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Relations between the Positive Inertia Index of a $\mathbb{T}$-Gain Graph and That of Its Underlying Graph

Sai WANG[1,2], **Dengyin WANG**[1,*], **Fenglei TIAN**[3]

1. *School of Mathematics, China University of Mining and Technology, Jiangsu* 221116*, P. R. China;*
2. *Xuhai College, China University of Mining and Technology, Jiangsu* 221116 *P. R. China;*
3. *School of Management, Qufu Normal University, Shandong* 276826*, P. R. China*

Abstract Let $\mathbb{T}$ be the subgroup of the multiplicative group $\mathbb{C}^{\times}$ consisting of all complex numbers z with $|z| = 1$. A $\mathbb{T}$-gain graph is a triple $\Phi = (G, \mathbb{T}, \varphi)$ (or short for (G, φ)) consisting of a simple graph $G = (V, E)$, as the underlying graph of (G, φ), the circle group $\mathbb{T}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(v_i v_j) = \overline{\varphi(v_j v_i)}$ for any adjacent vertices v_i and v_j. Let $i_+(G, \varphi)$ (resp., $i_+(G)$) be the positive inertia index of (G, φ) (resp., G). In this paper, we prove that

$$-c(G) \le i_+(G, \varphi) - i_+(G) \le c(G),$$

where $c(G)$ is the cyclomatic number of G, and characterize all the corresponding extremal graphs.

Keywords complex unit gain graphs; inertia index

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1. Introduction

All graphs considered in this article are simple graphs. The gain graph is a graph whose edges are labeled orientably by elements of a group S. This means that, if an edge e in one direction has label $s \in S$, then in the other direction it has label s^{-1}, the inverse element of $s \in S$. The group S is called the gain group. A gain graph is a generalization of a signed graph where the gain group S has only two elements 1 and -1 (see [1] for detail).

A complex unit gain graph (also named as $\mathbb{T}$-gain graph) is a special gain graph whose gain group is the subgroup of all complex units in $\mathbb{C}^{\times}$, where $\mathbb{C}^{\times}$ is the multiplicative group of all T nonzero complex numbers. More definitely, a $\mathbb{T}$-gain graph, with gain group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, is a triple $\Phi = (G, \mathbb{T}, \varphi)$ (or short for (G, φ)) consisting of a graph $G = (V, E)$, as the underlying graph of (G, φ), the gain group $\mathbb{T}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(v_i v_j) = \varphi(v_j v_i)^{-1} = \overline{\varphi(v_j v_i)}$ for any pair adjacent vertices v_i and v_j. The gain set of a gain graph (G, φ) refers to the set $\{\varphi(e) : e \in \overrightarrow{E}\}$. A simple graph is equivalent to a $\mathbb{T}$-gain graph with gain set $\{1\}$, and a signed graph is equivalent to a $\mathbb{T}$-gain graph with gain set $\{1, -1\}$. Thus, the concept of $\mathbb{T}$-gain graphs is an extension of both simple graphs and signed graphs.

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* Corresponding author
E-mail address: 27624181@qq.com (Sai WANG); wongdein@163.com (Dengyin WANG)

The adjacency matrix $A(G,\varphi)$ of a $\mathbb{T}$-gain graph (G,φ) of order n is an $n\times n$ complex matrix (a_{ij}), where $a_{ij}=\overline{a_{ji}}=\varphi(v_iv_j)$ if v_i is adjacent to v_j and $a_{ij}=0$ if otherwise. Obviously, the adjacency matrix $A(G,\varphi)$ of a $\mathbb{T}$-gain graph (G,φ) is Hermitian and its eigenvalues are real. The rank of the $\mathbb{T}$-gain graph (G,φ) is defined to be the rank of the matrix $A(G,\varphi)$, denoted by $r(G,\varphi)$. Let $i_+(G,\varphi)$, $i_-(G,\varphi)$ be the numbers of positive and negative eigenvalues of (G,φ), called positive and negative inertia indices of (G,φ). It is obvious that $r(G,\varphi)=i_+(G,\varphi)+i_-(G,\varphi)$. Let $A(G)$ be the adjacency matrix of the simple graph G. The numbers of nonzero, positive and negative eigenvalues of G are called rank, positive inertia index and negative inertia index of G, denoted by $r(G)$, $i_+(G)$ and $i_-(G)$, respectively.

Let $G=(V,E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. $G-u$ (resp., $G-uv$) is the graph obtained form G by deleting the vertex $u\in V(G)$ (resp., $uv\in E(G)$). This notation is naturally extended if more than one vertex or edge are deleted. The value $c(G)=|E(G)|-|V(G)|+\omega(G)$ is called the cyclomatic number of a graph G, where $\omega(G)$ is the number of connected components of G. Denote by P_n and C_n a path and a cycle on n vertices, respectively.

An induced subgraph (U,φ_U) (short for (U,φ)) of (G,φ) is a $\mathbb{T}$-gain graph obtained from (G,φ) by deleting some vertices and the incident edges and the gain function φ_U is obtained by restricting φ to the edge set of this subgraph U. For a vertex u of (G,φ), we denote by $(G,\varphi)-u$ the induced subgraph obtained from (G,φ) by deleting the vertex u and all edges incident with u. Given a gain function, it is clear that $(G,\varphi)-(U,\varphi)=(G-U,\varphi)$.

In recent years, the complex unit gain graph attracts more and more researchers' attention. Reff [2] defined the adjacency, incidence, and Laplacian matrices of a complex unit gain graph, and obtained some eigenvalue bounds for the adjacency and Laplacian matrices of such graphs. Lu et al. [3] characterized the $\mathbb{T}$-gain connected bicyclic graphs with rank $2,3$ or 4. Lu et al. [4] obtained the relation between the rank of a $\mathbb{T}$-gain graph and the rank of its underlying graph. He et al. [5] studied the rank of a complex unit gain graph in terms of the matching number. Wang et al. [6] provided a combinatorial description of $\det(L(G))$, where $L(G)$ is the Laplacian matrix of a complex unit gain graph.

The study on the positive and negative inertia indices has been a popular subject in the graph theory. Ma and Geng [7] studied the difference between the positive and negative inertia indices of graph G. Li and Sun [8] determined some upper and lower bounds on the difference for weighted graphs. Fan and Wang [9] presented the bounds for $i_+(G)$ and $i_-(G)$ in terms of the matching number and dimension of cycle space of G and characterized the graphs achieving these upper and lower bounds. For more results on inertia of graphs, one can see [10–13]. Recently, Yu et al. [14] obtained some properties of inertia indexes of $\mathbb{T}$-gain graphs and they characterized the $\mathbb{T}$-gain unicyclic graphs with small positive or negative index. Motivated by this line, we are to study the relationship between the positive (resp., the negative) inertia index of a $\mathbb{T}$-gain graph and that of its underlying graph.

In order to show the main results of this paper, we give some necessary definitions.

The gain of a cycle $C:v_1v_2\cdots v_nv_1$, written as $\varphi(C)$, is the product $\varphi(v_1v_2)\varphi(v_2v_3)\cdots\varphi(v_nv_1)$. Note that for the same cycle C, if we write it as $C^*:v_1v_nv_{n-1}\cdots v_2v_1$, it has gain $\varphi(C^*)=$

$\varphi(v_1v_n)\varphi(v_nv_{n-1})\cdots\varphi(v_2v_1)$. It is obvious that $\varphi(C)$ and $\varphi(C^*)$ are conjugate numbers in $\mathbb{T}$.

Each $\mathbb{T}$-gain cycle (C_n, φ) is in one of the five types [14] defined below:

$$\begin{cases} \text{Type A} & \text{if } n \text{ is even and } \varphi(C_n) = (-1)^{n/2}; \\ \text{Type B} & \text{if } n \text{ is even and } \varphi(C_n) \neq (-1)^{n/2}; \\ \text{Type C} & \text{if } n \text{ is odd and } \mathrm{Re}((-1)^{\frac{n-1}{2}}\varphi(C_n)) > 0; \\ \text{Type D} & \text{if } n \text{ is odd and } \mathrm{Re}((-1)^{\frac{n-1}{2}}\varphi(C_n)) < 0; \\ \text{Type E} & \text{if } n \text{ is odd and } \mathrm{Re}(\varphi(C_n)) = 0, \end{cases}$$

where $Re(z)$ is the real part of the complex number z.

Assume that G is a graph with pairwise vertex-disjoint cycles. Two cycles in G are adjacent if there exists a vertex on one cycle adjacent to one vertex on another. Let T_G be the graph obtained from G by contracting each even cycle C into a vertex v_c and contracting each odd cycle C' into an edge $e_{c'}$ (according to (i)–(vi) as the following). Suppose that $w, s \in G$ do not lie on any cycle, then

(i) w and s are adjacent in T_G if and only if they are adjacent in G;

(ii) w and v_c are adjacent in T_G if and only if w is adjacent to one vertex on C in G;

(iii) w is adjacent to an endpoint of $e_{c'}$ in T_G if and only if w is adjacent to one vertex on C' in G;

(iv) v_{c_1} is adjacent to v_{c_2} in T_G if and only if the two even cycles C_1 and C_2 are adjacent in G;

(v) v_c is adjacent to an endpoint of $e_{c'}$ in T_G if and only if the even cycle C and the odd cycle C' are adjacent in G;

(vi) An endpoint of $e_{c'_1}$ is adjacent to one endpoint of $e_{c'_2}$ in T_G if and only if the two odd cycles C'_1 and C'_2 are adjacent in G.

Clearly, T_G is a forest. Note that given a graph G with pairwise vertex-disjoint cycles, T_G is not unique due to the contraction of odd cycles. The set of all acyclic graphs obtained from G by the transformation above is denoted by $\mathcal{T}_G$. Let $[T_G] = T_G - \{v_c, c \subseteq G\}$ and $[\mathcal{T}_G] = \{[T_G], T_G \in \mathcal{T}_G\}$. Let $n_e(G)$ (resp., $n_o(G)$) be the number of even cycle (resp., odd cycle).

A cycle C of a graph G is a pendant cycle if C has exactly one vertex with degree 3 and all degrees of other vertices on the cycle are 2 in G. Assume that C_l is a pendant cycle. If l is even, denote the graph formed from G by contracting C_l into a pendant vertex by $G\langle C_l\rangle$. The gain function of the $\mathbb{T}$-gain graph $(G\langle C_l\rangle, \varphi)$ is obtained by restricting that of (G, φ) to the edge set of $G\langle C_l\rangle$. If l is odd, denote the graph formed from G by contracting C_l into a pendant edge e_{C_l} by $G\langle C_l\rangle$. The gain function of the $\mathbb{T}$-gain graph $(G\langle C_l\rangle, \varphi)$ is obtained by restricting that of (G, φ) to the edge set of $G\langle C_l\rangle - e_{c_l}$ and setting e_{c_l} with an arbitrary gain.

We are ready to announce the main results of this paper.

Theorem 1.1 *Let (G, φ) be a $\mathbb{T}$-gain graph with underlying graph G. Then*

$$-c(G) \leq i_+(G, \varphi) - i_+(G) \leq c(G). \tag{1.1}$$

A $\mathbb{T}$-gain graph (G,φ) is called i_+-lower optimal if $i_+(G,\varphi)-i_+(G)=-c(G)$, whereas (G,φ) is called i_+-upper optimal if $i_+(G,\varphi)-i_+(G)=c(G)$. The next main results characterize the $\mathbb{T}$-gain graphs achieving the lower and upper bounds in (1.1).

Theorem 1.2 *Let (G,φ) be a $\mathbb{T}$-gain graph.*

(i) (G,φ) is i_+-lower optimal if and only if all the following conditions hold:

(a) G is a graph with pairwise vertex-disjoint cycles;

(b) Each $\mathbb{T}$-gain cycle (C_l,φ) of (G,φ) satisfies either $l\equiv 2 \pmod 4$, (C_l,φ) is Type A, or $l\equiv 1 \pmod 4$, (C_l,φ) is Type D or E;

(c) $i_+(G,\varphi)=i_+(T_G)+\sum_{C\subseteq G} i_+(C,\varphi)-n_o(G)$, $i_+(G)=i_+([T_G])+\sum_{C\subseteq G} i_+(C)-n_o(G)$ and $i_+(T_G)=i_+([T_G])$ for all $T_G\in\mathcal{T}_G$.

(ii) (G,φ) is i_+-upper optimal if and only if all the following conditions hold:

(a) G is a graph with pairwise vertex-disjoint cycles;

(b) Each $\mathbb{T}$-gain cycle (C_l,φ) of (G,φ) satisfies either $l\equiv 0 \pmod 4$, (C_l,φ) is Type B, or $l\equiv 3 \pmod 4$, (C_l,φ) is Type C;

(c) $i_+(G,\varphi)=i_+(T_G)+\sum_{C\subseteq G} i_+(C,\varphi)-n_o(G)$, $i_+(G)=i_+([T_G])+\sum_{C\subseteq G} i_+(C)-n_o(G)$ and $i_+(T_G)=i_+([T_G])$ for all $T_G\in\mathcal{T}_G$.

The rest of this paper is organized as follows. In Section 2, we recall some known results and give proof for Theorem 1.1. In Section 3, we present proof of Theorem 1.2. In Section 4, we give some conclusion remarks which extend the main results of this paper to the negative inertia index.

2. Elementary lemmas and Proof for Theorem 1.1

We begin with some known results as follows.

Lemma 2.1 ([15]) *Let M be an Hermitian matrix of order s, and let N be a principal submatrix of M with order t. If $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_s$ are the eigenvalues of M and $\mu_1\ge\mu_2\ge\cdots\ge\mu_t$ of N, then $\lambda_i\ge\mu_i\ge\lambda_{s-t+i}$ for $1\le i\le t$.*

Lemma 2.2 *Let (G,φ) be a $\mathbb{T}$-gain graph with $u\in G$. Then*

$$i_+(G,\varphi)-1\le i_+((G,\varphi)-u)\le i_+(G,\varphi).$$

Proof Assume that G contains n vertices. The eigenvalues of (G,φ) and $(G,\varphi)-u$ are

$$\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n \quad \text{and} \quad \mu_1\ge\mu_2\ge\cdots\ge\mu_{n-1}$$

respectively. Since $A((G,\varphi)-u)$ is the principal submatrix of $A(G,\varphi)$, by Lemma 2.1, we have

$$\mu_{i_+(G,\varphi)-1}\ge\lambda_{i_+(G,\varphi)}\ge\mu_{i_+(G,\varphi)}\ge\lambda_{i_+(G,\varphi)+1}\ge\mu_{i_+(G,\varphi)+1}.$$

Note that $\lambda_{i_+(G,\varphi)}>0$ and $\lambda_{i_+(G,\varphi)+1}\le 0$, we have $\mu_{i_+(G,\varphi)-1}>0$ and $\mu_{i_+(G,\varphi)+1}\le 0$. Hence , $i_+(G,\varphi)-1\le i_+((G,\varphi)-u)\le i_+(G,\varphi)$. □

Lemma 2.3 ([14]) *Let (G,φ) be a $\mathbb{T}$-gain graph with a pendant vertex v with the unique* *neighbor u. Then $i_+(G,\varphi) = i_+((G,\varphi) - u - v) + 1$ and $i_-(G,\varphi) = i_-((G,\varphi) - u - v) + 1$.*

Lemma 2.4 ([14]) *Let (T,φ) be a $\mathbb{T}$-gain tree. Then $A(T,\varphi)$ and $A(T)$ have the same spectrum.*

Lemma 2.5 ([14]) *Let $(G,\varphi) = (G_1,\varphi)\cup(G_2,\varphi)\cup\cdots\cup(G_t,\varphi)$, where $(G_1,\varphi),(G_2,\varphi),\ldots,(G_t,\varphi)$ are connected components of (G,φ). Then*

$$i_+(G,\varphi) = \sum_{i=1}^{t} i_+(G_i,\varphi), \quad i_-(G,\varphi) = \sum_{i=1}^{t} i_-(G_i,\varphi).$$

Lemma 2.6 ([14]) *Let (C_l,φ) be a $\mathbb{T}$-gain cycle. Then*

$$(i_+(C_l,\varphi), i_-(C_l,\varphi)) = \begin{cases} \left(\frac{l-2}{2},\frac{l-2}{2}\right), & \text{if } (C_l,\varphi) \text{ is of Type A,} \\ \left(\frac{l}{2},\frac{l}{2}\right), & \text{if } (C_l,\varphi) \text{ is of Type B,} \\ \left(\frac{l+1}{2},\frac{l-1}{2}\right), & \text{if } (C_l,\varphi) \text{ is of Type C,} \\ \left(\frac{l-1}{2},\frac{l+1}{2}\right), & \text{if } (C_l,\varphi) \text{ is of Type D,} \\ \left(\frac{l-1}{2},\frac{l-1}{2}\right), & \text{if } (C_l,\varphi) \text{ is of Type E.} \end{cases}$$

Lemma 2.7 ([9]) *Let C_l be a cycle with order l. Then*

$$(i_+(C_l), i_-(C_l)) = \begin{cases} \left(\frac{l-2}{2},\frac{l-2}{2}\right), & \text{if } l \equiv 0 \pmod 4, \\ \left(\frac{l}{2},\frac{l}{2}\right), & \text{if } l \equiv 2 \pmod 4, \\ \left(\frac{l+1}{2},\frac{l-1}{2}\right), & \text{if } l \equiv 1 \pmod 4, \\ \left(\frac{l-1}{2},\frac{l+1}{2}\right), & \text{if } l \equiv 3 \pmod 4. \end{cases}$$

Lemma 2.8 ([16]) *Let G be a graph with $u \in G$.*

(i) If u lies outside any cycle of G $c(G) = c(G-u)$;

(ii) If u lies on a cycle, then $c(G-u) \le c(G) - 1$;

(iii) If G contains cycles sharing common vertices, then there exists a common vertex u of cycles such that $c(G-u) \le c(G) - 2$;

(iv) If the cycles of G are pairwise vertex-disjoint, then $c(G)$ precisely equals the number of cycles in G.

With the above lemmas in hand, we are ready to give a proof for Theorem 1.1.

Proof of Theorem 1.1 We proceed by induction on $c(G)$. If $c(G) = 0$, then G is a forest. By Lemmas 2.4 and 2.5, the inequalities hold both in (1.1). Assume that the result holds for any $\mathbb{T}$-gain graph (G_1,φ), with $C(G_1) < C(G)$. Next we consider the case $c(G) \ge 1$, i.e., there is at least one cycle in G.

Let u be a vertex lying on a cycle in G. By Lemma 2.2, we have

$$i_+(G,\varphi) - 1 \le i_+((G,\varphi) - u) \le i_+(G,\varphi),$$

$$i_+(G) - 1 \le i_+(G-u) \le i_+(G).$$

From Lemma 2.8, we have $c(G-u) \le c(G) - 1 < c(G)$. The induction on hypothesis to $(G,\varphi) - u$ means that,

$$i_+(G-u) - c(G-u) \le i_+((G,\varphi) - u) \le i_+(G-u) + c(G-u).$$

Combining the results above, we obtain

$$\begin{aligned} i_+(G,\varphi) &\geq i_+((G,\varphi)-u) \geq i_+(G-u) - c(G-u) \\ &\geq (i_+(G)-1) - (c(G)-1) \geq i_+(G) - c(G), \end{aligned}$$

and

$$i_+(G,\varphi) \leq i_+((G,\varphi)-u) + 1 \leq i_+(G-u) + c(G-u) + 1 \leq i_+(G) + c(G).$$

Hence, we complete the proof of Theorem 1.1. □

In order to characterize all $\mathbb{T}$-gain graphs which are i_+-lower optimal and i_+-upper optimal, we give some useful results in the sequel of this section.

Proposition 2.9 *Let (G,φ) be a $\mathbb{T}$-gain graph with vertex u on some $\mathbb{T}$-gain cycle. If (G,φ) is i_+-lower optimal, then*

(i) $i_+(G,\varphi) = i_+((G,\varphi)-u)$;

(ii) $(G,\varphi)-u$ *is i_+-lower optimal;*

(iii) $i_+(G) = i_+(G-u)+1$;

(iv) $c(G-u) = c(G)-1$;

(v) *u lies on just one cycle of G and u is not a quasi-pendant vertex of G.*

Proof Since (G,φ) is i_+-lower optimal, from the proof of Theorem 1.1, all inequalities turn into equalities. Thus (i)–(iv) hold. Due to the arbitrariness of u, by Lemma 2.8 and (iv), u lies on just one cycle of G. Assume that u is a quasi-pendant vertex of u and v is the pendant neighbour of u. By Lemma 2.3, we have

$$\begin{aligned} i_+(G,\varphi) &= i_+((G,\varphi)-u-v)+1 \\ &= i_+((G,\varphi)-u) + i_+(\{v\}) + 1 \\ &= i_+((G,\varphi)-u)+1, \end{aligned}$$

a contradiction. □

The following similar Propositions 2.10 also holds, the proof of which goes parallel to that of Proposition 2.9, thus omitted.

Proposition 2.10 *Let (G,φ) be a $\mathbb{T}$-gain graph with vertex u on some $\mathbb{T}$-gain cycle. If (G,φ) is i_+-upper optimal, then*

(i) $i_+(G,\varphi) = i_+((G,\varphi)-u)$;

(ii) $(G,\varphi)-u$ *is i_+-upper optimal;*

(iii) $i_+(G) = i_+(G-u)+1$;

(iv) $c(G-u) = c(G)-1$;

(v) *u lies on just one cycle of G and u is not a quasi-pendant vertex of G.*

The following Propositions 2.11–2.14 hold for both i_+-lower optimal and i_+-upper optimal. Here we only show the case of i_+-lower optimal.

Proposition 2.11 *Suppose that (G,φ) is a $\mathbb{T}$-gain graph and v is a pendant vertex with unique* *neighbour u. Then (G,φ) is i_+-lower optimal if and only if $(G-u-v,\varphi)$ is i_+-lower optimal and u is not on any cycle of G.*

Proof Assume that (G,φ) is i_+-lower optimal, i.e., $i_+(G,\varphi)=i_+(G)-c(G)$. Since u is a quasi-pendant vertex, by Proposition 2.9, u is not on any cycle of G. From Lemmas 2.3 and 2.8, we have

$$\begin{aligned} i_+((G,\varphi)-u-v) &= i_+(G,\varphi)-1 = i_+(G)-c(G)-1 \\ &= i_+(G-u-v)+1-c(G-u-v)-1 \\ &= i_+(G-u-v)-c(G-u-v). \end{aligned}$$

Hence $(G-u-v,\varphi)$ is i_+-lower optimal.

Conversely, since $(G-u-v,\varphi)$ is i_+-lower optimal, we have

$$i_+((G,\varphi)-u-v) = i_+(G-u-v)-c(G-u-v).$$

Note that u is not on any cycle of G, in view of Lemmas 2.3 and 2.8, we can get

$$\begin{aligned} i_+(G,\varphi) &= i_+((G,\varphi)-u-v)+1 \\ &= i_+(G-u-v)-c(G-u-v)+1 \\ &= i_+(G)-1-c(G)+1 = i_+(G)-c(G), \end{aligned}$$

i.e., (G,φ) is i_+-lower optimal. □

Proposition 2.12 *Assume that (G,φ) is a $\mathbb{T}$-gain graph with underlying graph G which contains a pendant C_l. If (G,φ) is i_+-lower optimal, then $(G-C_l,\varphi)$ is i_+-lower optimal.*

Proof Let u be the unique vertex on C_l of degree 3. By Lemmas 2.3 and 2.4, we have

$$\begin{aligned} i_+((G,\varphi)-u) &= i_+(G-C_l,\varphi)+i_+(P_{l-1},\varphi) \\ &= i_+(G-C_l,\varphi)+i_+(P_{l-1}), \\ i_+(G-u) &= i_+(G-C_l)+i_+(P_{l-1}). \end{aligned}$$

Consequently, by Proposition 2.9, $(G-u,\varphi)$ is i_+-lower optimal, one has

$$\begin{aligned} i_+(G-C_l,\varphi)-i_+(G-C_l) &= i_+((G,\varphi)-u)-i_+(G-u) \\ &= -c(G-u) = -c(G-C_l). \end{aligned}$$

Hence we complete the proof. □

Proposition 2.13 *Assume that (G,φ) is a $\mathbb{T}$-gain graph with underlying graph G which contains a pendant C_l. If (G,φ) is i_+-lower optimal, then $(G\langle C_l\rangle,\varphi)$ is i_+-lower optimal.*

Proof Let u be the unique vertex on C_l of degree 3 and v be a neighbour of u on the cycle C_l. Then $c(G(\langle C_l\rangle)=c(G-v)=c(G)-1$. Since (G,φ) is i_+-lower optimal, one has $i_+(G,\varphi)=i_+(G)-c(G)$.

Case 1. l is even, then cycle C_l is contracted into a pendant vertex. Using Lemma 2.3

repeatedly, we have

$$i_+((G,\varphi)-v)=i_+(G\langle C_l\rangle,\varphi)+\frac{l-2}{2} \ \text{ and } \ i_+(G-v)=i_+(G\langle C_l\rangle)+\frac{l-2}{2}.$$

Thus

$$\begin{aligned}i_+(G\langle C_l\rangle,\varphi)&=i_+((G,\varphi)-v)-\frac{l-2}{2}\\&=i_+(G,\varphi)-\frac{l-2}{2}=i_+(G)-c(G)-\frac{l-2}{2}\\&=i_+(G-v)+1-c(G)-\frac{t-2}{2}\\&=(i_+(G\langle C_l\rangle)+\frac{t-2}{2})+1-(c(G\langle C_l\rangle)+1)-\frac{l-2}{2}\\&=i_+(G\langle C_l\rangle)-c(G\langle C_l\rangle).\end{aligned}$$

Case 2. l is odd, then cycle C_l is contracted into a pendant edge. Using Lemma 2.3 repeatedly, we have

$$i_+((G,\varphi)-v)=i_+(G\langle C_l\rangle,\varphi)+\frac{l-3}{2} \ \text{ and } \ i_+(G-v)=i_+(G\langle C_l\rangle)+\frac{l-3}{2}.$$

By the similar discussion as that in Case 1, $i_+(G\langle C_l\rangle,\varphi)=i_+(G\langle C_l\rangle)-c(G\langle C_l\rangle)$.

Hence $(G\langle C_l\rangle,\varphi)$ is i_+-lower optimal.

Proposition 2.14 *If a $\mathbb{T}$-gain graph (G,φ) is i_+-lower optimal, then*

$$i_+(T_G)=i_+(T'_G) \ \text{ and } \ i_+([T_G])=i_+([T'_G]),$$

for all $T_G,T'_G\in\mathcal{T}_G$ and all $[T_G],[T'_G]\in[\mathcal{T}_G]$.

Proof Assume that (G,φ) is i_+-lower optimal, then G is a graph with pairwise vertex-disjoint cycles (Proposition 2.9 (v)). Let T_G,T'_G be any two graphs in $\mathcal{T}_G$. Then $[T_G],[T'_G]$ are any two graphs in $[\mathcal{T}_G]$. Let $E^*(G)$ be set of edges which are not on the cycles of G. We proceed by induction on $|E^*(G)|$. If $|E^*(G)|=0$, then G consists of disjoint union of several cycles and isolated vertices. Hence, T_G,T'_G are both isomorphism to the disjoint union n_oK_2 and some isolated vertices, which leads to $[T_G]\cong[T'_G]$. The results hold.

Next we consider the case $|E^*(G)|\ge 1$. Assume that the results hold for i_+-lower optimal $\mathbb{T}$-gain graph (G_1,φ), with $|E^*(G_1)|<|E^*(G)|$.

Case 1. G has a pendant vertex v. Suppose that u is the unique neighbour of vertex v. By Proposition 2.9 (v), u is not on a cycle of G. then uv is a pendant edge of T_G and $[T_G]$ (resp., T'_G and $[T'_G]$). It is clear that

$$\begin{aligned}&T_G-u-v=T_{G-u-v}\in\mathcal{T}_G,\ T'_G-u-v=T'_{G-u-v}\in\mathcal{T}_G,\\&[T_G]-u-v=[T_{G-u-v}]\in[\mathcal{T}_G],\ [T'_G]-u-v=[T'_{G-u-v}]\in[\mathcal{T}_G].\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}&i_+(T_G)=i_+(T_G-u-v)+1, i_+(T'_G)=i_+(T'_G-u-v)+1,\\&i_+([T_G])=i_+([T_G]-u-v)+1, i_+([T'_G])=i_+([T'_G]-u-v)+1.\end{aligned}$$

By Proposition 2.11, $(G-u-v,\varphi)$ is i_+-lower optimal. And $|E^*(G-u-v)|<|E^*(G)|$, by induction on $(G-u-v,\varphi)$, it follows that

$$i_+(T_{G-u-v}) = i_+(T'_{G-u-v}) \text{ and } i_+([T_{G-u-v}]) = i_+([T'_{G-u-v}]).$$

Consequently, we infer that

$$\begin{aligned} i_+(T_G) &= i_+(T_G-u-v)+1 = i_+(T_{G-u-v})+1 \\ &= i_+(T'_{G-u-v})+1 = i_+(T'_G-u-v)+1 = i_+(T'_G) \end{aligned}$$

and

$$\begin{aligned} i_+([T_G]) &= i_+([T_G]-u-v)+1 = i_+([T_{G-u-v}])+1 \\ &= i_+(T'_{G-u-v})+1 = i_+([T'_G-u-v])+1 = i_+([T'_G]). \end{aligned}$$

Case 2. G contains no pendant vertices. Observe that $|E^*(G)|\geq 1$ and G is a graph with pairwise vertex-disjoint cycles, then G must contain a pendant cycle C_l. It can easily be seen that $\mathcal{T}_G = \mathcal{T}_{G\langle C_l\rangle}$ according to the contraction rules. Then for any $T_G, T'_G \in \mathcal{T}_G$, there exist $T_{G\langle C_l\rangle}, T'_{G\langle C_l\rangle} \in \mathcal{T}_{G\langle C_l\rangle}$, such that

$$T_G \cong T_{G\langle C_l\rangle} \text{ and } T'_G \cong T'_{G\langle C_l\rangle}, \tag{2.1}$$

which also hold for $[T_G]$ and $[T'_G]$. We proceed with the parity of l.

Subcase 2.1. l is even, by Proposition 2.13, $(G\langle C_l\rangle,\varphi)$ is i_+-lower optimal. Notice that $|E^*((G\langle C_l\rangle)| = |E^*(G)|$ and the the graph $G\langle C_l\rangle$ contains a pendant vertex, proceeding as in the proof of Case 1, we have

$$i_+(T_{G\langle C_l\rangle}) = i_+(T'_{G\langle C_l\rangle}), \quad i_+([T_{G\langle C_l\rangle}]) = i_+([T'_{G\langle C_l\rangle}]).$$

Combining with (2.1), we obtain

$$i_+(T_G) = i_+(T'_G) \text{ and } i_+([T_G]) = i_+([T'_G]).$$

Subcase 2.2. l is odd. Assume that C_l is contracted to a pendant edge e_{c_l}, then the graph $G-C_l$ can be obtained from $G\langle C_l\rangle$ by deleting the two endpoints of e_{c_l}. Alternatively, T_{G-C_l} (resp., $[T_{G-C_l}]$) can be obtained from some $T_{G\langle C_l\rangle}$ (resp., $[T_{G\langle C_l\rangle}]$) by deleting the two endpoints of e_{c_l}, so do the graphs T'_{G-C_l} (resp., $[T'_{G-C_l}]$). By Lemma 2.3, we have

$$i_+(T_{G\langle C_l\rangle}) = i_+(T_{G-C_l})+1, i_+(T'_{G\langle C_l\rangle}) = i_+(T'_{G-C_l})+1; \tag{2.2}$$

$$i_+([T_{G\langle C_l\rangle}]) = i_+([T_{G-C_l}])+1, i_+([T'_{G\langle C_l\rangle}]) = i_+([T'_{G-C_l}])+1. \tag{2.3}$$

By Proposition 2.12, $(G-C_l,\varphi)$ is i_+-lower optimal. And $|E^*(G-C_l)|<|E^*(G)|$, by induction on $(G-C_l,\varphi)$, it follows that

$$i_+(T_{G-C_l}) = i_+(T'_{G-C_l}) \text{ and } i_+([T_{G-C_l}]) = i_+([T'_{G-C_l}]). \tag{2.4}$$

Combining (2.1)–(2.4), we have

$$i_+(T_G) = i_+(T'_G) \text{ and } i_+([T_G]) = i_+([T'_G]).$$

The proof is completed. □

Remark 2.15 Proposition 2.14 shows that if a $\mathbb{T}$-gain graph (G,φ) is i_+-lower optimal (resp., i_+-upper optimal), then the positive inertia of T_G and $[T_G]$ are irrelevant to the contraction of cycles.

3. Proof for Theorem 1.2

In this section, we give proof for Theorem 1.2. We begin with several Lemmas for latter use.

Lemma 3.1 *Let (G,φ) be a $\mathbb{T}$-gain graph with exactly one cycle C_l. If (G,φ) is i_+-lower optimal, then either $l\equiv 2 \pmod 4$, (C_l,φ) is Type A, or $l\equiv 1 \pmod 4$, (C_l,φ) is Type D or E.*

Proof Let $E^*(G)$ be set of edges which are not on the cycles of G. We proceed by induction on $|E^*(G)|$. If $|E^*(G)|=0$, then G consists of disjoint union of exactly one cycle and isolated vertices.

If $l\equiv 0 \pmod 4$, by Lemma 2.7, $i_+(C_l)=\frac{l-2}{2}$. Since (G,φ) is i_+-lower optimal, which leads to

$$i_+(G,\varphi)=i_+(G)-c(G)=i_+(C_l)-c(C_l)=\frac{l-2}{2}-1.$$

According to Lemma 2.6, (C_l,φ) does not exist.

If $l\equiv 1 \pmod 4$, by Lemma 2.7, $i_+(C_l)=\frac{l+1}{2}$. Since (G,φ) is i_+-lower optimal, then

$$i_+(G,\varphi)=i_+(G)-c(G)=i_+(C_l)-c(C_l)=\frac{l+1}{2}-1.$$

In view of 2.6, (C_l,φ) is Type D or E.

If $l\equiv 2 \pmod 4$, by Lemma 2.7, $i_+(C_l)=\frac{l}{2}$. Since (G,φ) is i_+-lower optimal, then

$$i_+(G,\varphi)=i_+(G)-c(G)=i_+(C_l)-c(C_l)=\frac{l}{2}-1.$$

In view of 2.6, (C_l,φ) is Type A.

If $l\equiv 3 \pmod 4$, by similar argument, (C_l,φ) does not exist. Hence, the result holds for $|E^*(G)|=0$.

It remains to show the case (G,φ) with exactly one C_l and $|E^*(G)|\geq 1$. Then there exists a pendant vertex $v\in V(G)$ with unique neighbour u. Since (G,φ) is i_+-lower optimal, by Proposition 2.11, $(G-u-v,\varphi)$ is i_+-lower optimal and u does not lie on the cycle C_l. By induction, the result holds for the $\mathbb{T}$-gain graph $(G-u-v,\varphi)$ with only one cycle and $|E^*(G-u-v)|<|E^*(G)|$. Hence, $(G-u-v,\varphi)$ satisfies either $l\equiv 2 \pmod 4$, (C_l,φ) is Type A, or $l\equiv 1 \pmod 4$, (C_l,φ) is Type D or E, so does in (G,φ).

This completes the proof. □

Lemma 3.2 *Let (G,φ) be a $\mathbb{T}$-gain graph in which each $\mathbb{T}$-gain cycle, say (C_t,φ), satisfies either $l\equiv 2 \pmod 4$, (C_t,φ) is Type A, or $t\equiv 1 \pmod 4$, (C_t,φ) is Type D or E. If (G,φ) is i_+-lower optimal, then for all $T_G\in\mathcal{T}_G$*

$$i_+(G,\varphi)=i_+(T_G)+\sum_{C\subseteq G} i_+(C,\varphi)-n_o(G),$$

$$i_+(G) = i_+([T_G]) + \sum_{C \subseteq G} i_+(C) - n_o(G).$$

Proof Since (G, φ) is i_+-lower optimal, then G is a graph with pairwise vertex-disjoint cycles. By Proposition 2.14, the positive inertia of T_G and $[T_G]$ are irrelevant to the contraction of cycles. It suffices to choose any T_G in $\mathcal{T}_G$. Let $E^*(G)$ be set of edges which are not on the cycles of G. We proceed by induction on $|E^*(G)|$.

If $|E^*(G)| = 0$, then G consists of disjoint union of several cycles and isolated vertices. Then

$$i_+(G, \varphi) = \sum_{C \subseteq G} i_+(C, \varphi), \quad i_+(G) = \sum_{C \subseteq G} i_+(C).$$

By the definitions of T_G and $[T_G]$, we have

$$T_G \cong n_o(G) K_2 \cup (|V(T_G)| - 2n_o(G)) K_1,$$

$$[T_G] \cong n_o(G) K_2 \cup (|V(T_G)| - 2n_o(G) - n_e(G)) K_1.$$

Then $i_+(T_G) = i_+([T_G]) = n_o(G)$. Hence

$$i_+(G, \varphi) = i_+(T_G) + \sum_{C \subseteq G} i_+(C, \varphi) - n_o(G),$$

$$i_+(G) = i_+([T_G]) + \sum_{C \subseteq G} i_+(C) - n_o(G).$$

Assume the result holds for i_+-lower optimal $\mathbb{T}$-gain graph (G, φ), whose each cycle satisfies the condition and $|E^*(G_1) < |E^*(G)|$. Next we consider the case (G, φ) with $|E^*(G)| \geq 1$, in which all cycles satisfy the hypothesis. We divide our proof in two cases.

Case 1. G has a pendant vertex v. Suppose that u is the unique neighbour of vertex v. By Proposition 2.9 (v), u is not on a cycle of G. then uv is a pendant edge of T_G and $[T_G]$. It is clear that

$$T_G - u - v = T_{G-u-v} \in \mathcal{T}_G, \quad [T_G] - u - v = [T_{G-u-v}] \in [\mathcal{T}_G]. \tag{3.1}$$

By Lemma 2.3, we have

$$i_+(T_G) = i_+(T_G - u - v) + 1, i_+(G, \varphi) = i_+(G - u - v, \varphi) + 1, \tag{3.2}$$

$$i_+([T_G]) = i_+([T_G] - u - v) + 1, i_+(G) = i_+(G - u - v) + 1. \tag{3.3}$$

By Proposition 2.11, $(G - u - v, \varphi)$ is i_+-lower optimal. And $|E^*(G - u - v)| < |E^*(G)|$, by induction on $(G - u - v, \varphi)$, it follows that

$$i_+(G - u - v, \varphi) = i_+(T_{G-u-v}) + \sum_{C \subseteq G-u-v} i_+(C, \varphi) - n_o(G - u - v), \tag{3.4}$$

$$i_+(G - u - v) = i_+([T_{G-u-v}]) + \sum_{C \subseteq G-u-v} i_+(C) - n_o(G - u - v). \tag{3.5}$$

Then combining (3.1)–(3.5), we have

$$\begin{aligned} i_+(G, \varphi) &= i_+((G, \varphi) - u - v) + 1 = i_+(G - u - v, \varphi) + 1 \\ &= i_+(T_{G-u-v}) + \sum_{C \subseteq G-u-v} i_+(C, \varphi) - n_o(G - u - v) + 1 \end{aligned}$$

$$= i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G)$$

and

$$i_+(G) = i_+(G-u-v)+1 = i_+([T_{G-u-v}]) + \sum_{C\subseteq G-u-v} i_+(C) - n_o(G-u-v)+1$$
$$= i_+([T_G]) + \sum_{C\subseteq G} i_+(C) - n_o(G).$$

Case 2. G contains no pendant vertices. Observe that $|E^*(G)|\geq 1$ and G is a graph with pairwise vertex-disjoint cycles, then G must contain a pendant cycle C_l. Let u be the unique vertex on C_l of degree 3. Let v be a neighbour of u on cycle C_l. Then it is sufficient to consider the following two subcases.

Subcase 2.1. $l\equiv 1 \pmod 4$, (C_l,φ) is Type D or E. Recall the definitions of T_G and $[T_G]$, then each T_{G-C_l} (resp., $[T_{G-C_l}]$) can be obtained from T_G (resp., $[T_G]$) by deleting two endpoints of the pendant edge which is contracted by C_l. By Lemma 2.3,

$$i_+(T_G) = i_+(T_{G-C_l})+1,\quad i_+([T_G]) = i_+([T_{G-C_l}])+1. \tag{3.6}$$

Since $l\equiv 1 \pmod 4$, (C_l,φ) is Type D or E, in view of Lemmas 2.6 and 2.7,

$$i_+(C_l,\varphi) = \frac{l-1}{2},\quad i_+(C_l) = \frac{l+1}{2}. \tag{3.7}$$

Note that $(G-C_l,\varphi)$ is i_+-lower optimal (Proposition 2.11), and $|E^*(G-C_l)|<|E^*(G)|$. Furthermore, each $\mathbb{T}$-gain cycle (C_t,φ) in (G,φ) satisfies either $t\equiv 2 \pmod 4$, (C_t,φ) is Type A, or $t\equiv 1 \pmod 4$, (C_t,φ) is Type D or E, so does in $(G-C_l,\varphi)$. By induction on $(G-C_l,\varphi)$, then for all $T_{G-C_l}\in \mathbb{T}_{G-C_l}$, we have

$$i_+(G-C_l,\varphi) = i_+(T_{G-C_l}) + \sum_{C\subseteq G-C_l} i_+(C,\varphi) - n_o(G-C_l), \tag{3.8}$$

$$i_+(G-C_l) = i_+([T_{G-C_l}]) + \sum_{C\subseteq G-C_l} i_+(C) - n_o(G-C_l). \tag{3.9}$$

Since (G,φ) is i_+-lower optimal, then by Proposition 2.9 and Lemma 2.3, one has

$$i_+(G,\varphi) = i_+(G-v,\varphi) = i_+(G-C_l,\varphi) + \frac{l-1}{2}, \tag{3.10}$$

$$i_+(G) = i_+(G-v)+1 = i_+(G-C_l) + \frac{l-1}{2} + 1. \tag{3.11}$$

Observe that C_l is the pendant cycle, combining (3.6)–(3.11) yields

$$i_+(G,\varphi) = i_+(G-C_l,\varphi) + \frac{l-1}{2}$$
$$= i_+(T_{G-C_l}) + \sum_{C\subseteq G-C_l} i_+(C,\varphi) - n_o(G-C_l) + \frac{l-1}{2}$$
$$= i_+(T_{G-C_l}) + \sum_{C\subseteq G-C_l} i_+(C,\varphi) - (n_o(G)-1) + i_+(C_l,\varphi)$$
$$= i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G)$$

and

$$\begin{aligned} i_+(G) &= i_+(G - C_l, \varphi) + \frac{l+1}{2} \\ &= i_+([T_{G-C_l}]) + \sum_{C \subseteq G - C_l} i_+(C) - n_o(G - C_l) + \frac{l+1}{2} \\ &= i_+([T_{G-C_l}]) + \sum_{C \subseteq G - C_l} i_+(C) - (n_o(G) - 1) + i_+(C_l) \\ &= i_+([T_G]) + \sum_{C \subseteq G} i_+(C) - n_o(G). \end{aligned}$$

Subcase 2.2. $l \equiv 2$ (mod4), (C_l, φ) is Type A. Notice that $|E^*(G)| \geq 1$ and G is a graph with pairwise vertex-disjoint cycles, then G must contain a pendant cycle C_l. It can easily be seen that $\mathcal{T}_G = \mathcal{T}_{G\langle C_l\rangle}$ according to the contraction rules. Then there exists $T_{G\langle C_l\rangle} \in \mathcal{T}_{G\langle C_l\rangle}$, such that

$$T_G \cong T_{G\langle C_l\rangle} \ \text{ and } \ [T_G] \cong [T_{G\langle C_l\rangle}]. \tag{3.12}$$

By Lemmas 2.6 and 2.7,

$$i_+(C_l, \varphi) = \frac{l-2}{2}, \ \ i_+(C_l) = \frac{l}{2}. \tag{3.13}$$

By Proposition 2.13, $(G\langle C_l\rangle, \varphi)$ is i_+-lower optimal. Notice that $|E^*((G\langle C_l\rangle)| = |E^*(G)|$ and the graph $(G\langle C_l\rangle, \varphi)$ contains a pendant vertex. Furthermore, each $\mathbb{T}$-gain cycle (C_t, φ) in (G, φ) satisfies either $t \equiv 2$ (mod4), (C_t, φ) is Type A, or $t \equiv 1$ (mod4), (C_t, φ) is Type D or E, so does in $(G\langle C_l\rangle, \varphi)$. Proceeding as in the proof of Case 1, we have

$$i_+(G\langle C_l\rangle, \varphi) = i_+\left(T_{G\langle C_l\rangle}\right) + \sum_{C \subseteq G\langle C_l\rangle} i_+(C, \varphi) - n_o(G\langle C_l\rangle), \tag{3.14}$$

$$i_+(G\langle C_l\rangle) = i_+\left(\left[T_{G\langle C_l\rangle}\right]\right) + \sum_{C \subseteq G\langle C_l\rangle} i_+(C) - n_o(G\langle C_l\rangle). \tag{3.15}$$

Since (G, φ) is i_+-lower optimal, by Lemma 2.3 and Proposition 2.9,

$$i_+(G, \varphi) = i_+(G - v, \varphi) = i_+(G\langle C_l\rangle, \varphi) + \frac{l-2}{2}, \tag{3.16}$$

$$i_+(G) = i_+(G - v) + 1 = i_+(G\langle C_l\rangle) + \frac{l-2}{2} + 1. \tag{3.17}$$

Bearing in mind C_l is a pendant cycle, combining (3.12)–(3.17), we have

$$\begin{aligned} i_+(G, \varphi) &= i_+(G\langle C_l\rangle, \varphi) + \frac{l-2}{2} \\ &= i_+(T_{G\langle C_l\rangle}) + \sum_{C \subseteq G\langle C_l\rangle} i_+(C, \varphi) - n_o(G\langle C_l\rangle) + \frac{l-2}{2} \\ &= i_+(T_{G\langle C_l\rangle}) + \sum_{C \subseteq G\langle C_l\rangle} i_+(C, \varphi) - n_o(G) + i_+(C_l, \varphi) \\ &= i_+(T_G) + \sum_{C \subseteq G} i_+(C, \varphi) - n_o(G) \end{aligned}$$

and

$$\begin{aligned}i_+(G) &= i_+(G\langle C_l\rangle) + \frac{l}{2} = i_+([T_{G\langle C_l\rangle}]) + \sum_{C\subseteq G\langle C_l\rangle} i_+(C) - n_o(G\langle C_l\rangle) + \frac{l}{2}\\ &= i_+([T_{G\langle C_l\rangle}]) + \sum_{C\subseteq G\langle C_l\rangle} i_+(C) - n_o(G) + i_+(C_l)\\ &= i_+([T_G]) + \sum_{C\subseteq G} i_+(C) - n_o(G).\end{aligned}$$

This completes the proof of Lemma 3.2. □

Arguments similar to those used in Lemmas 3.1 and 3.2 show that the following Lemmas hold, the proof will be omitted.

Lemma 3.3 *Let (G,φ) be a $\mathbb{T}$-gain graph with exactly one cycle C_l. If (G,φ) is i_+-upper optimal, then either $l \equiv 0 \pmod 4$, (C_l,φ) is Type B, or $l \equiv 3 \pmod 4$, (C_l,φ) is Type C.*

Lemma 3.4 *Let (G,φ) be a $\mathbb{T}$-gain graph in which each $\mathbb{T}$-gain cycle, say (C_t,φ), satisfies either $l \equiv 0 \pmod 4$, (C_t,φ) is Type B, or $t \equiv 3 \pmod 4$, (C_t,φ) is Type C. If (G,φ) is i_+-upper optimal, then for all $T_G \in \mathcal{T}_G$*

$$\begin{aligned}i_+(G,\varphi) &= i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G),\\ i_+(G) &= i_+([T_G]) + \sum_{C\subseteq G} i_+(C) - n_o(G).\end{aligned}$$

With the above Lemmas in hand, we are ready to give the proof of Theorem 1.2.

Proof Theorem 1.2 (i) Let (G,φ) be a $\mathbb{T}$-gain graph satisfying (a)–(c). Note that G is a graph with pairwise vertex-disjoint cycles, by Proposition 2.14 and (c), we have

$$\begin{aligned}i_+(G,\varphi) - i_+(G) &= i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G) - i_+[T_G] - \sum_{C\subseteq G} i_+(C) + n_o(G)\\ &= \sum_{C\subseteq G}(i_+(C,\varphi) - i_+(C)).\end{aligned} \tag{3.18}$$

By Lemmas 2.6 and 2.7, then $i_+(C,\varphi) - i_+(C) = -1$ for each $\mathbb{T}$-gain cycle $C \subseteq G$. In view of (2.18),

$$i_+(G,\varphi) - i_+(G) = -c(G).$$

Hence (G,φ) is i_+-lower optimal.

Conversely, assume that (G,φ) is i_+-lower optimal. By Proposition 2.9 (v), G is a graph with pairwise vertex-disjoint cycles, then (a) holds. For any $(C_l,\varphi) \subseteq G$, by deleting any vertex on each $\mathbb{T}$-gain cycle different from (C_l,φ) of (G,φ), one obtains the graph (G',φ). Then (G',φ) contains exactly one $\mathbb{T}$-gain cycle $(C_l,\varphi) \subseteq (G,\varphi)$. By Proposition 2.9, (G',φ) is i_+-lower optimal. By Lemma 3.1, (G',φ) satisfies (b). The arbitrariness of $(C_l,\varphi) \subseteq G$ yields the result satisfying (b).

By Lemma 3.2, one has for all $T_G \in \mathcal{T}_G$

$$i_+(G,\varphi) = i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G),$$
$$i_+(G) = i_+([T_G]) + \sum_{C\subseteq G} i_+(C) - n_o(G).$$

Since i_+-lower optimal, we obtain

$$\begin{aligned}-c\,(G) &= i_+\,(G,\varphi) - i_+\,(G)\\ &= i_+\,(T_G) + \sum_{C\subseteq G} i_+\,(C,\varphi) - n_o\,(G) - i_+[T_G] - \sum_{C\subseteq G} i_+\,(C) + n_o\,(G)\\ &= i_+\,(T_G) - i_+[T_G] + \sum_{C\subseteq G} \left(i_+\,(C,\varphi) - i_+\,(C)\right).\end{aligned}$$

Combining with (a) and (b), by Lemmas 2.6 and 2.7, then $i_+(C,\varphi) - i_+(C) = -1$ for each cycle $C \subseteq G$. Thus,

$$-c(G) = i_+(T_G) - i_+[T_G] - c(G).$$

Hence (G,φ) satisfies (c). □

Proof of Theorem 1.2 (ii) Let (G,φ) be a $\mathbb{T}$-gain graph satisfying (a)–(c). Note that G is a graph with pairwise vertex-disjoint cycles, by Proposition 2.14 and (c), we have

$$\begin{aligned}i_+\,(G,\varphi) - i_+\,(G) &= i_+\,(T_G) + \sum_{C\subseteq G} i_+\,(C,\varphi) - n_o\,(G) - i_+[T_G] - \sum_{C\subseteq G} i_+\,(C) + n_o\,(G)\\ &= \sum_{C\subseteq G} (i_+\,(C,\varphi) - i_+\,(C)).\end{aligned}$$

By Lemmas 2.6 and 2.7, then $i_+(C,\varphi) - i_+(C) = 1$ for each $\mathbb{T}$-gain cycle $C \subseteq G$. In view of (3.18),

$$i_+(G,\varphi) - i_+(G) = c(G).$$

Hence (G,φ) is i_+-upper optimal.

Conversely, assume that (G,φ) is i_+-upper optimal. By Proposition 2.10 (v), G is a graph with pairwise vertex-disjoint cycles, then (a) holds. For any $C_l \subseteq G$, by deleting any vertex on each $\mathbb{T}$-gain cycle different from $(C_l,\varphi) \subseteq (G,\varphi)$, one obtains the graph (G',φ). Then (G',φ) contains exactly one $\mathbb{T}$-gain cycle $(C_l,\varphi) \subseteq (G,\varphi)$. By Proposition 2.10, (G',φ) is i_+-upper optimal. By Lemma 3.3, (G',φ) satisfies (b). The arbitrariness of $(C_l,\varphi) \subseteq G$ yields the result satisfying (b).

By Lemma 3.4, one has for all $T_G \in \mathcal{T}_G$

$$i_+(G,\varphi) = i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G),$$
$$i_+(G) = i_+([T_G]) + \sum_{C\subseteq G} i_+(C) - n_o(G).$$

Since i_+-upper optimal, we obtain

$$c\,(G) = i_+\,(G,\varphi) - i_+\,(G)$$

$$= i_+(T_G) + \sum_{C\subseteq G} i_+(C,\varphi) - n_o(G) - i_+[T_G] - \sum_{C\subseteq G} i_+(C) + n_o(G)$$
$$= i_+(T_G) - i_+[T_G] + \sum_{C\subseteq G} (i_+(C,\varphi) - i_+(C)).$$

Combining with (a) and (b), by Lemmas 2.6 and 2.7, then $i_+(C,\varphi) - i_+(C) = 1$ for each cycle $C \subseteq G$. Thus,

$$c(G) = i_+(T_G) - i_+[T_G] + c(G).$$

Hence (G,φ) satisfies (c). □

4. Conclusion remarks

In this paper, we show the sharp bounds on the difference between the positive inertia index of a $\mathbb{T}$-gain graph with that of its underlying graph. By the similar discussion, one can extend the results to the negative inertia index as the following, we omit the proofs here.

Theorem 4.1 *Let (G,φ) be a $\mathbb{T}$-gain graph with underlying graph G. Then*

$$-c(G) \le i_-(G,\varphi) - i_-(G) \le c(G).$$

Theorem 4.2 *Let (G,φ) be a $\mathbb{T}$-gain graph.*

(i) (G,φ) is i_--lower optimal if and only if all the following conditions hold:

(a) G is a graph with pairwise vertex-disjoint cycles;

(b) Each $\mathbb{T}$-gain cycle (C_l,φ) of (G,φ) satisfies either $l \equiv 2 \pmod 4$, (C_l,φ) is Type A, or $l \equiv 3 \pmod 4$, (C_l,φ) is Type C or E;

(c) $i_-(G,\varphi) = i_-(T_G) + \sum_{C\subseteq G} i_-(C,\varphi) - n_o(G)$, $i_-(G) = i_-([T_G]) + \sum_{C\subseteq G} i_-(C) - n_o(G)$ and $i_-(T_G) = i_-([T_G])$ for all $T_G \in \mathcal{T}_G$.

(ii) (G,φ) is i_--upper optimal if and only if all the following conditions hold:

(a) G is a graph with pairwise vertex-disjoint cycles;

(b) Each $\mathbb{T}$-gain cycle (C_l,φ) of (G,φ) satisfies either $l \equiv 0 \pmod 4$, (C_l,φ) is Type B, or $l \equiv 3 \pmod 4$, (C_l,φ) is Type D;

(c) $i_-(G,\varphi) = i_-(T_G) + \sum_{C\subseteq G} i_-(C,\varphi) - n_o(G)$, $i_-(G) = i_-([T_G]) + \sum_{C\subseteq G} i_-(C) - n_o(G)$ and $i_-(T_G) = i_-([T_G])$ for all $T_G \in \mathcal{T}_G$.

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