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A Class of Dually Flat Spherically Symmetric Finsler Metrics

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Abstract In this paper, we study spherically symmetric Finsler metrics. By analysing the solution of the spherically symmetric dually flat equation, we construct several new families of dually flat spherically symmetric Finsler metrics.

Keywords spherically symmetric; Finsler metrics; dually flat

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1. Introduction

Dually flat Finsler metrics on Riemann-Finsler manifolds were introduced by Amari-Nagaoka and Shen [1,2]. Recently the study of dually flat Finsler metrics has attached a lot of attention [3–6]. It has wide applications in information geometry, superstring theory and theory of relativity. For a Finsler metric $F = F(x, y)$ on a manifold M, the geodesics $c = c(t)$ of F in local coordinates (x^i) are characterized by

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + 2G^i(x, \frac{\mathrm{d}x}{\mathrm{d}t}) = 0,$$

where

$$G^i = \frac{1}{4}\mathrm{g}^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\},$$

$\mathrm{g}_{ij} = \frac{1}{2}[F^2]_{y^iy^j}$ and $(\mathrm{g}^{ij}) := (\mathrm{g}_{ij})^{-1}$. G^i are called the spray coefficients. A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point, there is a standard coordinate system (x^i) in TM such that

$$G^i = -\frac{1}{2}\mathrm{g}^{ij}H_{y^j},$$

where $H = H(x, y)$ is a locally scalar function on the tangent bundle TM which satisfies $H(x, \lambda y) = \lambda^3H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. In [2], Shen proved that a Finsler metric F on an open subset $\mathcal{U} \subset \mathbb{R}^k$ is dually flat

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if and only if it satisfies the following equations

$$(F^2)_{x^iy^j}y^i = 2(F^2)_{x^j}. \tag{1.1}$$

Besides, a Finster metric F is said to be spherically symmetric if F satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $A \in O(k)$ (see [7]).

In [8], the authors pointed out that a Finsler metric F on $\mathbb{B}^k(\nu)$ is spherically symmetric if and only if there exists a function $f : [0, \nu) \times \mathbb{R} \to \mathbb{R}$ such that

$$F = |y|\sqrt{f(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|})}.$$

Furthermore, in [6], the authors found a simple partial differential equation, which characterizes dually flat spherically symmetric Finsler metrics. Precisely, they showed

Theorem 1.1 ([6]) *Let $F = |y|\sqrt{f(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})}$ be a spherically symmetric function on $T\mathbb{B}^k(\nu)$. Then F is a solution of the following dually flat Eq.(1.1) if and only if*

$$sf_{ts} + f_{ss} - 2f_t = 0,$$

where $f = f(t, s)$.

Theorem 1.2 *Let $F : T\mathcal{U} \subseteq T\mathbb{R}^2 \to \mathbb{R}$ be a function defined by*

$$F(x^1, x^2, y^1, y^2) = \Big\{ \int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} (y^1 \cos\theta + y^2 \sin\theta)^{m+1}\zeta(x^1 \cos\theta + x^2 \sin\theta, \theta)\mathrm{d}\theta\Big\}^{\frac{1}{2}}, \tag{1.2}$$

where ζ is a positive continuous function and $\phi = \arg(y^1 + \sqrt{-1}y^2)$, $m \in \{1, 2, \ldots\}$. Then the function F satisfies the dually flat Eq. (1.1).

We will prove Theorem 1.2 in Section 2. As its application, we explicitly construct some new two-dimensional dually flat spherically symmetric metrics. Combining these Finsler metrics with Theorem 1.1, we obtain some new families of dually flat spherically symmetric Finsler metrics in Section 3.

2. Two-dimensional case

A Finsler metric $\Theta = \Theta(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^k$ is said to be projectively flat if all geodesics are straight in $\mathcal{U}$, equivalently, if it satisfies the following Hamel's equation [4]:

$$\Theta_{x^iy^j}y^i = \Theta_{x^j}, \tag{2.1}$$

where $x = (x^1, \ldots, x^k) \in \mathcal{U}$ and $y = y^j \frac{\partial}{\partial x^j} \in T_x\mathcal{U}$.

Lemma 2.1 ([9]) *Assume $\Theta : T\mathcal{U} \to \mathbb{R}$ is positively homogeneous of degree one with respect to y. Then Θ is a solution of projectively flat equation (2.1) if and only if it satisfies the following*

$$\Theta_{x^iy^j} = \Theta_{x^jy^i}. \tag{2.2}$$

Lemma 2.2 *Let*

$$\Theta(x,y)=\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}(y^1\cos\theta+y^2\sin\theta)^m f(x,\theta)\mathrm{d}\theta, \tag{2.3}$$

where f is a function on $\mathcal{U}\times\mathbb{R}$, $y=(y^1,y^2)$, $m\in\{1,2,\ldots\}$ and $\phi=\arg(y^1+\sqrt{-1}y^2)$. Then

$$\frac{\partial\Theta}{\partial y^j}=\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\frac{\partial}{\partial y^j}[(y^1\cos\theta+y^2\sin\theta)^m f(x,\theta)]\mathrm{d}\theta,$$

where $j\in\{1,2\}$.

Proof We can express $y=(y^1,y^2)$ in the polar coordinate system:

$$y^1=|y|\cos\phi,\quad y^2=|y|\sin\phi. \tag{2.4}$$

From (2.4), we have

$$\frac{\partial|y|}{\partial y^1}=\cos\phi,\ \frac{\partial|y|}{\partial y^2}=\sin\phi,\ \frac{\partial\phi}{\partial y^1}=-\frac{\sin\phi}{|y|},\ \frac{\partial\phi}{\partial y^2}=\frac{\cos\phi}{|y|}. \tag{2.5}$$

$$\Theta(x,y)=|y|^m\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}\frac{\partial\Theta}{\partial y^1}=&m|y|^{m-1}\cos\phi\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta+\\&|y|^m\frac{\partial}{\partial y^1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{\partial\Theta}{\partial y^2}=&m|y|^{m-1}\sin\phi\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta+\\&|y|^m\frac{\partial}{\partial y^2}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta.\end{aligned}$$

Let

$$g(x,\phi,\theta)=\int\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta.$$

It follow that

$$\frac{\partial g}{\partial\phi}=-m\int f(x,\theta)\sin(\phi-\theta)\cos^{m-1}(\phi-\theta)\mathrm{d}\theta, \tag{2.8}$$

$$\frac{\partial g}{\partial\theta}=f(x,\theta)\cos^m(\phi-\theta). \tag{2.9}$$

Thus, we have

$$\begin{aligned}&\frac{\partial}{\partial y^1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\phi-\theta)f(x,\theta)\mathrm{d}\theta=\frac{\partial g}{\partial y^1}(x,\phi,\phi+\frac{\pi}{2})-\frac{\partial g}{\partial y^1}(x,\phi,\phi-\frac{\pi}{2})\\&=\frac{\partial g}{\partial\phi}(x,\phi,\phi+\frac{\pi}{2})\frac{\partial\phi}{\partial y^1}+\frac{\partial g}{\partial\theta}(x,\phi,\phi+\frac{\pi}{2})\frac{\partial\theta}{\partial y^1}-\\&\quad\frac{\partial g}{\partial\phi}(x,\phi,\phi-\frac{\pi}{2})\frac{\partial\phi}{\partial y^1}-\frac{\partial g}{\partial\theta}(x,\phi,\phi-\frac{\pi}{2})\frac{\partial\theta}{\partial y^1}\end{aligned}$$

$$= \Big[\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial y^1} - \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y^1}\Big]\Big|_{\theta=\phi-\frac{\pi}{2}}^{\theta=\phi+\frac{\pi}{2}}. \tag{2.10}$$

From (2.5), (2.7) and (2.8)–(2.10), we have

$$\begin{aligned}\frac{\partial \Theta}{\partial y^1} =& m|y|^{m-1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \big[\cos\phi\cos^m(\phi-\theta)+\sin\phi\sin(\phi-\theta)\cos^{m-1}(\phi-\theta)\big]f(x,\theta)\mathrm{d}\theta\\ =& m|y|^{m-1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \cos\theta\cos^{m-1}(\phi-\theta)f(x,\theta)\mathrm{d}\theta\\ =& \int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \frac{\partial}{\partial y^1}\big[(y^1\cos\theta+y^2\sin\theta)^m f(x,\theta)\big]\mathrm{d}\theta.\end{aligned}$$

Similarly, we have

$$\frac{\partial \Theta}{\partial y^2} = \int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \frac{\partial}{\partial y^2}\big[(y^1\cos\theta+y^2\sin\theta)^m f(x,\theta)\big]\mathrm{d}\theta.$$

This completes the proof of Lemma 2.2. □

Corollary 2.3 *Let $\Theta=\Theta(x,y)$ denote the function on $T\mathcal{U}$ defined by (2.3). The function Θ is a solution of (2.2) if and only if*

$$\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \cos^{m-1}(\phi-\theta)\big[(\cos\theta)\frac{\partial f}{\partial x^2}-(\sin\theta)\frac{\partial f}{\partial x^1}\big]\mathrm{d}\theta=0 \tag{2.11}$$

for any ϕ.

Proof Lemma 2.1 gives that

$$\begin{aligned}&\frac{\partial^2\Theta}{\partial x^2\partial y^1}-\frac{\partial^2\Theta}{\partial x^1\partial y^2} = \frac{\partial}{\partial x^2}\Big(\frac{\partial\Theta}{\partial y^1}\Big)-\frac{\partial}{\partial x^1}\Big(\frac{\partial\Theta}{\partial y^2}\Big)\\ &= m|y|^{m-1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \cos^{m-1}(\phi-\theta)\big[\frac{\partial}{\partial x^2}f(x,\theta)\cos\theta-\frac{\partial}{\partial x^1}f(x,\theta)\sin\theta\big]\mathrm{d}\theta\\ &= m|y|^{m-1}\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}} \cos^{m-1}(\phi-\theta)\big[\frac{\partial f}{\partial x^2}\cos\theta-\frac{\partial f}{\partial x^1}\sin\theta\big]\mathrm{d}\theta.\end{aligned}$$

Thus (2.2) is equivalent to (2.11). □

Corollary 2.4 *Let $\Theta: T\mathcal{U}\to\mathbb{R}$ be a function defined by*

$$\Theta(x^1,x^2,y^1,y^2)=\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}(y^1\cos\theta+y^2\sin\theta)^m\rho(x^1\cos\theta+x^2\sin\theta,\theta)\mathrm{d}\theta, \tag{2.12}$$

where $\phi=\arg(y^1+\sqrt{-1}y^2)$ and $m\in\{1,2,\ldots\}$. Then the function Θ is a solution of (2.2).

Proof First of all, we consider the following equation

$$(\cos\theta)\frac{\partial f}{\partial x^2}-(\sin\theta)\frac{\partial f}{\partial x^1}=0. \tag{2.13}$$

Noticing that

$$\frac{\partial f}{\partial x^i}=\frac{\partial f}{\partial s}\frac{\partial s}{\partial x^i},$$

combining this with (2.13), we have

$$f(s,\theta)=\rho(x^1\cos\theta+x^2\sin\theta,\theta)$$

is the solution of (2.13), where ρ is a continuously differentiable function. It follows that Θ is a solution of (2.2) from Corollary 2.3. □

Lemma 2.5 ([5]) *Let $\Theta : T\mathcal{U}\to\mathbb{R}$ satisfy projectively flat Eq. (2.1) where $\mathcal{U}$ is an open subset in $\mathbb{R}^2$. Then*

$$F:=\sqrt{\Theta_{x^i}y^i} \tag{2.14}$$

is a solution of the dually flat Eq. (1.1).

The proof of Theorem 1.2 Let us consider the function Θ defined in (2.12), then

$$\frac{\partial\Theta}{\partial x^1}=\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}(y^1\cos\theta+y^2\sin\theta)^m\cos\theta\zeta(s,\theta)\mathrm{d}\theta, \tag{2.15}$$

and

$$\frac{\partial\Theta}{\partial x^2}=\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}(y^1\cos\theta+y^2\sin\theta)^m\sin\theta\zeta(s,\theta)\mathrm{d}\theta, \tag{2.16}$$

where $\zeta(s,\theta)=\frac{\partial\rho(s,\theta)}{\partial s}$. Plugging (2.15) and (2.16) into (2.14),

$$\begin{aligned}F=\sqrt{\Theta_{x^i}y^i}=&\Big\{\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}[(y^1\cos\theta+y^2\sin\theta)^m y^1\cos\theta\zeta(s,\theta)+\\&(y^1\cos\theta+y^2\sin\theta)^m y^2\sin\theta\zeta(s,\theta)]\mathrm{d}\theta\Big\}^{\frac{1}{2}}\\=&\Big\{\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}[(y^1\cos\theta+y^2\sin\theta)^{m+1}\zeta(s,\theta)]\mathrm{d}\theta\Big\}^{\frac{1}{2}}.\end{aligned}$$

Noting that $s=x^1\cos\theta+x^2\sin\theta$, we can derive (1.2). Combining this with Corollary 2.4 and Lemma 2.5, we complete the proof of Theorem 1.2. □

3. New dually flat spherically symmetric Finsler metrics

In this section, we are going to construct some new families of dually flat spherically symmetric Finsler metrics. Precisely, we obtain following

Theorem 3.1 *On $T\mathbb{B}^k(\nu)$, the following Finsler metric*

$$\begin{aligned}F:=|y|\Big\{\epsilon+\frac{|x|^{2l-1}|y|^{2k+2}}{2^{2(k+l-1)}}\Big[l\cos\beta\sum_{j=0}^{k}a_1(k,l,j)+\\\sum_{i=0}^{l-1}\sum_{j=0}^{k}\sum_{s=0}^{l-i}a_1'(k,l,i,j,s)(l-s-i\sin^2\beta)\cos^{2(l-i-s)-1}\beta\sin^{2s}\beta\Big]\Big\}^{\frac{1}{2}}\end{aligned}$$

is dually flat, with $k\in\{0,1,\ldots\}$, $l\in\{1,2,\ldots\}$ and

$$\cos\beta=\frac{\langle x,y\rangle}{|x||y|},\quad a_1(k,l,j)=\frac{C_{2k+1}^{j}C_{2l}^{l}(-1)^{k-j}}{2(k-j)+1},$$

$$a'_1(k,l,i,j,s)=\frac{(k-j+\frac{1}{2})(-1)^{k-j+l-i+s}C_{2l}^{i}C_{2k+1}^{j}C_{2(l-i)}^{2s}}{(k-j+\frac{1}{2})^2-(l-i)^2}.$$

Remark 3.2 If $k=0$, then Theorem 3.1 reduces to [5, Theorem 1.3].

Proof In (2.12), we take $\rho(s,\theta)=s^n$, where $n\in\{1,2,\ldots\}$. Then we obtain

$$\rho(x^1\cos\theta+x^2\sin\theta,\theta)=(x^1\cos\theta+x^2\sin\theta)^n. \tag{3.1}$$

We can express $x=(x^1,x^2)$ in the polar coordinate system

$$x^1=|x|\cos\varphi,\quad x^2=|x|\sin\varphi. \tag{3.2}$$

From (2.4), (2.12), (3.1) and (3.2), we have

$$\begin{aligned}\Theta(x,y)=&|x|^n|y|^m\int_{\phi-\frac{\pi}{2}}^{\phi+\frac{\pi}{2}}\cos^m(\theta-\phi)\cos^n(\theta-\varphi)\mathrm{d}\theta\\ =&|x|^n|y|^m\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^m\alpha\cos^n(\alpha+\beta)\mathrm{d}\alpha,\end{aligned} \tag{3.3}$$

where

$$\alpha:=\theta-\phi,\ \beta:=\phi-\varphi,\ (x,y)=(x^1,x^2,y^1,y^2). \tag{3.4}$$

Let $m=2k+1$, $n=2l$, where $k\in\{0,1,\ldots\}$, $l\in\{1,2.\ldots\}$. We have

$$\cos^{2l}(\alpha+\beta)=\frac{1}{2^{2l-1}}\Big[\frac{1}{2}C_{2l}^{l}+\sum_{i=0}^{l-1}C_{2l}^{i}\cos[(2l-2i)(\alpha+\beta)]\Big], \tag{3.5}$$

$$\cos^{2k+1}\alpha=\frac{1}{2^{2k}}\sum_{j=0}^{k}C_{2k+1}^{j}\cos[(2k+1-2j)\alpha]. \tag{3.6}$$

It follows that

$$\begin{aligned}\cos^{2k+1}\alpha\cos^{2l}(\alpha+\beta)=&\frac{1}{2^{2(k+l)}}\Big\{\sum_{j=0}^{k}C_{2l}^{l}C_{2k+1}^{j}\cos[(2k+1-2j)\alpha]+\\ &\sum_{i=0}^{l-1}\sum_{j=0}^{k}C_{2l}^{i}C_{2k+1}^{j}\cos[(2k+1-2j+2l-2i)\alpha+(2l-2i)\beta]+\\ &\sum_{i=0}^{l-1}\sum_{j=0}^{k}C_{2l}^{i}C_{2k+1}^{j}\cos[(2k+1-2j-2l+2i)\alpha-(2l-2i)\beta]\Big\}.\end{aligned} \tag{3.7}$$

By simple calculations, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2k+1-2j)\alpha]\mathrm{d}\alpha=\frac{2(-1)^{k-j}}{2(k-j)+1}, \tag{3.8}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2k+1-2j+2l-2i)\alpha+2(l-i)\beta]\mathrm{d}\alpha=\frac{2(-1)^{k-j+l-i}}{(2k+1-2j)+2(l-i)}\cos[2(l-i)\beta], \tag{3.9}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2k+1-2j-2l+2i)\alpha-2(l-i)\beta]\mathrm{d}\alpha=\frac{2(-1)^{k-j-l+i}}{(2k+1-2j)-2(l-i)}\cos[2(l-i)\beta]. \tag{3.10}$$

Plugging (3.7) into (3.3) and using (3.8)–(3.10), we have

$$\Theta = \frac{|x|^{2l}|y|^{2k+1}}{2^{2(k+l)-1}}\Big\{C_{2l}^{l}\sum_{j=0}^{k}\frac{C_{2k+1}^{j}(-1)^{k-j}}{2(k-j)+1}+ \sum_{i=0}^{l-1}\sum_{j=0}^{k}C_{2l}^{i}C_{2k+1}^{j}\frac{(k-j+\frac{1}{2})(-1)^{k-j+l-i}}{(k-j+\frac{1}{2})^2-(l-i)^2}\cos(2l-2i)\beta\Big\}. \tag{3.11}$$

A direct computation gives

$$\cos[(2l-2i)\beta] = \sum_{s=0}^{l-i}(-1)^s C_{2(l-i)}^{2s}\cos^{2(l-i-s)}\beta(1-\cos^2\beta)^s. \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\Theta = \frac{|x|^{2l}|y|^{2k+1}}{2^{2(k+l)-1}}\Big\{\sum_{j=0}^{k}a_1(k,l,j)+ \sum_{i=0}^{l-1}\sum_{j=0}^{k}\sum_{s=0}^{l-i}a_1'(k,l,i,j,s)\cos^{2(l-i-s)}\beta(1-\cos^2\beta)^s\Big\}, \tag{3.13}$$

where

$$a_1(k,l,j) = \frac{C_{2k+1}^{j}C_{2l}^{l}(-1)^{k-j}}{2(k-j)+1},\ a_1'(k,l,i,j,s) = \frac{(k-j+\frac{1}{2})(-1)^{k-j+l-i+s}C_{2l}^{i}C_{2k+1}^{j}C_{2(l-i)}^{2s}}{(k-j+\frac{1}{2})^2-(l-i)^2}.$$

From (2.4), (3.2) and (3.4), we have

$$\cos\beta = \frac{\langle x,y\rangle}{|x||y|},$$

where $|\ |$ and $\langle\ \rangle$ denote the standard Euclidean norm and inner product in $\mathbb{R}^2$. Now we compute $\Theta_{x^i}y^i$. By simple computations, we have

$$(|x|^{2l})_{x^i}y^i = 2l|x|^{2l-2}\langle x,y\rangle = 2l|x|^{2l-1}|y|\cos\beta, \tag{3.14}$$

$$(\cos^{2(l-i-s)}\beta)_{x^i}y^i = \frac{|y|}{|x|}2(l-i-s)\cos^{2(l-i-s)-1}\beta\sin^2\beta, \tag{3.15}$$

$$(1-\cos^2\beta)^s_{x^i}y^i = -\frac{|y|}{|x|}2s\cos\beta\sin^{2s}\beta. \tag{3.16}$$

From (3.14)–(3.16), we have

$$\begin{aligned}\Theta_{x^i}y^i =& \frac{|y|^{2k+1}}{2^{2(k+l)-1}}2l|x|^{2l-1}|y|\cos\beta\Big\{\sum_{j=0}^{k}a_1(k,l,j)+\sum_{i=0}^{l-1}\sum_{j=0}^{k}\sum_{s=0}^{l-i}a_1'(k,l,i,j,s)\times \\ &\cos^{2(l-i-s)}\beta(1-\cos^2\beta)^s\Big\}+\frac{|x|^{2l}|y|^{2k+1}}{2^{2(k+l)}}\sum_{i=0}^{l-1}\sum_{j=0}^{k}\sum_{s=0}^{l-i}a_1'(k,l,i,j,s)\times \\ &\Big\{2(l-i-s)\frac{|y|}{|x|}\cos^{2(l-i-s)-1}\beta\sin^{2+2s}\beta-2s\frac{|y|}{|x|}\cos^{2(l-i-s)+1}\beta\sin^{2s}\beta\Big\} \\ =& \frac{|x|^{2l-1}|y|^{2k+2}}{2^{2(k+l-1)}}\Big\{l\cos\beta\sum_{j=0}^{k}a_1(k,l,j)+\sum_{i=0}^{l-1}\sum_{j=0}^{k}\sum_{s=0}^{l-i}a_1'(k,l,i,j,s)\times\end{aligned}$$

$$(l-s-i\sin^2\beta)\cos^{2(l-i-s)-1}\beta\sin^{2s}\beta\Big\}.$$

Combining this with Corollary 2.4 and Lemma 2.5, we complete the proof of Theorem 3.1. □

Theorem 3.3 *The following spherically symmetric Finsler metric is dually flat,*

(i) $l\geq k$*:*

$$F:=|y|\Big\{\epsilon+\frac{\pi|x|^{2l}|y|^{2k+2}}{2^{2(k+l)+1}}\sum_{j=0}^{k}\sum_{s=0}^{k-j}a_2(k,l,j,s)\times [2l-2s+1-2(l-k+j)\sin^2\beta]\cos^{2(k-j-s)}\beta\sin^{2s}\beta\Big\}^{\frac{1}{2}},$$

(ii) $l<k$*:*

$$F:=|y|\Big\{\epsilon+\frac{\pi|x|^{2l}|y|^{2k+2}}{2^{2(k+l)+1}}\sum_{i=0}^{l}\sum_{s=0}^{l-i}a_2'(k,l,i,s)\times (2l-2s+1-2i\sin^2\beta)\cos^{2(l-i-s)}\beta\sin^{2s}\beta\Big\}^{\frac{1}{2}},$$

where $k,l\in\{0,1,\ldots\}$ *and*

$$a_2(k,l,j,s)=(-1)^sC_{2k+1-2j}^{2s}C_{2l+1}^{l-k+j}C_{2k+1}^{j},\ \ a_2'(k,l,i,s)=(-1)^sC_{2l+1-2i}^{2s}C_{2l+1}^{i}C_{2k+1}^{k-l+i}.$$

Remark 3.4 If $k=0$, then Theorem 3.3 reduces to Theorem 4.1 in [5].

Proof Let $m=2k+1$, $n=2l+1$, where $k,l\in\{0,1,2,\ldots\}$. From (3.3) and (3.6) we have

$$\begin{aligned}&\cos^{2k+1}\alpha\cos^{2l+1}(\alpha+\beta)\\&=\frac{1}{2^{2(k+l)+1}}\sum_{i=0}^{l}\sum_{j=0}^{k}C_{2l+1}^{i}C_{2k+1}^{j}\Big\{\cos[(2l-2i+2k-2j+2)\alpha+(2l+1-2i)\beta]+\\&\quad\cos[(2l-2i-2k+2j)\alpha+(2l+1-2i)\beta]\Big\}.\end{aligned}\tag{3.17}$$

By simple calculations, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2l-2i+2k-2j+2)\alpha+(2l+1-2i)\beta]\mathrm{d}\alpha=0,\tag{3.18}$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2l-2i-2k+2j)\alpha+(2l+1-2i)\beta]\mathrm{d}\alpha=\begin{cases}0, & \text{if }\ k-j\neq l-i\\ \pi\cos(2l+1-2i)\beta, & \text{if } k-j=l-i.\end{cases}\tag{3.19}$$

Plugging (3.17) into (3.3) and using (3.18), (3.19), we have

$$\Theta=\begin{cases}\frac{\pi|x|^{2l+1}|y|^{2k+1}}{2^{2(k+l)+1}}\sum\limits_{j=0}^{k}C_{2l+1}^{l-k+j}C_{2k+1}^{j}\cos(2k+1-2j)\beta, & l\geq k,\\ \frac{\pi|x|^{2l+1}|y|^{2k+1}}{2^{2(k+l)+1}}\sum\limits_{i=0}^{l}C_{2l+1}^{i}C_{2k+1}^{k-l+i}\cos(2l+1-2i)\beta, & l<k.\end{cases}\tag{3.20}$$

A direct computation gives

$$\cos[(2l+1-2i)\beta]=\sum_{s=0}^{l-i}(-1)^sC_{2l+1-2i}^{2s}\cos^{2(l-i-s)+1}\beta(1-\cos^2\beta)^s.\tag{3.21}$$

Substituting (3.21) into (3.20), we have

$$\Theta = \begin{cases} \frac{\pi|x|^{2l+1}|y|^{2k+1}}{2^{2(k+l)+1}} \sum_{j=0}^{k}\sum_{s=0}^{k-j} a_2(k,l,j,s)\cos^{2(l-i-s)+1}\beta(1-\cos^2\beta)^s, & l \geq k, \\ \frac{\pi|x|^{2l+1}|y|^{2k+1}}{2^{2(k+l)+1}} \sum_{i=0}^{l}\sum_{s=0}^{l-i} a_2'(k,l,i,s)\cos^{2(l-i-s)+1}\beta(1-\cos^2\beta)^s, & l < k, \end{cases} \tag{3.22}$$

where

$$a_2(k,l,j,s) = (-1)^s C_{2k+1-2j}^{2s} C_{2l+1}^{l-k+j} C_{2k+1}^{j},\ \ a_2'(k,l,i,s) = (-1)^s C_{2l+1-2i}^{2s} C_{2l+1}^{i} C_{2k+1}^{k-l+i}.$$

Now we compute $\Theta_{x^i}y^i$. First, we have

$$(|x|^{2l+1})_{x^i}y^i = (2l+1)|x|^{2l}|y|\cos\beta, \tag{3.23}$$

$$(\cos^{2(l-i-s)+1}\beta)_{x^i}y^i = \frac{|y|}{|x|}(2l-2i-2s+1)\cos^{2(l-i-s)}\beta\sin^2\beta. \tag{3.24}$$

From (3.16) and (3.22)–(3.24), we have

(i) $l \geq k$:

$$\Theta_{x^i}y^i = \frac{\pi|x|^{2l}|y|^{2k+2}}{2^{2(k+l)+1}} \sum_{j=0}^{k}\sum_{s=0}^{k-j} a_2(k,l,j,s)\times \\ [2l-2s+1-2(l-k+j)\sin^2\beta]\cos^{2(k-j-s)}\beta\sin^{2s}\beta,$$

(ii) $l < k$:

$$\Theta_{x^i}y^i = \frac{\pi|x|^{2l}|y|^{2k+2}}{2^{2(k+l)+1}} \sum_{i=0}^{l}\sum_{s=0}^{l-i} a_2'(k,l,i,s)\times \\ (2l-2s+1-2i\sin^2\beta)\cos^{2(l-i-s)}\beta\sin^{2s}\beta.$$

Combining these with Corollary 2.4 and Lemma 2.5, we complete the proof of Theorem 3.3. □

Theorem 3.5 *The following spherically symmetric Finsler metric*

(i) $l \geq k$:

$$F := |y|\Big\{\epsilon + \frac{\pi|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}\Big[lC_{2l}^{l}C_{2k}^{k}\cos\beta + \sum_{j=0}^{k-1}\sum_{s=0}^{k-j} a_3(k,l,j,s)\times \\ [l-s-(l-k+j)\sin^2\beta]\cos^{2(k-j-s)-1}\beta\sin^{2s}\beta\Big]\Big\}^{\frac{1}{2}},$$

(ii) $l < k$:

$$F := |y|\Big\{\epsilon + \frac{\pi|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}\Big[lC_{2l}^{l}C_{2k}^{k}\cos\beta + \sum_{i=0}^{l-1}\sum_{s=0}^{l-i} a_3'(k,l,i,s)\times \\ (l-s-i\sin^2\beta)\cos^{2(l-i-s)-1}\beta\sin^{2s}\beta\Big]\Big\}^{\frac{1}{2}}$$

is dually flat, with $k, l \in \{1, 2, \ldots\}$ *and*

$$a_3(k,l,j,s) = 2(-1)^s C_{2l}^{l-k+j} C_{2k}^{j} C_{2(k-j)}^{2s},\ \ a_3'(k,l,i,s) = 2(-1)^s C_{2l}^{i} C_{2k}^{k-l+i} C_{2(l-i)}^{2s}.$$

Proof Let $m=2k, n=2l$, where $k,l\in\{1,2,\ldots\}$. From (3.3) and (3.5), we have

$$\begin{aligned}&\cos^{2k}\alpha\cos^{2l}(\alpha+\beta)\\&=\frac{1}{2^{2(k+l)-1}}\Big\{\frac{1}{2}C_{2k}^kC_{2l}^l+\sum_{j=0}^{k-1}C_{2l}^lC_{2k}^j\cos(2k-2j)\alpha+\sum_{i=0}^{l-1}C_{2k}^kC_{2l}^i\cos[(2l-2i)(\alpha+\beta)]+\\&\sum_{i=0}^{l-1}\sum_{j=0}^{k-1}C_{2l}^iC_{2k}^j\cos[(2l-2i+2k-2j)\alpha+(2l-2i)\beta]+\\&\sum_{i=0}^{l-1}\sum_{j=0}^{k-1}C_{2l}^iC_{2k}^j\cos[(2l-2i-2k+2j)\alpha+(2l-2i)\beta]\Big\}.\end{aligned}\tag{3.25}$$

By simple calculations, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(2k-2j)\alpha\mathrm{d}\alpha=0,\tag{3.26}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(2k-2j)(\alpha+\beta)\mathrm{d}\alpha=0,\tag{3.27}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[(2l-2i+2k-2j)\alpha+(2l-2i)\beta]\mathrm{d}\alpha=0.\tag{3.28}$$

Plugging (3.25) into (3.3) and using (3.19) and (3.26)–(3.28), we have

$$\Theta=\begin{cases}\frac{|x|^{2l}|y|^{2k}}{2^{2(k+l)-1}}[\frac{\pi}{2}C_{2l}^lC_{2k}^k+\pi\sum\limits_{j=0}^{k-1}C_{2l}^{l-k+j}C_{2k}^j\cos(2k-2j)\beta], & l\ge k,\\ \frac{|x|^{2l}|y|^{2k}}{2^{2(k+l)-1}}[\frac{\pi}{2}C_{2l}^lC_{2k}^k+\pi\sum\limits_{i=0}^{l-1}C_{2l}^iC_{2k}^{k-l+i}\cos(2l-2i)\beta], & l<k.\end{cases}\tag{3.29}$$

Substituting (3.12) into (3.29), we have

$$\Theta=\begin{cases}\frac{\pi|x|^{2l}|y|^{2k}}{2^{2(k+l)}}[C_{2l}^lC_{2k}^k+\sum\limits_{j=0}^{k-1}\sum\limits_{s=0}^{k-j}a_3(k,l,j,s)\cos^{2(k-j-s)}\beta(1-\cos^2\beta)^s], & l\ge k,\\ \frac{\pi|x|^{2l}|y|^{2k}}{2^{2(k+l)}}[C_{2l}^lC_{2k}^k+\sum\limits_{i=0}^{l-1}\sum\limits_{s=0}^{l-i}a_3'(k,l,i,s)\cos^{2(l-i-s)}\beta(1-\cos^2\beta)^s], & l<k,\end{cases}\tag{3.30}$$

where

$$a_3(k,l,j,s)=2(-1)^sC_{2l}^{l-k+j}C_{2k}^jC_{2(k-j)}^{2s},\quad a_3'(k,l,i,s)=2(-1)^sC_{2l}^iC_{2k}^{k-l+i}C_{2(l-i)}^{2s}.$$

Now we compute $\Theta_{x^i}y^i$. From (3.14)–(3.16) and (3.30), we have

(i) $l\ge k$:

$$\begin{aligned}\Theta_{x^i}y^i=&\frac{\pi l|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}C_{2l}^lC_{2k}^k\cos\beta+\frac{\pi|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}\sum_{j=0}^{k-1}\sum_{s=0}^{k-j}a_3(k,l,j,s)\times\\&[l-s-(l-k+j)\sin^2\beta]\cos^{2(k-j-s)-1}\beta\sin^{2s}\beta\end{aligned}$$

(ii) $l<k$:

$$\Theta_{x^i}y^i=\frac{\pi l|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}C_{2l}^lC_{2k}^k\cos\beta+\frac{\pi|x|^{2l-1}|y|^{2k+1}}{2^{2(k+l)-1}}\sum_{i=0}^{l-1}\sum_{s=0}^{l-i}a_3'(k,l,i,s)\times$$

$$(l-s-i\sin^2\beta)\cos^{2(l-i-s)-1}\beta\sin^{2s}\beta.$$

Combining this with Corollary 2.4 and Lemma 2.5, we complete the proof of Theorem 3.5. □

Theorem 3.6 *The following spherically symmetric Finsler metric*

$$F := |y|\Big\{\epsilon + \frac{|x|^{2l}|y|^{2k+1}}{2^{2(k+l)-1}}\sum_{i=0}^{l}\sum_{s=0}^{l-i} a_4(k,l,i,j,s)\times$$
$$(2l-2s+1-2i\sin^2\beta)\cos^{2(l-i-s)}\beta\sin^{2s}\beta\Big\}^{\frac{1}{2}}$$

is dually flat, with $k \in \{1,2,\ldots\}$, $l \in \{0,1,\ldots\}$ *and*

$$a_4(k,l,i,j,s) = \frac{C_{2l-2i+1}^{2s}C_{2k}^{k}C_{2l+1}^{i}(-1)^{l-i+s}}{2(l-i)+1} + \sum_{j=0}^{k-1}\frac{C_{2l-2i+1}^{2s}C_{2l+1}^{i}C_{2k}^{j}(k-j+\frac{1}{2})(-1)^{l-i+k-j+s}}{(l-i+\frac{1}{2})^2-(k-j)^2}.$$

Proof Let $m = 2k$, $n = 2l+1$, where $l \in \{0,1,\ldots\}$, $k \in \{1,2,\ldots\}$. From (3.3), (3.5) and (3.6), we have

$$\cos^{2k}\alpha\cos^{2l+1}(\alpha+\beta) = \frac{1}{2^{2(k+l)}}\Big\{\sum_{i=0}^{l} C_{2k}^{k}C_{2l+1}^{i}\cos(2l+1-2i)(\alpha+\beta) +$$
$$\sum_{i=0}^{l}\sum_{j=0}^{k-1} C_{2l+1}^{i}C_{2k}^{j}\cos[(2l+1-2i+2k-2j)\alpha+(2l+1-2i)\beta] +$$
$$\sum_{i=0}^{l}\sum_{j=0}^{k-1} C_{2l+1}^{i}C_{2k}^{j}\cos[(2l+1-2i-2k+2j)\alpha+(2l+1-2i)\beta]\Big\}. \tag{3.31}$$

Plugging (3.31) into (3.3) and using (3.9), (3.10), we have

$$\Theta = \frac{|x|^{2l+1}|y|^{2k}}{2^{2(k+l)-1}}\Big\{\sum_{i=0}^{l}\Big[\frac{C_{2k}^{k}C_{2l+1}^{i}(-1)^{l-i}}{2(l-i)+1} + \sum_{j=0}^{k-1}\frac{C_{2l+1}^{i}C_{2k}^{j}(l-i+\frac{1}{2})(-1)^{l-i+k-j}}{(l-i+\frac{1}{2})^2-(k-j)^2}\Big]\times$$
$$\cos(2l+1-2i)\beta\Big\}. \tag{3.32}$$

Substituting (3.21) into (3.32), we have

$$\Theta = \frac{|x|^{2l+1}|y|^{2k}}{2^{2(k+l)-1}}\Big\{\sum_{i=0}^{l}\sum_{s=0}^{l-i} a_4(k,l,i,j,s)\cos^{2(l-i-s)+1}\beta(1-\cos^2\beta)^s\Big\}, \tag{3.33}$$

where

$$a_4(k,l,i,j,s) = \frac{C_{2l-2i+1}^{2s}C_{2k}^{k}C_{2l+1}^{i}(-1)^{l-i+s}}{2(l-i)+1} + \sum_{j=0}^{k-1}\frac{C_{2l-2i+1}^{2s}C_{2l+1}^{i}C_{2k}^{j}(l-i+\frac{1}{2})(-1)^{l-i+k-j+s}}{(l-i+\frac{1}{2})^2-(k-j)^2}.$$

Now we compute $\Theta_{x^i}y^i$. From (3.16), (3.23), (3.24) and (3.33), we have

$$\Theta_{x^i}y^i = \frac{|x|^{2l}|y|^{2k+1}}{2^{2(k+l)-1}} \sum_{i=0}^{l} \sum_{s=0}^{l-i} a_4(k,l,i,j,s)(2l+1-2s-2i\sin^2\beta)\cos^{2(l-i-s)}\beta\sin^{2s}\beta.$$

Combining this with Corollary 2.4 and Lemma 2.1, we complete the proof of Theorem 3.6. □

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