Journal of Mathematical Research with Applications
Mar., 2021, Vol. 41, No. 2, pp. 150–160
DOI:10.3770/j.issn:2095-2651.2021.02.004
Http://jmre.dlut.edu.cn

Global Well-Posedness of Solutions for the Sixth Order Convective Cahn-Hilliard Equation

Xiaopeng ZHAO[1,*], **Fengnan LIU**[2], **Haichao MENG**[3]
1. *College of Sciences, Northeastern University, Liaoning* 110819*, P. R. China;*
2. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
3. *School of Science, Jiangnan University, Jiangsu* 214122*, P. R. China*

Abstract In this paper, we consider the global well-posedness of smooth solutions for the Cauchy problem of a sixth order convective Cahn-Hilliard equation with small initial data. We first construct a local smooth solution, then by combining some a priori estimates, continuity argument, the local smooth solution is extended step by step to all $t > 0$ provided that the L^1 norm of initial data is suitably small and the smooth nonlinear functions $f(u)$ and $g(u)$ satisfy certain local growth conditions at some fixed point $\bar{u} \in \mathbb{R}$.

Keywords Global smooth solution; sixth order convective Cahn-Hilliard equation; Cauchy problem; local existence

MR(2020) Subject Classification 35K55; 35A01; 35B45

1. Introduction

Consider the Cauchy problem of the sixth order convective Cahn-Hilliard equation [1–4]

$$\begin{cases} \frac{\partial u}{\partial t} - \delta g(u)_x - (u_{xx} - f(u))_{xxxx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(t,x) = h_x(t,x)$ is the slope of a $1+1$D (one dimension in space, one in time) surface $h(t,x)$, δ is proportional to the deposition strength of an atomic flux and the overall convective term $\delta g(u)_x$ stems from the normal impingement of the deposited atoms [2]. Moreover, the sixth order linear term results from a curvature dependent regularization, and all other terms represent the anisotropy of the surface energy under surface diffusion. In this paper, for convenience, we set $\delta = 1$.

The sixth order convective Cahn-Hilliard equation can be used to describe the faceting of a growing surface with small slopes. There are many papers devoted to the well-posedness theory of the initial boundary value problem for it. In [5], by an extension of the method of matched asymptotic expansions that retains exponentially small terms, Korzec, Evans, Münch and Wagner derived new types of stationary solutions for the sixth order convective Cahn-Hilliard equation.

Received March 6, 2020; Accepted September 7, 2020
Supported by the Fundamental Research Funds for the Central Universities (Grant No. N2005031).
* Corresponding author
E-mail address: zhaoxiaopeng@jiangnan.edu.cn (Xiaopeng ZHAO); liufengnan@dlut.edu.cn (Fengnan LIU); 1148372879@qq.com (Haichao MENG)

Moreover, by using Galerkin techniques, Korzec and Rybka [2] investigated the existence and uniqueness of weak solutions of the sixth order convective Cahn-Hilliard equation with periodic boundary conditions. The authors also used numerical methods to study how the long-time behavior of solutions depends on the parameter δ, and they presented numerical evidence that typical solutions stop coarsening before reaching a trivial state. Latterly, Korzec, Nayar and Rybka [1] established the existence of global-in-time weak solutions and exponential-in-time a priori estimates on the H^2 norm of solutions for the 2D sixth order convective Cahn-Hilliard equation together with periodic boundary conditions. Very recently, the long time behavior of solutions for the 1D and 2D sixth order convective Cahn-Hilliard equation was studied by Korzec, Nayar and Rybka [3]. Applying the ideas from the theory of infinite dimensional dynamical systems combined with the available results on convective Cahn-Hilliard equation, the authors proved the existence of global attractor for such equations with periodic boundary conditions.

The goal of this paper is to investigate the global well-posedness for the Cauchy problem of one-dimensional sixth order convective Cahn-Hilliard equation. We prove the existence and uniqueness of global smooth solutions for problem (1.1) by using Hoff and Smoller's method [6–9] with a slight modification. More precisely, the result can be stated as follows.

Theorem 1.1 *Let $r > 0$ be any given constant. Suppose that $\bar{u} \in \mathbb{R}$ is some fixed constant. Assume that $u_0(x) - \bar{u} \in L^1(\mathbb{R}, \mathbb{R})$ with $\|u_0(x) - \bar{u}\|_{L^\infty} \le r$ and $\|u_0(x) - \bar{u}\|_{L^1}$ sufficiently small and the nonlinear functions $f(u), g(u) \in C^7(\bar{B}(\bar{u}, 2r), \mathbb{R})$ satisfy $f(u) = O(1)|u - \bar{u}|^3$ and $g(u) = O(1)|u - \bar{u}|^6$ as $u \to \bar{u}$. Then, there exists a unique global smooth solution $u(t, x)$ for the Cauchy problem (1.1) such that*

$$\|u(t, x) - \bar{u}\|_{L^\infty} \le 2r, \quad t \ge 0. \tag{1.2}$$

We remark that the proof of Theorem 1.1 is greatly inspired from the work of Liu, Wang and Zhao [10] for the fourth order Cahn-Hilliard equaton. In [10], the authors used the Hoff and Smoller's method with a slight modification to prove the existence of global smooth solutions for the Cauchy problem of Cahn-Hilliard equation. Comparing with Liu, Wang and Zhao [10], our main difficulty is how to deal with the relation between the convective term $g(u)_x$ and the term $f(u)_{xx}$. In this paper, under the assumptions $g(u) = O(1)u^6$ and $f(u) = O(1)u^3$, we prove the local existence of smooth solution, establish the L^1-norm estimate for the problem (1.1), and extend up the local solution to all $t > 0$ by induction. In fact, a simple calculation shows that the relation between the dimension N and the orders of $f(u)$ and $g(u)$ are:

$$g(u) = O(1)u^{1+\frac{5}{N}} \quad \text{and} \quad f(u) = O(1)u^{1+\frac{2}{N}}.$$

In other words, Theorem 1.1 can be generalized to the N-dimensional case if $f(u)$ and $g(u)$ satisfy the above equalities. However, because of the physically relevant (the equation we consider models behavior of a crystal surface, thus even three dimensional model does not have any physical content), we only consider one-dimensional in this paper.

Remark 1.2 On the basis of Lemma 2.4, for problem (1.1), we can easily obtain $\|u - \bar{u}\|_{L^1} \le C$

and $\|u-\bar{u}\|_{L^6} \le Ct^{-\frac{5}{36}}$. By using Sobolev's embedding theorem, we easily obtain

$$\|u-\bar{u}\|_{L^2} \le \|u-\bar{u}\|_{L^1}^{\frac{2}{5}} \|u-\bar{u}\|_{L^6}^{\frac{3}{5}},$$

hence $\|u-\bar{u}\|_{L^2} \le Ct^{-\frac{1}{12}}$. Comparing with Miao, Yuan and Zhang [11]'s result on the generalized heat equations, we found that this decay rate is optimal.

The rest of this paper is organized as follows. In the next section, we introduce some preliminary results. The proof of Theorem 1.1 is postponed in the final section of this paper.

2. Preliminaries

The following three lemmas will be used in this section.

Lemma 2.1 ([12]) *If $1 \le p \le r \le q \le \infty$ and $u \in L^p(\mathbb{R}^N) \bigcap L^q(\mathbb{R}^N)$, then $u \in L^r(\mathbb{R}^N)$ with*

$$\|u\|_{L^r} \le \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \ \text{where} \ \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

Lemma 2.2 ([8]) *Suppose that $M(t)$ is a nonnegative continuous function of t. $M(t)$ satisfies*

$$M(t) \le d_1 + d_2 M(t)^r$$

in some interval containing 0, where d_1 and d_2 are positive constants and $r > 1$. If $M(0) \le d_1$ and

$$d_1 d_2 < (1 - r^{-1}) r^{-(r-1)^{-1}},$$

then in the same interval

$$M(t) \le \frac{d_1}{1 - r^{-1}}.$$

Lemma 2.3 ([10]) *Assume that $g(t)$ is a nonnegative continuous function defined on $[s, T]$ and satisfies*

$$g(t) \le N_1(t-b)(t-a)^{-\alpha} + N_2(t-b) \int_a^t (t-s)^{-\alpha} g(s) \mathrm{d}s,$$

where s, α, a and b are positive constants satisfying

$$0 < \alpha < 1, \quad s > \max\{a, b\},$$

and $N_i(t-b)$ $(i = 1, 2)$ are continuous increasing functions of t. Then,

$$g(t) \le (t-a)^{-\alpha} N(t-a, t-b) < \infty, \quad s \le t \le T$$

with

$$N(t-a, t-b) = N_1(t-b)\Big\{1 + \sum_{j=1}^{\infty} \frac{\Gamma(1-\alpha)}{\Gamma((j+1)(1-\alpha))} \times [\Gamma(1-\alpha) N_2(t-b)(t-a)^{1-\alpha}]^j\Big\}.$$

It is easy to see that $N(t-a, t-b)$ is a continuous increasing function of t.

In order to prove Theorem 1.1, we give the $L^p(\mathbb{R}^N, \mathbb{R})$-estimate on the fundamental solution to the sixth order convective Cahn-Hilliard equations.

Lemma 2.4 *Suppose that c_p and $c_{p,k}$ are positive constants with $c_1 = 0$ and $\mathcal{F}^{-1}$ is the inverse*

Fourier transformation with respect to ξ. *Assume that* $k(t,x)=\mathcal{F}^{-1}(e^{-|\xi|^6 t})$, *where* $\xi, x\in\mathbb{R}$ *and* $t>0$, *then*

$$\|k(t)\|_{L^p}\le c_p t^{-\frac{N}{6}(1-\frac{1}{p})}, \tag{2.1}$$

$$\|D^s k(t)\|_{L^p}\le c_{s,p} t^{-\frac{N}{6}(1-\frac{1}{p})-\frac{s}{6}},\quad s=1,2,\ldots. \tag{2.2}$$

Proof We set $\xi=\eta t^{-\frac{1}{6}}$. Hence

$$k(t,x)=\int_{\mathbb{R}^3} e^{-|\xi|^6 t}e^{ix\cdot\xi}\mathrm{d}\xi=t^{-\frac{N}{6}}\int_{\mathbb{R}^3} e^{-|\eta|^6}e^{ix\cdot\eta t^{-\frac{1}{6}}}\mathrm{d}\eta.$$

Let $G(y)=\int_{\mathbb{R}^3} e^{-|\eta|^6}e^{iy\cdot\eta}d\eta$. Clearly, $G(y)$ is a rapidly decreasing function. Then

$$\begin{aligned}\Big(\int_{\mathbb{R}^3}|k(t,x)|^p\mathrm{d}x\Big)^{\frac{1}{p}} &=t^{-\frac{N}{6}}\Big(\int_{\mathbb{R}^3}|G(xt^{-\frac{1}{6}}|^p\mathrm{d}x\Big)^{\frac{1}{p}}\\ &=t^{-\frac{N}{6}}t^{\frac{N}{6p}}\Big(\int_{\mathbb{R}^3}|G(z)|^p\mathrm{d}z\Big)^{\frac{1}{p}}=C_p t^{-\frac{N}{6}(1-\frac{1}{p})}.\end{aligned}$$

We complete the proof of (2.1). Moreover, the following equality holds:

$$D^k k(t,x)=t^{-\frac{N+k}{6}}D_y^k G(xt^{-\frac{1}{6}}).$$

By simple calculations, we obtain (2.2). The proof is completed. □

3. Proof of Theorem 1.1

In this section, we consider the global existence of smooth solutions for the Cauchy problem of the sixth order convective Cahn-Hilliard equation.

We first give the local existence result.

Lemma 3.1 *Suppose that* $\max_{u\in\bar{B}(\bar{u},2r)}\sum_{k=1}^{7}(|D^k f(u)|+|D^k g(u)|)=b$ *and the conditions listed in Theorem 1.1 are satisfied. Then, there exists a unique smooth solution* $u(t,x)$ *on the strip* $\Pi_{t_1}=\{(t,x):0<t\le t_1, x\in\mathbb{R}\}$ *which satisfies*

$$\|u(t,x)-\bar{u}\|_{L^\infty}\le 2r,\quad 0\le t\le t_1, \tag{3.1}$$

where $t_1=\min\{1,\frac{1}{1728b^3c_{1,4}^3},(\frac{5}{24bc_{1,1}})^{\frac{6}{5}}\}$. *Moreover, for each* $0<s_1'<s_2'<\cdots<s_7'<t\le t_1$, *we have*

$$\|D^k u(t,x)\|_{L^\infty}\le (t-s_k')^{-\frac{k}{6}}C_k(r,s_k'-s_1',t-s_k'),\quad k=1,2,\ldots,7, \tag{3.2}$$

where C_k *is a continuous increasing function of* $t-s_k'$.

Proof Since $u(t,x)$ is a smooth solution of problem (1.1), it should satisfy

$$\begin{aligned}u(t,x)=&\int k(t,x-y)u_0\mathrm{d}y-\int_0^t\mathrm{d}s\int D^4k(x-y,t-s)f(u(s,y))\mathrm{d}y+\\ &\int_0^t\mathrm{d}s\int Dk(x-y,t-s)g(u(s,y))\mathrm{d}y.\end{aligned} \tag{3.3}$$

In order to prove Lemma 3.1, we first prove that there is a sufficiently small $t_1>0$ such that there exists a unique continuous solution $u(t,x)$ for (3.3) on the strip Π_{t_1}. Then, if we can show that the solution obtained above is a smooth solution, such a $u(t,x)$ is indeed a local smooth solution to problem (1.1). Suppose that $T(t)u = k(t,x) * u(t,x)$. Hence, (3.3) is equivalent to

$$u(t,x) = T(t)u_0 - \int_0^t D^4T(t-s)f(u(s))\mathrm{d}s + \int_0^t DT(t-s)g(u(s))\mathrm{d}s. \tag{3.4}$$

Since $T(t)1 = 1$, to prove that there exists a local smooth solution for (3.3), we also need to show that there exists a local smooth solution

$$u(t,x) - \bar{u} = T(t)(u_0 - \bar{u}) - \int_0^t D^4T(t-s)f(u(s))\mathrm{d}s + \int_0^t DT(t-s)g(u(s))\mathrm{d}s. \tag{3.5}$$

By using the standard method of successive approximations [9, 10]: Set $u^{(0)}(t,x) = u_0(x)$ and for $n \geq 1$,

$$u^{(n)}(t,x)-\bar{u} = T(t)(u^{(0)}-\bar{u})-\int_0^t D^4T(t-s)f(u^{(n-1)}(s))\mathrm{d}s+\int_0^t DT(t-s)g(u^{(n-1)}(s))\mathrm{d}s. \tag{3.6}$$

It is easy to see that $u^{(n)}(t,x)$ is well defined on $[0,\infty)\times\mathbb{R}$ for each $n \geq 0$. Set $v^{(n)}(t,x) = u^{(n)}(t,x) - \bar{u}$ and

$$\P\chi\P = \sup_{(t,x)\in\Pi_{t_1}} |\chi(t,x)|.$$

By induction, we prove that if $t_1 = \min\{1, \frac{1}{1728b^3c_{1,4}^3}, (\frac{5}{24bc_{1,1}})^{\frac{6}{5}}\}$, then

$$\P v^{(n)}\P \leq 2r. \tag{3.7}$$

For the case $n = 0$, (3.7) holds from the assumption we imposed on the initial data. As to the case $n = 1$, applying Hausdorff-Young's inequality and the assumptions on the initial data, we derive that

$$\Big|\int k(t,x-y)(u_0(y)-\bar{u})\mathrm{d}y\Big| \leq \|u_0-\bar{u}\|_{L^\infty} \leq r. \tag{3.8}$$

It then follows from (3.6) and (3.8) that

$$\begin{aligned} |v^{(1)}(t,x)| \leq & r + bc_{1,4}\int_0^t (t-s)^{-\frac{2}{3}}\P v^{(0)}\P\mathrm{d}s + bc_{1,1}\int_0^t (t-s)^{-\frac{1}{6}}\P v^{(0)}\P\mathrm{d}s \\ \leq & r + 6rbc_{1,4}(t_1)^{\frac{1}{3}} + \frac{12}{5}rbc_{1,1}(t_1)^{\frac{5}{6}} \leq 2r. \end{aligned} \tag{3.9}$$

Hence, (3.7) is right for $n = 1$. In addition, assume that (3.7) holds for $n \leq m-1$, where $m \geq 1$ is a positive integer. We now show that (3.7) also holds for $n = m$. Applying Hausdorff-Young's inequality and (3.6), we deduce that

$$\begin{aligned} |v^{(m)}| \leq & r + b\int_0^t \|D^4k(t-s)\|_{L^1}\P v^{(m-1)}\P\mathrm{d}s + b\int_0^t \|Dk(t-s)\|_{L^1}\P v^{(m-1)}\P\mathrm{d}s \\ \leq & r + bc_{1,4}\P v^{(m-1)}\P\int_0^t (t-s)^{-\frac{2}{3}}ds + bc_{1,1}\P v^{(m-1)}\P\int_0^t (t-s)^{-\frac{1}{6}}\mathrm{d}s \\ \leq & r + 6rbc_{1,4}(t_1)^{\frac{1}{3}} + \frac{12}{5}rbc_{1,1}(t_1)^{\frac{5}{6}} \leq 2r. \end{aligned} \tag{3.10}$$

Therefore, (3.7) holds for $n = m$ and by induction, we show that (3.7) is true for any $n \geq 0$. In the following, we prove that $v^{(n)}(t,x)$ satisfies

$$\P v^{(n)} - v^{(n-1)}\P \le \frac{(\mathcal{C}_1(t_1)^{\frac{1}{3}})^{n-1}}{\Gamma(\frac{n-1}{2}+\frac{1}{2})}M_1 + \frac{(\mathcal{C}_2(t_1)^{\frac{5}{6}})^{n-1}}{\Gamma(\frac{n-1}{2}+\frac{1}{2})}M_2 \le \frac{\mathcal{C}_1^{n-1}M_1 + \mathcal{C}_2^{n-1}M_2}{\Gamma(\frac{n-1}{2}+\frac{1}{2})}, \quad n \ge 1, \tag{3.11}$$

where $M_1 = 3rbc_{1,4}\sqrt{\pi}$, $\mathcal{C}_1 = bc_{1,4}\sqrt{\pi}$, $M_2 = \frac{6}{5}rbc_{1,1}\sqrt{\pi}$ and $\mathcal{C}_2 = bc_{1,1}\sqrt{\pi}$. We show the estimate (3.11) by induction. For $n = 1$, we can obtain from (3.6) that

$$\begin{aligned}
&|v^{(1)}(t,x) - v^{(0)}(t,x)| \\
&\le \int_0^t \|D^4k(t-s)\|_{L^1}\P f(u^{(0)} - f(\bar{u})\P \mathrm{d}s + \int_0^t \|Dk(t-s)\|_{L^1}\P g(u^{(0)}) - g(\bar{u})\P \mathrm{d}s \\
&\le rbc_{1,4}\int_0^t (t-s)^{-\frac{2}{3}}\mathrm{d}s + rbc_{1,1}\int_0^t (t-s)^{-\frac{1}{6}}\mathrm{d}s \\
&\le 4bc_{1,4}(t_1)^{\frac{1}{3}} + \frac{6}{5}rbc_{1,1}(t_1)^{\frac{5}{6}} \le \frac{M_0 + M_1}{\sqrt{\pi}},
\end{aligned}$$

which implies that the estimate (3.11) is right for $n = 1$. Suppose that (3.11) holds for $n \le m-1$, where $m \ge 2$ is a positive integer. Then, we can get from (3.6) that

$$\begin{aligned}
&|v^{(m)}(t,x) - v^{(m-1)}(t,x)| \\
&\le \int_0^t \|D^4k(t-s)\|_{L^1}\P f(u^{(m-1)}) - f(u^{(m-2)})\P \mathrm{d}s + \\
&\quad \int_0^t \|Dk(t-s)\|_{L^1}\P g(u^{(m-1)}) - g(u^{(m-2)})\P \mathrm{d}s \\
&\le bc_{1,4}\int_0^t (t-s)^{-\frac{2}{3}}\frac{(\mathcal{C}_1 s^{\frac{1}{3}})^{m-2}}{\Gamma(\frac{m-2}{2}+\frac{1}{2})}M_1\mathrm{d}s + bc_{1,1}\int_0^t (t-s)^{-\frac{5}{6}}\frac{(\mathcal{C}_2 s^{\frac{5}{6}})^{m-2}}{\Gamma(\frac{m-2}{2}+\frac{1}{2})}M_2\mathrm{d}s \\
&\le bc_{1,4}\sqrt{\pi}\frac{\Gamma(\frac{m-2}{2}+1)}{\Gamma(\frac{m-2}{2}+\frac{3}{2})}(t_1)^{\frac{m-1}{3}}\frac{(\mathcal{C}_1)^{m-2}M_1}{\Gamma(\frac{m-2}{2}+\frac{1}{2})} + bc_{1,1}\sqrt{\pi}\frac{\Gamma(\frac{m-2}{2}+1)}{\Gamma(\frac{m-2}{2}+\frac{3}{2})}(t_1)^{\frac{5(m-1)}{6}}\frac{(\mathcal{C}_2)^{m-2}M_2}{\Gamma(\frac{m-2}{2}+\frac{1}{2})} \\
&\le \frac{(\mathcal{C}_1)^{m-1}M_1 + (\mathcal{C}_2)^{m-1}M_2}{\Gamma(\frac{m-2}{2}+\frac{1}{2})},
\end{aligned}$$

which implies that (3.11) is true for $n = m$. Then, by induction again, we derive that the estimate (3.11) holds for any $n \ge 1$. It is easy to see that $\sum_{n=0}^{\infty}\frac{(\mathcal{C}_1)^{m-1}M_1 + (\mathcal{C}_2)^{m-1}M_2}{\Gamma(\frac{m-2}{2}+\frac{1}{2})}$ is convergent. Then, by (3.11), we obtain $v^{(n)}(t,x)$ converges uniformly to $v(t,x) = u(t,x) - \bar{u}$ on the strip Π_{t_1}. It is clear that the unique limit $u(t,x)$ is a continuous solution of (3.3) on Π_{t_1}. In order to show that such a $u(t,x)$ obtained above is indeed a smooth solution of problem (1.1) on Π_{t_1}, we also need to obtain the regularity of $u(t,x)$. To do so, we need to derive the following estimates: For each $1 \le k \le 7$, $n \ge 1$, there exists a C_k which is a continuous increasing function of $t - s'_k$ such that

$$\|D^k u^{(n)}(t)\|_{L^\infty} \le (t - s'_k)^{-\frac{k}{6}} C_k(r, s'_k - s'_1, t - s'_k), \quad s'_k < t \le t_1. \tag{3.12}$$

By (3.6) and the semigroup property of $T(t)$, we derive that

$$u^{(n)}(t,x) - \bar{u} = T(t-\bar{t})(u^{(n)}(\bar{t},x) - \bar{u}) - \int_{\bar{t}}^t D^4T(t-s)f(u^{(n-1)}(s))\mathrm{d}s +$$

$$\int_{\bar{t}}^{t} DT(t-s)g(u^{(n-1)}(s))\mathrm{d}s. \tag{3.13}$$

For $k=1$, we have

$$Du^{(n)}(t,x) = DT(t-s_1')(u^{(n)}(s_1',x)-\bar{u}) - \int_{s_1'}^{t} D^5T(t-s)f(u^{(n-1)}(s))\mathrm{d}s + \int_{s_1'}^{t} D^2T(t-s)g(u^{(n-1)}(s)\mathrm{d}s.$$

On the basis of the Hausdorff-Young's inequality, we deduce that

$$\begin{aligned}\|Du^{(n)}(t)\|_{L^\infty} \le & c_{1,1}(t-\bar{s}_1')^{-\frac{1}{6}}r + bc_{1,5}\int_{a_1'}^{t}(t-s)^{-\frac{5}{6}}¶v^{(n-1)}¶\mathrm{d}s + bc_{1,2}\int_{s_1'}^{t}(t-s)^{-\frac{1}{3}}¶v^{(n-1)}¶\mathrm{d}s \\ \le & (t-\bar{s}_1')^{-\frac{1}{6}}C_1(r,t-s_1').\end{aligned}$$

For $k=2$, note that

$$D^2u^{(n)}(t,x) = D^2T(t-s_2')(u^{(n)}(s_2',x)-\bar{u}) - \int_{s_2'}^{t} D^5T(t-s)Df(u^{(n-1)}(s))\mathrm{d}s + \int_{s_2'}^{t} D^2T(t-s)Dg(u^{(n-1)})\mathrm{d}s.$$

Then, applying Hausdorff-Young's inequality and (3.12) with $k=1$ that

$$\begin{aligned}\|D^2u^{(n)}\|_{L^\infty} \le & c_{1,2}(t-s_2')^{-\frac{1}{3}}¶v^{(n)}¶ + bc_{1,5}\int_{s_2'}^{t}(t-s)^{-\frac{5}{6}}(s-s_1')^{-\frac{1}{6}}C_1(r,s-s_1')\mathrm{d}s + bc_{1,2}\int_{s_2'}^{t}(t-s)^{-\frac{1}{3}}(s-s_1')^{-\frac{1}{6}}C_1(r,s-s_1')\mathrm{d}s \\ \le & (t-s_2')^{-\frac{1}{3}}C_2(r,s_2'-s_1',t-s_2'),\end{aligned}$$

which implies that (3.12) holds for $k=2$ and $n\ge 1$. Now, suppose that (3.12) holds for $k\le m-1$ for some $3\le m\le 7$, i.e.,

$$\|D^ku^{(n)}\|_{L^\infty} \le (t-s_k')^{-\frac{k}{6}}C_k(r,s_k'-s_1',t-s_k'), \quad k=1,2,\ldots,m-1. \tag{3.14}$$

Therefore, combining (3.7), (3.12) and (3.14) together, applying Hausdorff-Young's inequality, we derive that

$$\begin{aligned}&\|D^mu^{(n)}\|_{L^\infty} \\ &\le c_{1,m}(t-s_m')^{-\frac{m}{6}}¶v^{(n)}¶ + bc_{1,5}\int_{s_m'}^{t}(t-s)^{-\frac{5}{6}}\sum_{\sum_{i=1}^m i\beta_i=m}\Pi_{i=1}^{m-1}\|(D^iu^{(n-1)}(s))^{\beta_i}\|_{L^\infty}\mathrm{d}s + \\ &\quad bc_{1,2}\int_{s_m'}^{t}(t-s)^{-\frac{1}{3}}\sum_{\sum_{i=1}^m i\beta_i=m}\Pi_{i=1}^{m-1}\|(D^iu^{(n-1)}(s))^{\beta_i}\|_{L^\infty}\mathrm{d}s \\ &\le 2rc_{1,m}(t-s_m')^{-\frac{m}{6}} + bc_{1,5}\int_{s_m'}^{t}(t-s)^{-\frac{5}{6}}C_1(r,s-s_1')\cdots C_{m-1}(r,s_{m-1}'-s_1',s-s_{m-1}')\mathrm{d}s+\end{aligned}$$

$$bc_{1,2}\int_{s'_m}^{t}(t-s)^{-\frac{1}{3}}C_1(r,s-s'_1)\cdots C_{m-1}(r,s'_{m-1}-s'_1,s-s'_{m-1})\mathrm{d}s$$
$$\leq (t-s'_m)^{-\frac{m}{6}}C_m(r,s'_m-s'_1,t-s'_m),$$

which implies that (3.12) is true for $k=m$ and $n\geq 1$. Consequently, by induction, we know that (3.12) holds for $1\leq k\leq 7$, $n\geq 1$. Since (3.12) is obtained, it is a routine matter to verify that for each $\varsigma>0$, $D^k u^{(n)}(t,x)$ converges uniformly to $D^k u(t,x)$ on $[\varsigma,t_1]\times\mathbb{R}$ for $k=1,2,\ldots,6$. Hence, we obtain $u(t,x)\in C^{1,6}([\varsigma,t_1]\times\mathbb{R})$ and since $\varsigma>0$ can be chosen sufficiently small, we get $u(t,x)\in C^{1,6}((0,t_1]\times\mathbb{R})$. Applying the above regularity result, we can show that the solution $u(t,x)$ is indeed a smooth solution to problem (1.1) on Π_{t_1} and (3.2) is the direct consequence of (3.13). Therefore, we complete the proof. □

In the following, we establish the certain $L^1(\mathbb{R},\mathbb{R})$ estimates on $u(t,x)$ on the time interval on which the smooth solutions exist.

Lemma 3.2 *Suppose that the solution $u(t,x)$ obtained in Lemma 3.1 has been extended up to time T ($T\geq t_1>0$) and the smooth properties and the a priori estimate (3.1) (and hence (3.2)) are kept unchanged. Then, for any $0<s'_1<\bar{s}'_1<t\leq T$, we have*

$$\|Du\|_{L^1}\leq (t-\bar{s}'_1)^{-\frac{1}{6}}\sup_{[0,t_1]}\|u-\bar{u}\|_{L^1}\mathcal{M}_1(r,t-\bar{s}'_1), \tag{3.15}$$

where $\mathcal{M}_k$ is a continuous increasing function of $t-\bar{s}'_1$.

Proof Note that

$$u(t,x)-\bar{u}=T(t-\bar{s}'_1)(u(\bar{s}'_1,x)-\bar{u})-\int_{\bar{s}'_1}^{t}D^4T(t-s)f(u(s))\mathrm{d}s+\int_{\bar{s}'_1}^{t}DT(t-s)g(u(s))\mathrm{d}s. \tag{3.16}$$

Then,

$$\|Du(t)\|_{L^1}\leq C_{1,1}(t-\bar{s}'_1)^{-\frac{1}{6}}\|u(\bar{s}'_1,x)-\bar{u}\|_{L^1}+bc_{1,4}\int_{\bar{s}'_1}^{t}(t-s)^{-\frac{2}{3}}\|Du(s)\|_{L^1}\mathrm{d}s+$$
$$bc_{1,1}\int_{\bar{s}'_1}^{t}(t-s)^{-\frac{1}{6}}\|Du(s)\|_{L^1}\mathrm{d}s.$$

On the basis of the singular Gronwall's inequality, we can easily derive that the estimate (3.15) holds. Then, the proof is completed. □

We also have the following lemma, which is concerned with the time-independent $L^1(\mathbb{R},\mathbb{R})$-a priori estimate on the solution $u(t,x)$. This estimate is very important in extending the local solution step by step to a global one.

Lemma 3.3 *Suppose that the assumptions listed in Lemma 3.2 are satisfied, then there exists a positive constant $C_1(r)$ depending only on r such that*

$$\|u-\bar{u}\|_{L^1}+t^{\frac{5}{36}}\|u-\bar{u}\|_{L^6}\leq C_1(r)\|u_0-\bar{u}\|_{L^1},\quad 0\leq t\leq T, \tag{3.17}$$

where the constant $C_1(r)$ is independent of T.

Proof Let

$$X=\{u(t,x):u(t,x)-\bar{u}\in C(0,T;L^1(\mathbb{R},\mathbb{R}),t^{\frac{5}{36}}(u(t,x)-\bar{u})\in C(0,T;L^6(\mathbb{R},\mathbb{R}))\},$$

with its norm defined by

$$\|u\|_X=\sup_{[0,T]}\{\|u\|_{L^1}+t^{\frac{5}{36}}\|u\|_{L^6}\}.$$

Then, it follows from (3.5) that

$$\begin{aligned}\|u-\bar{u}\|_X\le&\|T(t)(u_0-\bar{u})\|_X+\Big\|\int_0^t D^4T(t-s)f(u(s))\mathrm{d}s\Big\|_X+\Big\|\int_0^t DT(t-s)g(u(s))\mathrm{d}s\Big\|_X\\=&I_1+I_2+I_3.\end{aligned}$$

On the basis of the Hausdorff-Young's inequality, we have

$$\begin{aligned}I_1=&\sup_{[0,T]}\{\|T(t)(u_0-\bar{u})\|_{L^1}+t^{\frac{5}{36}}\|T(t)(u_0-\bar{u})\|_{L^6}\}\\\le&\sup_{[0,T]}\{\|k(t)\|_{L^1}\|u_0-\bar{u}\|_{L^1}+t^{\frac{5}{36}}\|K(t)\|_{L^6}\|u_0-\bar{u}\|_{L^1}\}\\\le&\sup_{[0,T]}\{(1+c_lt^{\frac{5}{36}-\frac{1}{6}(1-\frac{1}{6})}\})\|u_0-\bar{u}\|_{L^1}\}\\\le&(1+c_l)\|u_0-\bar{u}\|_{L^1}.\end{aligned}$$

For I_3, by employing a similar argument, we deduce that

$$\begin{aligned}I_3\le&\sup_{[0,T]}\Big\{\int_0^t\|DT(t-s)g(u(s))\|_{L^1}\mathrm{d}s+t^{\frac{5}{36}}\int_0^t\|DT(t-s)g(u(s))\|_{L^6}\mathrm{d}s\Big\}\\\le&C_2\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{1}{6}}\|u(s)-\bar{u}\|_{L^6}^6+t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{1}{6}}\|u(s)-\bar{u}\|_{L^6}^6\mathrm{d}s\Big)\\\le&C_2\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{1}{6}}s^{-\frac{5}{6}}\mathrm{d}s+t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{1}{6}}s^{-\frac{5}{6}}\mathrm{d}s\Big)\|u(s)-\bar{u}\|_X^6\\\le&C_3\|u-\bar{u}\|_X^6.\end{aligned}$$

As to I_2, we have

$$\begin{aligned}I_2\le&\sup_{[0,T]}\Big\{\int_0^t\|D^4T(t-s)f(u(s))\|_{L^1}\mathrm{d}s+t^{\frac{5}{36}}\int_0^t\|D^4T(t-s)f(u(s))\|_{L^6}\mathrm{d}s\Big\}\\\le&C_4\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{2}{3}}\|u(s)-\bar{u}\|_{L^3}^3+t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{2}{3}}\|u(s)-\bar{u}\|_{L^3}^3\mathrm{d}s\Big)\\\le&C_4\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{2}{3}}\|u(s)-\bar{u}\|_{L^1}^{\frac{3}{5}}\|u(s)-\bar{u}\|_{L^6}^{\frac{12}{5}}+\\&t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{2}{3}}\|u(s)-\bar{u}\|_{L^1}^{\frac{3}{5}}\|u(s)-\bar{u}\|_{L^6}^{\frac{12}{5}}\mathrm{d}s\Big)\\\le&C_4\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{2}{3}}s^{-\frac{1}{3}}\mathrm{d}s+t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{2}{3}}s^{-\frac{1}{3}}\mathrm{d}s\Big)\|u(s)-\bar{u}\|_{L^1}^{\frac{3}{5}}\|u(s)-\bar{u}\|_{L^6}^{\frac{12}{5}}\\\le&C_4\sup_{[0,T]}\Big(\int_0^t(t-s)^{-\frac{2}{3}}s^{-\frac{1}{3}}\mathrm{d}s+t^{\frac{5}{36}}\int_0^t(t-s)^{-\frac{1}{6}(1-\frac{1}{6})-\frac{2}{3}}s^{-\frac{1}{3}}\mathrm{d}s\Big)\|u(s)-\bar{u}\|_X^3\end{aligned}$$

$$\le C_5\|u-\bar{u}\|_X^3.$$

Summing up, we immediately conclude

$$\|u-\bar{u}\|_X \le (1+c_l)\|u-\bar{u}\|_{L^1} + C_5\|u-\bar{u}\|_X^3 + C_3\|u-\bar{u}\|_X^6.$$

Note that

$$C_5\|u-\bar{u}\|_X^3 \le \frac{1}{2}\|u-\bar{u}\|_X + C_6\|u-\bar{u}\|_X^6.$$

Combining the above two inequalities together gives

$$\|u-\bar{u}\|_X \le 2(1+c_l)\|u-\bar{u}\|_{L^1} + 2(C_3+C_6)\|u-\bar{u}\|_X^6.$$

If we suppose that $\|u_0-\bar{u}\|_{L^1}$ is sufficiently small, then we can get (3.17), immediately. The proof is completed. □

With the above preparations in hand, we now prove Theorem 1.1.

Proof of Theorem 1.1 Let β be a sufficiently small positive constant. Choose $0 < s_1 < \bar{s}_1 \le T$ sufficiently small such that $\bar{s}_1 \le t_1$ and

$$t_1 - \bar{s}_1 = \bar{s}_1 - s_1 = \beta. \tag{3.18}$$

It then follows form (3.15) and (3.17) that

$$\begin{cases} \|u(t)-\bar{u}\|_{L^1} \le C_1(r)\|u_0-\bar{u}\|_{L^1}, \quad 0\le t\le t_1, \\ \|u(t_1)-\bar{u}\|_{W^{1,1}} \le C_7(\beta,r,t_1)\sup_{[0,t_1]}\|u(t)-\bar{u}\|_{L^1}. \end{cases} \tag{3.19}$$

Assume that $\mathfrak{C}$ is the constant in Sobolev's embedding $\|u(t)-\bar{u}\|_{L^\infty} \le \mathfrak{C}\|u(t)-\bar{u}\|_{W^{1,1}}$, if we choose $\|u_0-\bar{u}\|_{L^1}$ sufficiently small such that

$$\mathfrak{C}C_1(r)C_7(\beta,r,t_1)\|u_0-\bar{u}\|_{L^1} \le \|u_0-\bar{u}\|_{L^\infty}. \tag{3.20}$$

Therefore,

$$\begin{aligned} \|u(t_1)-\bar{u}\|_{L^\infty} \le & \mathfrak{C}\|u(t_1)-\bar{u}\|_{W^{1,1}} \le \mathfrak{C}C_7(\beta,r,t_1)\sup_{[0,t_1]}\|u(t)-\bar{u}\|_{L^1} \\ \le & \mathfrak{C}C_1(r)C_7(\beta,r,t_1)\|u_0-\bar{u}\|_{L^1} \le \|u_0-\bar{u}\|_{L^\infty} \le r. \end{aligned}$$

On the basis of Lemmas 3.1 and 3.3, the solution $u(t,x)$ can be extended up to $2t_1$ and satisfies

$$\begin{cases} \|u(t)-\bar{u}\|_{L^\infty} \le 2r, & 0\le t\le 2t_1, \\ \|u(t)-\bar{u}\|_{L^1} \le C_1(r)\|u_0-\bar{u}\|_{L^1}, & 0\le t\le 2t_1. \end{cases} \tag{3.21}$$

Taking $t=2t_1$, $s_1' = s_1+t_1$ and $\bar{s}_1' = \bar{s}_1+t_1$ in (3.15) and (3.17), we derive that

$$\|u(2t_1)-\bar{u}\|_{W^{1,1}} \le C_7(\beta,r,t_1)\sup_{[0,2t_1]}\|u(t)-\bar{u}\|_{L^1}. \tag{3.22}$$

Now, assume that $u(t,x)$ has been defined up to kt_1 for some $k\in\mathbb{Z}_+$ such that

$$\begin{cases} \|u(t)-\bar{u}\|_{L^\infty} \le 2r, & 0\le t\le kt_1, \\ \|u(t)-\bar{u}\|_{L^1} \le C_1(r)\|u_0-\bar{u}\|_{L^1}, & 0\le t\le kt_1. \end{cases} \tag{3.23}$$

Taking $t = kt_1$, $s_1' = s_1 + (k-1)t_1$ and $\bar{s}_1' = \bar{s}_1 + (k-1)t_1$ in (3.15) and (3.17), we get

$$\|u(kt_1) - \bar{u}\|_{W^{1,1}} \le C_7(\beta, r, t_1) \sup_{[0,kt_1]} \|u(t) - \bar{u}\|_{L^1}. \tag{3.24}$$

It then follows from (3.20), (3.23) and (3.24) that

$$\begin{aligned}\|u(kt_1) - \bar{u}\|_{L^\infty} \le & \mathfrak{C}\|u(kt_1) - \bar{u}\|_{W^{1,1}} \le \mathfrak{C}C_7(\beta, r, t_1) \sup_{[0,t_1]} \|u(t) - \bar{u}\|_{L^1} \\ \le & \mathfrak{C}C_1(r)C_7(\beta, r, t_1)\|u_0 - \bar{u}\|_{L^1} \le \|u_0 - \bar{u}\|_{L^\infty} \le r.\end{aligned}$$

By using Lemmas 3.1 and 3.3 again, the solution $u(t, x)$ can be extended up to $(k+1)t_1$ and satisfies

$$\begin{cases} \|u(t) - \bar{u}\|_{L^\infty} \le 2r, & 0 \le t \le (k+1)t_1, \\ \|u(t) - \bar{u}\|_{L^1} \le C_1(r)\|u_0 - \bar{u}\|_{L^1}, & 0 \le t \le (k+1)t_1. \end{cases} \tag{3.25}$$

Proceeding inductively, we thus establish the existence of solution $u(x, t)$ in all $t > 0$. The proof is completed. $\square$

Acknowledgements We thank the referees for their suggestions.

References

[1] M. D. KORZEC, P. NAYAR, P. RYBKA. *Global weak solutions to a sixth order Cahn-Hilliard type equation*. SIAM J. Math. Anal., 2012, **44**(5): 3369–3387.

[2] M. D. KORZEC, P. RYBKA. *On a higher order convective Cahn-Hilliard-type equation*. SIAM J. Appl. Math., 2012, **72**: 1343–1360.

[3] M. D. KORZEC, P. NAYAR, P. RYBKA. *Global attractors of sixth order PDEs describing the faceting of growing surfaces*. J. Dynam. Differential Equations, 2016, **28**(1): 49–67.

[4] T. V. SAVINA, A. A. GOLOVIN, S. H. DAVIS, et al. *Faceting of a growing crystal surface by surface diffusion*. Phys Rev. E, 2003, **67**: 021606.

[5] M. D. KORZEC, P. L. EVANS, A. MÜNCH, et al. *Stationary solutions of driven fourth- and sixth-order Cahn-Hilliard-type equations*. SIAM J. Appl. Math., 2007, **69**: 348–374.

[6] D. HOFF, J. A. SMOOLER. *Global existence for parabolic conservation laws*. J. Differential Equations, 1987, **68**(2): 210–220.

[7] D. HOFF, J. A. SMOOLER. *Solutions in the large for certain nonlinear parabolic systems*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1985, **2**(3): 213–235.

[8] W. A. STRUSS. *Decay and asymptotic for $u_{tt} - \Delta u = F(u)$*. J. Funct. Anal., 1968, **2**: 409–457.

[9] Xiaxi DING, Jinghua WANG. *Global solution for a semilinear parabolic system*. Acta Math. Sci. (English Ed.), 1983, **3**(4): 397–414.

[10] Shuangqian LIU, Fei WANG, Huijiang ZHAO. *Global existence and asymptotics of solutions of the Cahn-Hilliard equation*. J. Differential Equations, 2007, **238**(2): 426–469.

[11] Changxing MIAO, Baoquan YUAN, Bo ZHANG. *Well-posedness of the Cauchy problem for the fractional power dissipative equations*. Nonlinear Anal., 2008, **68**(3): 461–484.

[12] J. C. ROBINSON, J. L. RODRIGO, W. SADOWSKI. *The Three-Dimensional Navier-Stokes Equations*. Cambridge University Press, Cambridge, 2016.