z) = \sum_{n=0}^{\infty} v_n z^n = \frac{d}{dz}(zM(z)) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2\sqrt{1 - 2z - 3z^2}}, \tag{1.3}$$

since any tree with n edges has $n+1$ vertices. Using the uplift principle [7], Cheon and Shapiro [1] obtained that the total number l_n of leaves in Motzkin trees with n edges is determined by the following generating function

$$L(z) = \sum_{n=0}^{\infty} l_n z^n = \frac{V(z)}{M(z)} = \frac{1}{\sqrt{1 - 2z - 3z^2}}. \tag{1.4}$$

Note that the numbers $\{l_0, l_1, \ldots, l_n, \ldots\}$ are the central trinomial coefficients [8]. Furthermore, the following asymptotic formulae were derived in [7]

$$\frac{l_{n+1}}{l_n} \sim 3, \quad \frac{v_n}{l_n} \sim 3. \tag{1.5}$$

A protected node in an ordered tree is a vertex that is not a leaf and is not adjacent to a leaf. Cheon and Shapiro [1] started the study of protected nodes in ordered trees. After this pioneering paper, many scholars have studied protected nodes in various classes of ordered trees. See Bóna [9] (binary search trees), Devroye and Janson [10] (random trees), Mahmoud and Ward [11] (binary search trees), Du and Prodinger [12] (digital search trees), Heuberger and Prodinger [13] (plane trees), Mansour [14] (k-ary trees). In particular, Cheon and Shapiro [1] showed that the proportion of protected nodes in Motzkin trees approaches 10/27. But they did not give the explicit formula of the number of Motzkin trees with n edges and k protected nodes. The main purpose of the present paper is to study the number of Motzkin trees with n edges and k protected nodes. We know that the number of Motzkin trees with n edges is the n-th Motzkin number m_n. Thus, for all arrays showed in this paper, the row sums are the Motzkin numbers.

It is easy to find that the vertices among Motzkin trees are partitioned into three classes: vertices adjacent to a leaf, leaves, and protected nodes. From Cheon and Shapiro [1], we can deduce that the asymptotic proportions of these three classes are $\frac{8}{27}$, $\frac{9}{27}$, $\frac{10}{27}$, respectively. Furthermore, the protected nodes are partitioned into two subclasses: branch protected nodes and lonely protected nodes. We will show that the asymptotic proportions of these two subclasses are $\frac{4}{27}$, $\frac{6}{27}$, respectively. On the other hand, the vertices among Motzkin trees may also be partitioned into leaves, branch nodes, and lonely nodes. We will show that these three classes are asymptotic equinumerous, i.e., all of the asymptotically proportions of these three classes are $\frac{1}{3}$.

This paper is organized as follows. In Section 2, we will study the number of leaves and the number of vertices adjacent to a leaf in all Motzkin trees of size n. In Section 3, we will give the formula for the number of Motzkin trees with n edges and k protected nodes. In Section 4, we will enumerate the Motzkin trees according to the number of edges and branch (resp., protected branch) nodes. In Section 5, we will enumerate the Motzkin trees according to the number of edges and lonely (resp., protected lonely) nodes.

2. Leaves and vertices adjacent to a leaf

Let $L(t, z)$ be the generating function for the number of Motzkin trees according to the number of leaves, where z marks edges and t marks leaves.

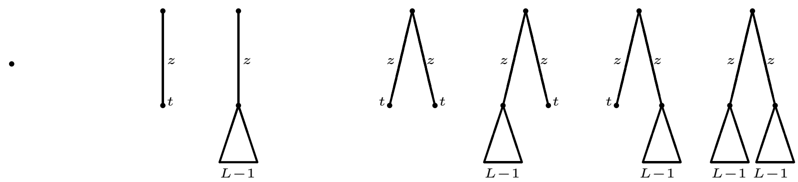

Figure 2 Decomposition of the Motzkin trees with leaves marked

Making use of the symbolic method [15] and the decomposition of Motzkin trees illustrated in Figure 2, we get an equation for $L(t, z)$ as follows

$$L = 1 + z(t + L - 1) + z^2(t + L - 1)^2, \tag{2.1}$$

which is equivalent to

$$z(t + L - 1) = z(t + z(t + L - 1) + z^2(t + L - 1)^2). \tag{2.2}$$

We can now prove the following result.

In the following, we will study the number of Motzkin trees of size n containing k leaves. If $n = 0$, then $k = 0$. This is the trivial tree. Thus we discuss $n, k \geq 1$.

Theorem 2.1 *For all $n, k \geq 1$, the number of Motzkin trees of size n containing k leaves is given by*

$$l_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{k-1}.$$

Proof Let $y(t, z) = z(t + L - 1)$. Then from (2.2), we can obtain that $y = z(t + y + y^2)$. For $n \geq 1$, using Lagrange inversion formula [16], we obtain that

$$\begin{aligned}[z^n]L(t,z) =& [z^{n+1}]y = \frac{1}{n+1}[x^n](t + x + x^2)^{n+1} \\ =& \frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\binom{n+1-k}{k-1}t^k.\end{aligned}$$

Thus, we have

$$[t^k z^n]L(t,z) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{k-1}. \quad \square$$

Hence, we can get the following three formulas:

$$m_n = \sum_{k=0}^{n} l_{n,k} = \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{k-1},$$
$$v_n = (n+1)\sum_{k=0}^{n} l_{n,k} = \sum_{k=0}^{n} \binom{n+1}{k}\binom{n+1-k}{k-1},$$
$$l_n = \sum_{k=0}^{n} k l_{n,k} = \sum_{k=0}^{n} \frac{k}{n+1}\binom{n+1}{k}\binom{n+1-k}{k-1}.$$

Let $A(t,z)$ be the generating function for the number of Motzkin trees according to the number of vertices adjacent to a leaf, where z marks edges and t marks vertices adjacent to a leaf. Thus, we can obtain that $A(t,z)$ satisfies the following equation

$$A = 1 + tz + z(A-1) + tz^2 + 2tz^2(A-1) + z^2(A-1)^2. \tag{2.3}$$

We now have the following result.

Theorem 2.2 *For all $n, k \geq 1$, the number of Motzkin trees of size n containing k vertices adjacent to a leaf is given by*

$$\sum_{m=1}^{n}\sum_{i=0}^{m} \frac{2^{2k+i-n-1}}{m}\binom{m}{i}\binom{i}{n-m}\binom{n-m}{k-1}\binom{k-1}{n-i-k}.$$

Proof Let $y(t,z) = A - 1$. Then, from (2.3), we can obtain that

$$y = z(t + y + tz + 2tzy + zy^2).$$

Using Lagrange inversion formula, we obtain

$$\begin{aligned}[z^m]A(t,z) =& [z^m]y(t,z) = \frac{1}{m}[x^{m-1}](t + x + tz + 2tzx + zx^2)^m \\ =& \frac{1}{m}\sum_{i=0}^{m}\sum_{j=0}^{i}\sum_{p=0}^{j}\binom{m}{i}\binom{i}{j}\binom{j}{p}\binom{p}{m+j-i-p-1}2^{2p-m-j+i+1}t^{p+1}z^j.\end{aligned}$$

It follows that

$$y(t,z) = \sum_{m=1}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{i}\sum_{p=0}^{j}\binom{m}{i}\binom{i}{j}\binom{j}{p}\binom{p}{m+j-i-p-1}\frac{1}{m}2^{2p-m-j+i+1}t^{p+1}z^{j+m},$$

and hence,

$$[t^k z^n]A(t,z) = [t^k z^n]y(t,z) = \sum_{m=1}^{n}\sum_{i=0}^{m}\binom{m}{i}\binom{i}{n-m}\binom{n-m}{k-1}\binom{k-1}{n-i-k}\frac{1}{m}2^{2k+i-n-1}. \ \square$$

The first terms of the array $(A_{n,k})$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 7 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 15 & 4 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 27 & 22 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 43 & 74 & 8 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.3 *For $n \geq 2$, the number of vertices adjacent to a leaf in all Motzkin trees of size n is given by $l_n - l_{n-2}$. Furthermore, the proportion of vertices adjacent to a leaf among all vertices of Motzkin trees with n edges is asymptotically $\frac{8}{27}$.*

Proof Differentiating both sides of equation (2.3) with respect to t and taking into account that $A(1, z) = M(z)$ leads to

$$\frac{\partial A}{\partial t}\Big|_{t=1} = \frac{1 - z^2 - \sqrt{1 - 2z - 3z^2}}{\sqrt{1 - 2z - 3z^2}} = L(z) - z^2 L(z) - 1. \tag{2.4}$$

Hence,

$$[z^n]\frac{\partial A}{\partial t}\Big|_{t=1} = l_n - l_{n-2}.$$

From (1.5), it follows that $l_n - l_{n-2} \sim \frac{8}{9} l_n$. Thus, the proportion of vertices adjacent to a leaf among all vertices of Motzkin trees with n edges is

$$\frac{l_n - l_{n-2}}{v_n} \sim \frac{8/9 l_n}{3 l_n} = \frac{8}{27}. \quad \square$$

The referee kindly supplied a proof using a bijective. The proof goes like this.

Each leaf v counted by l_n can be classified into the following two types:

(a) v is the first leaf adjacent to a vertex w, including the case that w is lonely and thus has only one descendent (i.e., v). Such v is then enumerated by the number of vertices in any tree of size n that are adjacent to a leaf.

(b) v is the second leaf adjacent to a vertex w. We can then delete the two descendents of w as well as the two connecting edges, to arrive at a tree of size $n - 2$ and having w as a leaf. Such v is therefore counted by l_{n-2}.

3. Protected nodes

Let $P(t, z)$ be the generating function for the number of Motzkin trees according to the number of protected nodes, where z marks edges and t marks protected nodes. By considering the root is protected or unprotected, we get that each Motzkin tree is one of the forms shown in

Figure 3. From this, we obtain that

$$P = 1 + z + tz(P-1) + z^2 + 2z^2(P-1) + z^2t(P-1)^2. \tag{3.1}$$

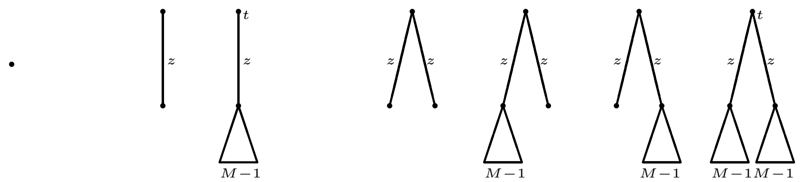

Figure 3 Decomposition of the Motzkin trees with protected nodes marked

Theorem 3.1 *For all $n, k \geq 1$, let $P_{n,k}$ be the number of Motzkin trees of size n with k protected nodes. Then*

$$P_{n,k} = \sum_{m=1}^{n}\sum_{j=0}^{n} \frac{2^{m+j-2k-1}}{m}\binom{m}{2m-n}\binom{2m-n}{j}\binom{n-m}{n-m+j-k}\binom{n-m+j-k}{m+j-2k-1}.$$

Proof Let $y(t,z) = P(t,z) - 1$. Then, from (3.1), we get

$$y = z(1+ty) + z^2(1+2y+ty^2).$$

By the Lagrange inversion formula it follows that

$$\begin{aligned}[z^m]y(z) &= \frac{1}{m}[x^{m-1}](1+tx+z(1+2x+tx^2))^m \\ &= \frac{1}{m}\sum_{i=0}^{m}\sum_{j=0}^{i}\sum_{q=0}^{m-i}\binom{m}{i}\binom{i}{j}\binom{m-i}{q}\binom{q}{2i-j+2q-m-1}2^{2i-j+2q-m-1}t^{m-i+j-q}z^{m-i}.\end{aligned}$$

Furthermore, after some manipulations we can obtain

$$y = \sum_{m=1}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{i}\sum_{q=0}^{m-i}\frac{1}{m}\binom{m}{i}\binom{i}{j}\binom{m-i}{q}\binom{q}{2i-j+2q-m-1}2^{2i-j+2q-m-1}t^{m-i+j-q}z^{2m-i}$$

and hence, we finally have

$$\begin{aligned}&[t^kz^n]P(t,z) = [t^kz^n]y(t,z) \\ &= \sum_{m=1}^{n}\sum_{j=0}^{n}\frac{1}{m}\binom{m}{2m-n}\binom{2m-n}{j}\binom{n-m}{n-m+j-k}\binom{n-m+j-k}{m+j-2k-1}2^{m+j-2k-1}. \ \square\end{aligned}$$

The first terms of the array $(P_{n,k})$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 5 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 9 & 1 & 1 & 0 & 0 & \cdots \\ 4 & 19 & 12 & 14 & 1 & 1 & 0 & \cdots \\ 8 & 24 & 53 & 20 & 20 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By Theorem 3.1, the triangle $(P_{n,k})$ corresponds to the sequence A143364 in [17], which was also given by Deutsch in [17].

Theorem 3.2 *For $n \geq 2$, the number of protected nodes in all Motzkin trees of size n is given by $v_n - 2v_{n-2} + l_{n-2} - l_{n-1} - l_n$. The proportion of protected nodes among all vertices for Motzkin trees with n edges approaches $\frac{10}{27}$.*

Proof Differentiating both sides of equation (3.1) with respect to t and taking into account that $M(1,z) = M(z)$ leads to

$$\frac{\partial M}{\partial t}\Big|_{t=1} = \frac{(1-2z^2)M(z) + z^2 - z - 1}{\sqrt{1-2z-3z^2}}.$$

Hence,

$$[z^n]\frac{\partial M}{\partial t}\Big|_{t=1} = [z^n]\frac{(1-2z^2)M(z) + z^2 - z - 1}{\sqrt{1-2z-3z^2}} = v_n - 2v_{n-2} + l_{n-2} - l_{n-1} - l_n.$$

From (1.5), it follows that

$$\frac{v_n - 2v_{n-2} + l_{n-2} - l_{n-1} - l_n}{v_n} \sim \frac{10}{27}. \quad \square$$

The generating function for the number of protected nodes in all Motzkin trees is also obtained in Cheon and Shapiro [1] by the uplift principle. The first few terms of the generating function are

$$z^2 + 3z^3 + 10z^4 + 31z^5 + 94z^6 + 281z^7 + 834z^8 + 2465z^9 + 7269z^{10} + \cdots.$$

4. Branch nodes and protected branch nodes

Let $G(t,z)$ be the generating function for the number of Motzkin trees according to the number of branch nodes, where z marks edges and t marks branch nodes. Using Figure 1, we get

$$G = 1 + zG + tz^2G^2. \tag{4.1}$$

Then we have

Theorem 4.1 *For all $n, k \geq 1$, the number of Motzkin trees with n edges and k branch nodes is equal to*

$$\frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{n-2k}.$$

Proof Let $y(t,z) = zG(t,z)$. Then from (4.1), we can deduce

$$y = z(1+y+ty^2).$$

Using Lagrange inversion formula we obtain

$$\begin{aligned}[z^n]G(t,z) = [z^{n+1}]y(t,z) &= \frac{1}{n+1}[z^n](1+z+tz^2)^{n+1} \\ &= \frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\binom{n+1-i}{n-2i}t^i.\end{aligned}$$

Thus, we have

$$\begin{aligned}[t^kz^n]G(t,z) &= [t^k]\frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\binom{n+1-i}{n-2i}t^i \\ &= \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{n-2k}. \quad \square\end{aligned}$$

The first terms of the array formed by the coefficients of $G(t,z)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 10 & 10 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 15 & 30 & 5 & 0 & 0 & 0 & \cdots \\ 1 & 21 & 70 & 35 & 0 & 0 & 0 & \cdots \\ 1 & 28 & 140 & 140 & 14 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 4.2 *For $n \geq 2$, the number of Motzkin trees with n edges having exactly one branch node is equal to $\binom{n}{2}$.*

Proof Let $k = 1$. $\square$

Corollary 4.3 *For $m \geq 1$, the number of Motzkin trees with $2m$ edges containing m branch nodes is equal to $C_m = \frac{1}{m+1}\binom{2m}{m}$.*

Proof Let $n = 2m$ and $k = m$. $\square$

Theorem 4.4 *For $n \geq 1$, the number of branch nodes in all Motzkin trees with n edges is equal to $v_n - v_{n-1} - l_n$. In particular, the proportion of branch nodes among all vertices of Motzkin trees with n edges is asymptotically $\frac{1}{3}$.*

Proof Differentiating both sides of equation (4.1) with respect to t and setting $t = 1$ gives that

$$\frac{\partial G}{\partial t}\Big|_{t=1} = \frac{(1-z)M(z) - 1}{\sqrt{1-2z-3z^2}},$$

where we have used the fact that $G(1, z) = M(z)$. From (1.5), it follows that

$$\frac{v_n - v_{n-1} - l_n}{v_n} \sim \frac{1}{3}. \quad \square$$

Let $g(t, z)$ be the generating function for the number of Motzkin trees according to the number of protected branch nodes, where z marks edges and t marks protected branch nodes. Then we have

$$g = 1 + zg + z^2(1 + 2(g-1) + (g-1)^2 t). \tag{4.2}$$

Precisely, we have

Theorem 4.5 *For all $n, k \geq 1$, the number of Motzkin trees with n edges and k protected branch nodes is*

$$\sum_{m=1}^{n} \sum_{j=0}^{2m-n} \frac{2^{m-2k-j-1}}{m} \binom{m}{n-m} \binom{2m-n}{j} \binom{n-m}{n-m-k} \binom{n-m-k}{m-2k-j-1}.$$

Proof Let $y(t, z) = g(t, z) - 1$. Then from (4.2), we get

$$y = z((y+1) + z(1 + 2y + y^2 t)).$$

Using Lagrange inversion formula, we obtain

$$\begin{aligned}
[z^m]g(t,z) = [z^m]y(t,z) &= \frac{1}{m}[x^{m-1}]((x+1) + z(1+2x+x^2 t))^m \\
&= \frac{1}{m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{p=0}^{i}\binom{m}{i}\binom{m-i}{j}\binom{i}{p}\binom{p}{m+2p-2i-j-1}2^{m+2p-2i-j-1}t^{i-p}z^i.
\end{aligned}$$

By some computations it follows that

$$g(t,z) = \sum_{m=1}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{p=0}^{i}\binom{m}{i}\binom{m-i}{j}\binom{i}{p}\binom{p}{m+2p-2i-j-1}\frac{1}{m}2^{m+2p-2i-j-1}t^{i-p}z^{m+i},$$

and hence,

$$[t^k z^n]g(t,z) = \sum_{m=1}^{n}\sum_{j=0}^{2m-n}\frac{1}{m}\binom{m}{n-m}\binom{2m-n}{j}\binom{n-m}{n-m-k}\binom{n-m-k}{m-2k-j-1}2^{m-2k-j-1}. \quad \square$$

The first terms of the array formed by the coefficients of $g(t,z)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 8 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 16 & 5 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 32 & 19 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 64 & 61 & 2 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

When we take $k=0$ and $k=1$ in Theorem 4.5, we can obtain sequences A000079 and A102841 in [17]. Thus, the first column and the second column of the matrix correspond to sequences A000079 and A102841 in [17], respectively.

Theorem 4.6 *For $n \geq 2$, the number of protected branch nodes in all Motzkin trees with n edges is equal to*

$$v_n - v_{n-1} - 2v_{n-2} + l_{n-2} - l_n.$$

The proportion of protected branch nodes among all vertices for trees with n edges is asymptotically $\frac{4}{27}$.

Proof Differentiating both sides of Eq. (4.2) with respect to t and taking into account that $g(1,z) = M(z)$, we obtain

$$\frac{\partial g}{\partial t}\Big|_{t=1} = \frac{(1-z-2z^2)M(z)+z^2-1}{\sqrt{1-2z-3z^2}}.$$

Thus,

$$[z^n]\frac{\partial g}{\partial t}\Big|_{t=1} = [z^n]\frac{(1-z-2z^2)M(z)+z^2-1}{\sqrt{1-2z-3z^2}} = v_n - v_{n-1} - 2v_{n-2} + l_{n-2} - l_n.$$

From (1.5), it follows that $v_n - v_{n-1} - 2v_{n-2} + l_{n-2} - l_n \sim \frac{4}{9} l_n$, and so

$$\frac{v_n - v_{n-1} - 2v_{n-2} + l_{n-2} - l_n}{v_n} \sim \frac{4}{27}. \quad \square$$

The first few terms of the generating function for the number of protected branch nodes in all Motzkin trees are (sequence A025568 [17]),

$$z^4 + 5z^5 + 19z^6 + 65z^7 + 211z^8 + 665z^9 + 2058z^{10} + \cdots.$$

5. Lonely nodes and protected lonely nodes

Let $H(t,z)$ be the generating function for the number of Motzkin trees according to the number of lonely nodes, where z marks edges and t marks lonely nodes. Using Figure 1, we get

$$H = 1 + tzH + z^2H^2. \tag{5.1}$$

Thus, we can obtain the following result.

Theorem 5.1 *For all $n, k \geq 1$, the number of Motzkin trees with n edges containing k lonely nodes is equal to*

$$\frac{1+(-1)^{n-k}}{2(n+1)}\binom{n+1}{\frac{n-k}{2}}\binom{\frac{n+k+2}{2}}{k}.$$

Proof Let $y(t,z) = zH(t,z)$. Then, from (5.1), we get

$$y = z(1 + ty + y^2).$$

By the Lagrange inversion formula it follows that

$$\begin{aligned}[z^n]H(t,z) = [z^{n+1}]y(t,z) &= \frac{1}{n+1}[z^n]\left(1+tz+z^2\right)^{n+1} \\ &= \frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\binom{n+1-i}{i+1}t^{n-2i}.\end{aligned}$$

Therefore, we have

$$[t^kz^n]H(t,z) = \frac{1+(-1)^{n-k}}{2(n+1)}\binom{n+1}{\frac{n-k}{2}}\binom{\frac{n+k+2}{2}}{k}. \quad \square$$

By Theorem 5.1, we know that the array formed by the coefficients of $H(t,z)$ are (A097610),

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 6 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 10 & 0 & 10 & 0 & 1 & 0 & \cdots \\ 5 & 0 & 30 & 0 & 15 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 5.2 *For $n \geq 1$, the number of Motzkin trees of size $2m$ containing no lonely nodes is equal to C_m, the mth Catalan number.*

Proof Let $k = 0$. □

Theorem 5.3 *For $n \geq 1$, the number of lonely nodes in all Motzkin trees with n edges is equal to v_{n-1}. In particular, the proportion of lonely nodes among all vertices of Motzkin trees with n edges is asymptotically $\frac{1}{3}$.*

Proof Differentiating both sides of Eq. (5.1) with respect to t and setting $t = 1$ gives that

$$\left.\frac{\partial H}{\partial t}\right|_{t=1} = \frac{1-z-\sqrt{1-2z-3z^2}}{2z\sqrt{1-2z-3z^2}} = zV(z),$$

where we have used the fact that $H(1,z) = M(z)$. Hence, the number of lonely nodes in all Motzkin trees of size n is

$$[z^n]\frac{\partial H}{\partial t}\Big|_{t=1} = [z^n]zV(z) = v_{n-1}.$$

The proportion of lonely nodes among all vertices for trees with n edges is $\frac{v_{n-1}}{v_n} \sim \frac{1}{3}$. □

Since $\frac{l_n}{v_n} \sim \frac{1}{3}$ and Theorem 4.4, we can also have that the proportion of lonely nodes among all vertices of Motzkin trees with n edges is asymptotically $\frac{1}{3}$.

Let $h(t,z)$ be the generating function for the number of Motzkin trees according to the number of protected lonely nodes, where z marks edges and t marks protected lonely nodes. Thus we obtain

$$h = 1 + z + tz(h-1) + z^2h^2, \tag{5.2}$$

which is equivalent to

$$h = 1 + tzh + (1-t)z + z^2h^2. \tag{5.3}$$

We now immediately deduce the following result.

Theorem 5.4 *For all $n, k \geq 1$, the number of Motzkin trees with n edges containing k protected lonely nodes is given by*

$$\sum_{m=0}^{n}\sum_{i=0}^{m+1}\frac{(-1)^{k+3m}}{m+1}\binom{m+1}{i}\binom{i}{2i+2n-3m-2}\binom{m+1-i}{n-m}\binom{n-m}{k+3m+2-2i-2n}.$$

Proof Let $y(t,z) = zh(t,z)$, then from (5.3), we can obtain that

$$y = z(1 + ty + (1-t)z + y^2).$$

Using Lagrange inversion formula, we obtain

$$\begin{aligned}
[z^m]h(t,z) &= [z^{m+1}]y = \frac{1}{m+1}[x^m](1 + tx + (1-t)z + x^2)^{m+1} \\
&= \frac{1}{m+1}\sum_{i=0}^{m+1}\sum_{p=0}^{m+1-i}\sum_{q=0}^{p}\binom{m+1}{i}\binom{i}{2i+2p-m-2}\binom{m+1-i}{p}\binom{p}{q}(-1)^q t^{2i+2p-m-2+q}z^p.
\end{aligned}$$

Furthermore, we deduce that

$$\begin{aligned}
h(t,z) = &\sum_{m=0}^{\infty}\sum_{i=0}^{m+1}\sum_{p=0}^{m+1-i}\sum_{q=0}^{p}\frac{1}{m+1}\binom{m+1}{i}\binom{i}{2i+2p-m-2}\binom{m+1-i}{p}\cdot \\
&\binom{p}{q}(-1)^q t^{2i+2p-m-2+q}z^{p+m}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&[t^kz^n]h(t,z) \\
&= \sum_{m=0}^{n}\sum_{i=0}^{m+1}\frac{(-1)^{k+3m}}{m+1}\binom{m+1}{i}\binom{i}{2i+2n-3m-2}\binom{m+1-i}{n-m}\binom{n-m}{k+3m+2-2i-2n}. \quad \square
\end{aligned}$$

The first terms of the array formed by the coefficients of $h(t,z)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 4 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 7 & 6 & 1 & 1 & 0 & 0 & \cdots \\ 11 & 18 & 12 & 8 & 1 & 1 & 0 & \cdots \\ 22 & 39 & 36 & 18 & 10 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

When we take $k = 0$ and $k = 1$ in Theorem 5.4, we should obtain sequences A007477 and A234269 in [17]. Thus, the first column and the second column of the matrix correspond to sequences A007477 and A234269 in [17], respectively.

By Theorems 3.2 and 4.6, we can obtain the following result. The result can be proved by the generating function.

Theorem 5.5 *For $n \geq 1$, the number of protected lonely nodes in all Motzkin trees with n edges is equal to $v_{n-1} - l_{n-1}$. In particular, the proportion of protected lonely nodes among all vertices of Motzkin trees with n edges approaches $\frac{2}{9}$.*

Proof Differentiating both sides of Eq. (5.3) with respect to t and setting $t = 1$ gives that

$$\frac{\partial h}{\partial t}\Big|_{t=1} = \frac{zM(z) - z}{\sqrt{1 - 2z - 3z^2}} = zV(z) - zL(z),$$

where we have used the fact that $h(1, z) = M(z)$. Hence, the number of protected lonely nodes in all Motzkin trees with n edges is given by

$$[z^n]\frac{\partial h}{\partial t}\Big|_{t=1} = [z^n]zV(z) - [z^n]zL(z) = v_{n-1} - l_{n-1}.$$

From the relations (1.5), it follows that $v_{n-1} - l_{n-1} \sim 2l_{n-1} \sim \frac{2}{3}l_n$, and the proportion of protected lonely nodes among all vertices of Motzkin trees with n edges is $\frac{v_{n-1}-l_{n-1}}{v_n} \sim \frac{\frac{2}{3}l_n}{3l_n} = \frac{2}{9}$. $\square$

The first few terms of the generating function for the number of protected lonely nodes in all Motzkin trees are (sequence A005774 [17]),

$$z^2 + 3z^3 + 9z^4 + 26z^5 + 75z^6 + 216z^7 + 623z^8 + 1800z^9 + 5211z^{10} + \cdots.$$

As future works, we will find bijective proofs for Theorems 3.2, 4.4, 4.6 and 5.5.

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