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Toeplitz Determinants for the Inverse of Starlike Functions Connected with the Sine Function

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Abstract Let $\mathcal{S}_s^*$ be the class of normalized functions f defined in the open unit $\mathbb{U}=\{z\in\mathbb{C}: |z|<1\}$ such that the quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane and satisfies the condition $\frac{zf'(z)}{f(z)}\prec 1+\sin z$ $(z\in\mathbb{U})$. In this paper, we aim to investigate Toeplitz determinants for the inverse of this function classes $\mathcal{S}_s^*$ associated with sine function.

Keywords inverse of starlike function; Toeplitz determinant; sine function; upper bound

MR(2020) Subject Classification 30C45; 30C50

1. Introduction

Let $\mathcal{H}$ denote the space of analytic functions in the unit disk $\mathbb{U}=\{z\in \mathbf{C}\colon |z|<1\}$ and $\mathcal{A}$ denote the class of functions f in $\mathcal{H}$ with Taylor series

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

The subclass S of $\mathcal{A}$ consisting of univalent functions has attracted much interest for over a century and is a central area of research in complex analysis. Let $\mathcal{P}$ denote the class of analytic functions of the form

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots,$$

such that $\mathcal{R}(p(z))>0$ in $\mathbb{U}$.

Recently, Cho et al. [1] introduced the following function class $\mathcal{S}_s^*$ which are associated with sin function:

$$\mathcal{S}_s^*:=\Big\{f\in\mathcal{A}\colon \frac{zf'(z)}{f(z)}\prec 1+\sin z\ (z\in\mathbb{U})\Big\},$$

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where $\prec$ stands for the subordination symbol (for details, see [2]) and also implies that the quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane. Zhang et al. [3] investigate the third-order Hankel determinant $H_3(1)$ and Toeplitz determinant $H_3(2)$ for this function class $\mathcal{S}_s^*$.

Hankel matrices and determinants play an important role in several branches of mathematics and have many applications [4]. A Toeplitz determinant is closely related to a Hankel determinant. Recently, Thomas and Halim [5] introduced the symmetric Toeplitz determinant $T_q(n)$ for analytic functions f of the form (1.1), defined by

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix},$$

where $a_1 = 1$ and $n,\ q = 1, 2, 3, \ldots$.

In recent years, some authors have studied the Toeplitz determinants [3], [5]–[7]. Inspired by the aforementioned works, in this paper, we aim to investigate Toeplitz determinants for the inverse of starlike functions connected with the sin function, and obtain the upper bounds of the above determinants. We recall the following results to prove our main theorems.

Lemma 1.1 ([8]) *Let $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}$. Then $|c_n| \le 2$ $n = 1, 2, \ldots$.*

Lemma 1.2 ([9]) *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}$, then for some complex valued x with $|x| \le 1$ and some complex valued ς with $|\varsigma| \le 1$,*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2c_1 x(4 - c_1^2) - c_1 x^2 (4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)\varsigma.$$

2. Main results

In this section, we investigate Toeplitz determinants for the inverse of starlike functions connected with the sin function.

Theorem 2.1 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n \omega^n$ is the inverse function of f, then*

$$|d_2| \le 1, \quad |d_3| \le \frac{3}{2}, \quad |d_4| \le \frac{47}{18}.$$

These bounds are the best possible.

Proof Let $f \in \mathcal{S}_s^*$. Then, in the form of the Schwarz function, we have

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)).$$

Furthermore, we easily get

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \cdots . \tag{2.1}$$

Define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

Clearly, we have $p(z) \in \mathcal{P}$ and

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

On the other hand,

$$1 + \sin(\omega(z)) = 1 + \frac{1}{2}c_1 z + (\frac{c_2}{2} - \frac{c_1^2}{4})z^2 + (\frac{5c_1^3}{48} + \frac{c_3 - c_1c_2}{2})z^3 + \cdots . \tag{2.2}$$

By comparing (2.1) and (2.2), we obtain

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{c_2}{4}, \quad a_4 = \frac{c_3}{6} - \frac{c_1c_2}{24} - \frac{c_1^3}{144}. \tag{2.3}$$

Since $f(f^{-1}(\omega)) = \omega$, equating coefficients gives

$$d_2 = -a_2, \quad d_3 = 2a_2^2 - a_3, \quad d_4 = -5a_2^3 + 5a_2a_3 - a_4. \tag{2.4}$$

From (2.3) and (2.4), we have

$$d_2 = -\frac{c_1}{2}, \tag{2.5}$$

$$d_3 = \frac{2c_1^2 - c_2}{4}, \tag{2.6}$$

$$d_4 = \frac{-24c_3 + 96c_1c_2 - 89c_1^3}{144}. \tag{2.7}$$

Applying Lemma 1.1 in (2.5), we get $|d_2| \le 1$. Next we note that from (2.6) and Lemma 1.2

$$|d_3| = \frac{1}{8}|3c_1^2 - x(4 - c_1^2)|.$$

Let $c_1 = c \in [0, 2]$. Then, using the triangle inequality, we get

$$|d_3| \le \frac{1}{8}[3c^2 + |x|(4 - c^2)] = \varphi(c, |x|).$$

Since

$$\frac{\partial\varphi}{\partial|x|} = \frac{4 - c^2}{8} > 0, \quad c \in (0, 2),$$

namely, $\varphi(c, |x|)$ is an increasing function on the closed interval $[0, 1]$ about $|x|$. This implies that the maximum value of $\varphi(c, |x|)$ occurs at $|x| = 1$, which is

$$\max \varphi(c, |x|) = \varphi(c, 1) = \frac{c^2 + 2}{4} \le \frac{3}{2}.$$

From (2.7) and Lemma 1.2, we get

$$|d_4| = \frac{1}{144}| - 47c_1^3 + 36c_1x(4 - c_1^2) + 6c_1x^2(4 - c_1^2) - 12(4 - c_1^2)(1 - |x|^2)\varsigma|.$$

Now, without loss of generality we can assume that $c_1 = c \in [0,2]$, and so using the triangle inequality and $|\varsigma| < 1$, we obtain

$$|d_4| \leq \frac{1}{144}[47c^3 + 36c|x|(4-c^2) + 6c|x|^2(4-c^2) + 12(4-c^2)(1-|x|^2)] = \varphi(c,|x|).$$

Thus we need to maximise $\varphi(c,|x|)$ over $\mathcal{T} = [0,2] \times [0,1]$. First assume that $\varphi(c,|x|)$ has a stationary point at all interior points (c_0, x_0) of $\mathcal{T}$. Then

$$\frac{\partial\varphi(c,|x|)}{\partial|x|}|_{(c_0,|x_0|)} = \frac{(4-c_0^2)[36c_0 + 12|x_0|(c_0-2)]}{144} = 0,$$

when $c_0 = 2$. Since we have assumed that $c_0 < 2$, we have a contradiction, and so $\varphi(c,|x|)$ attains its maximum on the boundary of $\mathcal{T}$.

When $c=2$, $\varphi(2,|x|) = \frac{47}{18}$. When $c=0$, $\varphi(0,|x|) = \frac{1-|x|^2}{3} \leq \frac{1}{3}$.

When $|x|=1$,

$$\varphi(c,1) = \frac{5c^3 + 168c}{144} = \varphi(c).$$

Next note that, $\forall c \in (0,2)$,

$$\varphi'(c) = \frac{15c^2 + 168}{144} > 0,$$

thus, $\varphi(c) \leq \varphi(2) = \frac{47}{18}$.

When $|x|=0$,

$$\varphi(c,0) = \frac{47c^3 - 12c^2 + 48}{144} = \varphi(c).$$

Then we have

$$\varphi'(c) = \frac{47c^2 - 8c}{48}, \quad \varphi''(c) = \frac{47c-4}{24}.$$

If $\varphi'(c) = 0$, then the roots are $c = 0, c = \frac{8}{47}$. In addition, since $\varphi''(0) = -\frac{1}{6} < 0, \varphi''(\frac{8}{47}) = \frac{1}{6} > 0$, the function $\varphi(c)$ can take the maximum value at $c=0$, which is $\varphi(c) \leq \varphi(0) = \frac{1}{3}$.

We note that $|d_2| \leq 1$ is sharp when $c_1 = 2$, the estimate $|d_3| \leq \frac{3}{2}$ is sharp when $c_1 = c_2 = 2$ and $|x|=1$, and $|d_4| \leq \frac{47}{18}$ is sharp, when $c_1 = c_2 = c_3 = 2$ and $|x|=1$. □

Theorem 2.2 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n\omega^n$ is the inverse function of f, then*

$$|d_2^2 - d_3^2| \leq \frac{5}{4}.$$

The result is sharp.

Proof In view of (2.5) and (2.6), a simple computation leads to

$$|d_2^2 - d_3^2| = \frac{1}{16}|4c_1^4 - 4c_2c_1^2 + c_2^2 - 4c_1^2|. \tag{2.8}$$

Using Lemma 1.2 in (2.8), we obtain

$$|d_2^2 - d_3^2| = \frac{1}{64}|9c_1^4 - 16c_1^2 - 6c_1^2x(4-c_1^2) + x^2(4-c_1^2)^2|.$$

As in the proof of Theorem 2.1, without loss of generality, we can write $c_1 = c \in [0,2]$, by using the triangle inequality,

$$|d_2^2 - d_3^2| \leq \frac{1}{64}[|9c^4 - 16c^2| + 6c^2|x|(4-c^2) + |x|^2(4-c^2)^2] = \varphi(c,|x|).$$

Therefore, we get, $\forall |x| \in (0,1), \forall c \in (0,2)$

$$\frac{\partial \varphi}{\partial |x|} = \frac{(4-c^2)(3c^2+|x|(4-c^2))}{32} > 0,$$

namely, $\varphi(c,|x|)$ is an increasing function on the closed interval [0,1] about $|x|$. This implies that the maximum value of $\varphi(c,|x|)$ occurs at $|x|=1$, which is

$$\max \varphi(c,|x|) = \varphi(c,1) = \frac{1}{64}(|9c^4-16c^2| - 5c^4 + 16c^2 + 16) = \varphi(c). \tag{2.9}$$

We consider two cases.

Case (i). $0 \le c \le \frac{4}{3}$.

In this case, (2.9) becomes

$$\varphi(c) = \frac{1}{32}(-7c^4 + 16c^2 + 8).$$

Then we have

$$\varphi'(c) = \frac{c(-7c^2+8)}{8}, \quad \varphi''(c) = \frac{-21c^2+8}{8}.$$

If $\varphi'(c) = 0$, then the roots are $c = 0, c = \sqrt{\frac{8}{7}}$. In addition, since

$$\varphi''(0) = 1 > 0, \quad \varphi''(\sqrt{\frac{8}{7}}) = -2 < 0,$$

the function $\varphi(c)$ can take the maximum value at $c = \sqrt{\frac{8}{7}}$, which is $\varphi(c) \le \varphi(\sqrt{\frac{8}{7}}) = \frac{15}{28}$.

Case (ii). $\frac{4}{3} \le c \le 2$.

In this case, (2.9) becomes $\frac{c^4+4}{16}$, which increases with c for $\frac{4}{3} \le c \le 2$. Thus $\varphi(c) \le \varphi(2) = \frac{5}{4}$.

We note that $|d_2^2 - d_3^2| \le \frac{5}{4}$ is sharp when $c_1 = c_2 = 2$ and $|x| = 1$. □

Theorem 2.3 If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n \omega^n$ is the inverse function of f, then $|d_2 d_4 - d_3^2| \le \frac{3}{8}$.

Proof From (2.5), (2.6) and (2.7), we have

$$d_2 d_4 - d_3^2 = \frac{1}{288}(17c_1^4 + 24c_3c_1 - 24c_2c_1^2 - 18c_2^2). \tag{2.10}$$

Using Lemma 1.2 in (2.10), we get

$$d_2 d_4 - d_3^2 = \frac{1}{576}(13c_1^4 - 18c_1^2 x(4-c_1^2) - 3(4-c_1^2)(12+c_1^2)x^2 + 24c_1(4-c_1^2)(1-|x|^2)\varsigma).$$

Using the triangle inequality and assuming that $c_1 = c \in [0,2]$, we get

$$|d_2d_4 - d_3^2| \le \frac{1}{576}[13c^4 + 18c^2|x|(4-c^2) + 3(4-c^2)(12+c^2)|x|^2 + 24c(4-c^2)(1-|x|^2)] = \varphi(c,|x|).$$

Consider

$$\frac{\partial \varphi}{\partial |x|} = \frac{4-c^2}{96}[3c^2 + |x|(c-2)(c-6)] > 0, \quad \forall c \in (0,2), \ \forall |x| \in (0,1).$$

φ is an increasing function of $|x|$ on the closed interval $[0,1]$. Hence

$$\varphi(c,|x|) \le \varphi(c,1) = \frac{-c^4 + 6c^2 + 18}{72} = \varphi(c).$$

Then we get

$$\varphi'(c) = \frac{-c^3+3c}{18}, \quad \varphi''(c) = \frac{-c^2+1}{6}.$$

If $\varphi'(c) = 0$, then the roots are $c = 0, c = \sqrt{3}$. In addition, since $\varphi'(0) = \frac{1}{6} > 0$ and $\varphi'(\sqrt{3}) = -\frac{1}{3} < 0$, the function $\varphi(c)$ can take the maximum value at $c = \sqrt{3}$, which is $\varphi(c) \le \varphi(\sqrt{3}) = \frac{3}{8}$. □

Theorem 2.4 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n\omega^n$ is the inverse function of f, then*

$$|d_2d_3 - d_3d_4| \le \frac{29}{12}.$$

The result is sharp.

Proof From (2.5), (2.6) and (2.7), we have

$$|d_2d_3 - d_3d_4| = \frac{1}{576}|178c_1^5 - 144c_1^3 + 72c_2c_1 - 24c_2c_3 + 48c_3c_1^2 - 281c_2c_1^3 + 96c_1c_2^2|.$$

Using Lemma 1.2 with $Y = 4 - c_1^2$ and $Z = (1 - |x|^2)\varsigma$, we obtain

$$\begin{aligned}|d_2d_3 - d_3d_4| =& \frac{1}{1152}|141c_1^5 - 216c_1^3 - 155c_1^3xY - 18c_1^3x^2Y + 36c_1x^2Y^2 + \\ & 36c_1^2YZ + 6c_1x^3Y^2 + 72c_1xY - 12xY^2Z|.\end{aligned}$$

Using the triangle inequality and assuming that $c_1 = c \in [0, 2]$, we get

$$\begin{aligned}|d_2d_3 - d_3d_4| \le& \frac{1}{1152}[|141c^5 - 216c^3| + 155c^3|x|Y + 18c^3|x|^2Y + 36c|x|^2Y^2 + \\ & 36c^2YZ + 6c|x|^3Y^2 + 72c|x|Y + 12|x|Y^2Z] = \varphi(c, |x|),\end{aligned}$$

where now $Y = 4 - c^2$ and $Z = 1 - |x|^2$. Thus we need to maximise $\varphi(c, |x|)$ over $\mathcal{T} = [0, 2] \times [0, 1]$.

First assume that $\varphi(c, |x|)$ has a stationary point at all interior points (c_0, x_0) of $\mathcal{T}$. Then differentiating $\varphi(c, |x|)$ with respect to $|x|$ and equalling it to 0 would imply that $c_0 = 2$, which is a contradiction. So $\varphi(c, |x|)$ attains its maximum on the boundary of $\mathcal{T}$. When $c = 2$,

$$\varphi(2, |x|) = \frac{29}{12}.$$

When $c = 0$,

$$\varphi(0, |x|) = \frac{|x| - |x|^3}{6} \le \frac{\sqrt{3}}{27}.$$

When $|x| = 1$,

$$\varphi(c, 1) = \frac{1}{1152}[|141c^5 - 216c^3| - 131c^5 + 284c^3 + 960c],$$

which has maximum value $\frac{29}{12}$ on $[0, 2]$. When $|x| = 1$,

$$\varphi(c, 0) = \frac{1}{1152}[|141c^5 - 216c^3| + 131c^2 - 36c^4]$$

which has maximum value $\frac{1}{1152}(-141c_0^5 - 36c_0^4 + 216c_0^3 + 144c_0^2) \approx 0.1606807$, where c_0 is the positive root of the polynomial $-705c^3 - 144c^2 + 648c + 288 = 0$.

We note that $|d_2d_3 - d_3d_4| \le \frac{29}{12}$ is sharp when $c_1 = c_2 = c_3 = 2$ and $x = 1$. □

Theorem 2.5 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n \omega^n$ is the inverse function of f, then*

$$|T_2(3)| = |d_4^2 - d_3^2| \leq \frac{370}{81}.$$

The result is sharp.

Proof In view of (2.6) and (2.7), a simple computation leads to

$$|d_4^2 - d_3^2| = \frac{1}{20736}|576c_3^2 - 4608c_3c_2c_1 + 9216c_1^2c_2^2 + 4272c_3c_1^3 - 17088c_1^4c_2 + 7921c_1^6 - 5184c_1^4 + 5184c_1^2c_2 - 1296c_2^2|.$$

Using Lemma 1.2, we obtain with $Y = 4 - c_1^2$ and $Z = (1 - |x|^2)\varsigma$,

$$|d_4^2 - d_3^2| = \frac{1}{20736}|2209c_1^6 - 2916c_1^4 - 3384c_1^4xY - 564c_1^4x^2Y + 1296c_1^2x^2Y^2 + 432c_1^2x^3Y^2 - 864c_1xY^2Z + 1128c_1^3YZ + 36c_1^2x^4Y^2 - 144c_1x^2Y^2Z + 144Y^2Z^2 + 1944c_1^2xY - 324x^2Y^2|.$$

Using the triangle inequality, we obtain

$$|d_4^2 - d_3^2| \leq \frac{1}{20736}[|2209c^6 - 2916c^4| + 3384c^4|x|Y + 564c^4|x|^2Y + 1296c^2|x|^2Y^2 + 432c^2|x|^3Y^2 + 864c|x|Y^2Z + 1128c^3YZ + 36c^2|x|^4Y^2 + 144c|x|^2Y^2Z + 144Y^2Z^2 + 1944c^2|x|Y + 324|x|^2Y^2] = \varphi(c, |x|),$$

where $Y = 4 - c^2$ and $Z = (1 - |x|^2)$.

Further, we maximize the function $\varphi(c, |x|)$ on the closed region $[0, 2] \times [0, 1]$. Differentiate $\varphi(c, |x|)$ partially with respect to c and $|x|$, respectively.

For the extreme values of $\varphi(c, |x|)$, consider

$$\frac{\partial \varphi}{\partial |x|} = 0 \text{ and } \frac{\partial \varphi}{\partial c} = 0. \tag{2.11}$$

In view of (2.11), we get the critical point for the function $\varphi(c, |x|)$ which lies in the closed region $[0, 2] \times [0, 1]$ is $(0, 0)$ only.

For $c = 2$, the function $\varphi(c, |x|)$ becomes $\varphi(|x|) = \frac{370}{81} \approx 4.5679$.

For $c = 0$, the function $\varphi(c, |x|)$ becomes $\varphi(|x|) = \frac{4|x|^4 + |x|^2 + 4}{36} \leq \frac{1}{4} = 0.25$.

For $|x| = 1$, the function $\varphi(c, |x|)$ becomes

$$\varphi(c) = \frac{1}{20736}[|2209c^6 - 2916c^4| - 2184c^6 + 60c^4 + 33408c^2 + 5184]. \tag{2.12}$$

We consider two cases.

Case (i). $\frac{2916}{2209} \leq c^2 \leq 4$.

In this case, (2.12) becomes $\varphi(c) = \frac{1}{20736}(25c^6 - 2856c^4 + 33408c^2 + 5184)$. Then we have

$$\varphi'(c) = \frac{25c^5 - 1904c^3 + 11136c}{3456} = \frac{c(c^2 - c_1^2)(c^2 - c_2^2)}{3456} > 0,$$

where $c_1^2 = \frac{1904 - 16\sqrt{9811}}{50} \approx 6.3838$, $c_2^2 = \frac{1904 + 16\sqrt{9811}}{50} \approx 69.76$. So the function $\varphi(c)$ can take the maximum value at $c = 2$, which is $\varphi(c) \leq \varphi(2) = \frac{370}{81}$.

Case (ii). $0 \le c^2 \le \frac{2916}{2209}$.

In this case, (2.12) becomes $\varphi(c) = \frac{1}{20736}(-4393c^6 + 2976c^4 + 33408c^2 + 5184)$. We obtain

$$\varphi'(c) = \frac{c(-4393c^4 + 1984c^2 + 11136)}{3456} \text{ and } \varphi''(c) = \frac{-21965c^4 + 5952c^2 + 11136}{3456}.$$

Setting $\varphi'(c) = 0$, we can easily get $c = 0$ and $c_0 = c^2 = \frac{992+8\sqrt{779758}}{4393} \approx \frac{8085}{4393} = 1.8338$. In addition, since $\varphi''(0) > 0$ and $\varphi''(c_0) < 0$, the function $\varphi(c)$ can take maximum value at c_0, which is

$$\varphi(c) \le \varphi(c_0) = \frac{1}{20736}(-4393c_0^3 + 2976c_0^2 + 33408c_0 + 5184) \approx \frac{49364}{20736} \approx 2.3806.$$

For $|x| = 0$, we have

$$\varphi(c) = \frac{1}{20736}[|2209c^6 - 2916c^4| - 1128c^5 + 144c^4 + 4512c^3 - 1152c^2 + 2304],$$

which has maximum value $\frac{1017}{10368}$ on $[0, 2]$, which completes the proof of the theorem.

We note that the result is sharp when $c_1 = c_2 = c_3 = 2$, $x = 1$. $\square$

Theorem 2.6 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n\omega^n$ is the inverse function of f, then*

$$|T_3(2)| = \left|\begin{pmatrix} d_2 & d_3 & d_4 \\ d_3 & d_2 & d_3 \\ d_4 & d_3 & d_2 \end{pmatrix}\right| \le \frac{281}{48}.$$

Proof Because

$$T_3(2) = d_2(d_2^2 - d_3^2) - d_3(d_2d_3 - d_3d_4) + d_4(d_3^2 - d_2d_4),$$

by using the triangle inequality, we have

$$|T_3(2)| \le |d_2||d_2^2 - d_3^2| + |d_3||d_2d_3 - d_3d_4| + |d_4||d_3^2 - d_2d_4|. \tag{2.11}$$

From (2.11), Theorems 2.1–2.4, we immediately get the desired assertion. $\square$

Theorem 2.7 *If $f \in \mathcal{S}_s^*$ and $f^{-1}(\omega) = \omega + \sum\limits_{n=2}^{\infty} d_n\omega^n$ is the inverse function of f, then*

$$|T_3(1)| = \left|\begin{pmatrix} 1 & d_2 & d_3 \\ d_2 & 1 & d_2 \\ d_3 & d_2 & 1 \end{pmatrix}\right| \le \frac{5}{4}.$$

The result is sharp.

Proof Expanding the determinant $T_3(1)$ by using (2.5), (2.6) and Lemma 2.2, we get

$$\begin{aligned}|T_3(1)| =& |1 + 2d_2^2(d_3 - 1) - d_3^2| = \frac{1}{16}|16 + 2c_1^2c_2 - 8c_1^2 - c_2^2| \\ =& \frac{1}{64}|64 + 3c_1^4 - 32c_1^2 + 2c_1^2x(4 - c_1^2) - x^2(4 - c_1^2)^2|.\end{aligned}$$

Using the triangle inequality and assuming that $c_1 = c \in [0, 2]$, we get

$$|T_3(1)| \le \frac{1}{64}[|64 + 3c^4 - 32c^2| + 2c^2|x|(4 - c^2) + |x|^2(4 - c^2)^2].$$

Consider

$$\frac{\partial\varphi}{\partial|x|}=\frac{4-c^2}{32}(c^2+|x|(4-c^2))>0 \quad (\forall c\in(0,2),\forall|x|\in(0,1)).$$

φ is an increasing function of $|x|$ on the closed interval $[0,1]$. Hence

$$\varphi(c,|x|)\leq\varphi(c,1)=\frac{|64+3c^4-32c^2|+16-c^4}{64}=\varphi(c),$$

which has maximum value $\frac{5}{4}$ at $c=0$.

We note that $|T_3(1)|\leq\frac{5}{4}$ is sharp when $c_1=0, c_2=2i$ and $|x|=1$. $\square$

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