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Estimation of Scale Transformation for Approximate Periodic Time Series with Long-Term Trend

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Abstract Approximate periodic time series means it has an approximate periodic trend. The so-called approximate periodicity refers that it looks like having periodicity, however the length of each period is not constant such as sunspot data. Approximate periodic time series has a wide application prospect in modelling social economic phenomenon. As for approximate periodic time series, the key problem is to depict its approximate periodic trend because it can be dealt as an ordinary time series only if its approximate periodic trend has been depicted. However, there is little study on depicting approximate periodic trend.

In the paper, the authors first establish some necessary theories, especially bring forward the concept of shape-retention transformation with lengthwise compression and obtain necessary and sufficient condition for linear shape-retention transformation with lengthwise compression, then basing on the theories the authors present a method to estimate scale transformation, which can model approximate periodic trend very clearly. At last, a simulated example is analyzed by this presented method. The results show that the presented method is very effective and very powerful.

Keywords time series; approximate periodicity; scale transformation; shape-retention transformation with lengthwise compression

MR(2020) Subject Classification 37M10

1. Introduction

Time series has been used in statistics, econometrics, mathematical finance, signal processing, weather forecasting and communication engineering [1], such as forecasting the demand for airline capacity, seasonal telephone demand, the movement of short-term interest rates, etc. Periodicity of time series is one of its important characters [2]. The earlier work on the periodic time series can be traced back to Schuster [3], who used the periodic graph model to study the periodic problem of sunspot series from 1750 to 1900. In recent years, periodic time series is still one of important research topics, see [4–8] and their references.

In human social life and in nature, there are lots of time series which have no strict periodicity. For example, the sunspot data during the 20^{th} century looks like having periodicity and its

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period length is about 11 years in Figure 1 (Data source: SILSO data/image, Royal Observatory of Belgium, Brussels), but the length of adjacent two epochs is not always 11 years.

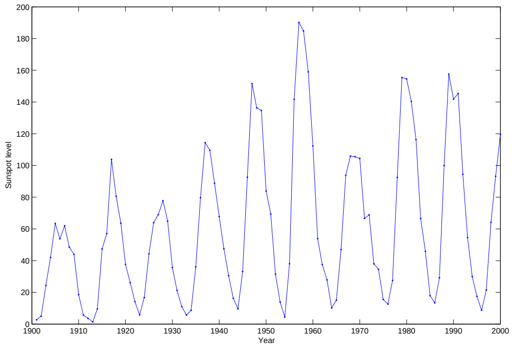


Figure 1 Sequential chart of sunspot data during the 20^{th} century

Figure 1 shows that the sunspot peak years during the 20^{th} century include 1905, 1917, 1928, 1937, 1947, 1957, 1968, 1979, 1989 and 2000. The lengths of adjacent two epochs are 12, 11, 9, 10, 10, 11, 11, 10 and 11. That is, the periods of sunspot are not any constant, which is really not periodic in strictly speaking. In fact, there are lots of time series which seem to have periodic trend but the length of each epoch is not any constant, such as consumption credit balance series, money supply M_0 series, lithium battery discharge cycle series, syphilis treatment rate series in China, photovoltaic power series, air temperature series, sunshine series, rainfall series, crop growth series, agricultural product supply series, epidemic number sequence, biological membrane potential oscillation series, heart rhythm series, fine cell cycle series, price series of risk securities, and traffic series on the highway, etc.

In order to analyze time series with non-strict period such as sunspot, mixed period model was brought forward. The so-called mixed period model is that multiple time series with the same period are mixed into one time series according to probability. In the last two decades, the mixed periodic model has developed rapidly. Shao proposed a mixed period autoregressive model [9], that is, multiple time series with the same period are mixed into one time series according to probability. Shao gave the robust estimation of multi-dimensional periodic autoregressive model [10]. Shao proposed a method based on local linear estimation to estimate the trend of periodic autoregressive model [11]. In order to reduce the problem of too many parameters to be estimated in the periodic time series model, Lund, Shao and Basawa proposed a reduced (parsimonious) periodic autoregressive moving average model [12]. Gong, Kiessler and Lund proposed a method to identify abnormal events in periodic time series based on residual sequence [13]. Bezandry and Diagana brought forward almost periodic stochastic processes [14], which means the moment of process has periodicity. In essence, almost periodic stochastic processes still belong to periodic stochastic processes, merely the periodicity exhibits on their moment. Dehay and Hurd studied the frequency determination of almost periodic time series [15]. However, all the proposed models are not very effective in analyzing the time series with non-strict period, such as sunspots in

Figure 1.

In order to effectively depict the time series with non-strict period, such as sunspots in Figure 1, Wu, Zhu and Yang brought forward the model of approximate periodic time series [16]. The so-called approximate periodicity refers that it looks like having periodicity, however the length of each epoch is not any constant such as sunspot data. Approximate periodic time series has a wide application prospect in modelling social economic phenomenon. In order to increase readability, we introduce the concept of approximate periodic time series as follows.

Definition 1.1 ([16]) *Let $\{S(t), t \geq 0\}$ be a real-valued function. If there exist a strictly increasing sequence $\{T_k | T_0 = 0, \lim_{k\to+\infty} T_k = +\infty\}$ and a strictly increasing continuous function $\{g(t), t \geq T_1\}$ satisfying $g(T_k) = T_{k-1}$ for all $k = 1, 2, \ldots$, such that for any $t \geq T_1$ it follows that*

$$S(t) = S(g(t)),$$

then $S(t)$ is called an approximate periodic function with scale transformation g, where $0 = T_0 < T_1 < \cdots < T_k < \cdots$ is called the dividing point series of approximate periodic function $\{S(t), t \geq 0\}$.

Proposition 1.2 ([16]) *$\{f(t), t \geq 0\}$ is an approximate periodic function if and only if there exists a strictly increasing continuous function $\{u(t), t \geq 0\}$ satisfying $u(0) = 0$ and $\lim_{t\to+\infty} u(t) = +\infty$ such that $\{f(u(t)), t \geq 0\}$ is a periodic function.*

Definition 1.3 ([16]) *If seasonal trend of a time series is some approximate periodic function with scale transformation g, then the time series is called an approximate periodic time series with scale transformation g.*

For an approximate periodic time series without long-term trend, that is,

$$x_t = S_t + \varepsilon_t, \quad t \geq 0, \tag{1.1}$$

where $\{S_t, t \geq 0\}$ is the approximate periodic trend of $\{x_t, t \geq 0\}$ and $\{\varepsilon_t, t \geq 0\}$ is a stationary time series with zero mean. Providing that the scale transformation g was known or could be estimated, Wu, Zhu and Yang first presented a method to extract approximate periodic trend for approximate periodic time series (1.1), then they brought forward a generalized difference operator to eliminate approximate periodic trend [16].

In practice, when we use approximate periodic time series to solve some problem, the scale transformation g is always unknown and the time series always has long-term trend. For example, Figure 2 depicts the balance data of personal consumer credit product of Ant Financial Services Group in China from Feb 1^{st} to Jul 13^{th}, 2015, where g is unknown and the balance data has both approximate periodicity and long-term trend.

If a time series has long-term trend and approximate periodicity, we could express its model as follows

$$x_t = f(h(t), S(t), \varepsilon_t), \quad t \geq 0, \tag{1.2}$$

where f is a function with three variables, $\{h(t), t \geq 0\}$ denotes the long-term trend, $\{S(t), t \geq 0\}$

denotes the approximate periodic trend with the scale transformation g, and $\{\varepsilon_t, t \geq 0\}$ is a stationary time series with zero mean.

For model (1.2), if we can obtain the scale transformation g, according to Proposition 1.2 there exists a strictly increasing function u such that $\{S(u(t)), t \geq 0\}$ becomes a periodic function, then the model (1.2) is similar to the ordinary time series model with long-term trend and periodicity, then we can use the classical method to model them. However, if we cannot obtain the scale transformation function g, then it is almost impossible to judge the function form of S, and it is very difficult to model (1.2). Thus, it is very important to estimate the scale transformation function g.

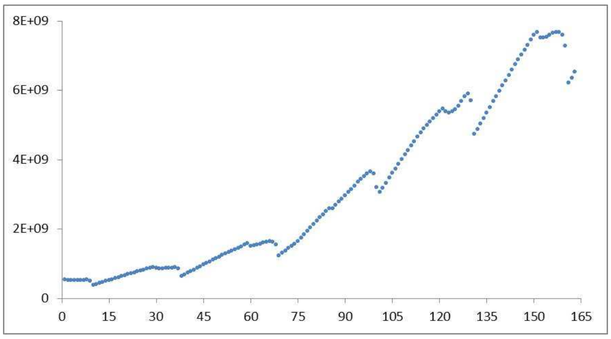


Figure 2 The balance data of personal consumer credit product

As for approximate periodic time series, the key problem is to depict its approximate periodic trend because it can be dealt as an ordinary time series only if its approximate periodic trend has been depicted. However, there is little study on depicting approximate periodic trend. In the paper, the authors first establish some necessary theories, especially bring forward the concept of shape-retention transformation with lengthwise compression and obtain necessary and sufficient condition for linear shape-retention transformation with lengthwise compression, then basing on the theories the authors present a method to estimate scale transformation, which can model approximate periodic trend very clearly. At last, a simulated example is analyzed by this presented method. The results show that the presented method is very effective and very powerful.

2. Difficulties in estimating scale transformation

For any approximate periodic time series samples $\{x_k, k = 1, 2, \ldots, n\}$, we cannot get any judgement on scale transformation g from the sequence chart of $\{x_k, k = 1, 2, \ldots, n\}$. Thus, it is very difficult to estimate g.

In fact, the difficulty in estimating scale transformation g mainly comes from two factors. In order to explain them clearly, we consider a continuous function f as follows

$$f(t) = \begin{cases} \cos(\frac{t}{4}\pi), & 0 \leq t < 8, \\ \cos(\frac{t-8}{7}\pi), & 8 \leq t \leq 22, \end{cases} \tag{2.1}$$

whose graph is presented in Figure 3 and the scale transformation $g(t) = \frac{t-8}{7}$, $8 \le t \le 22$, which has been given by (2.1). Of course, we can estimate g from Figure 3 by establishing the correspondence between t and $g(t)$.

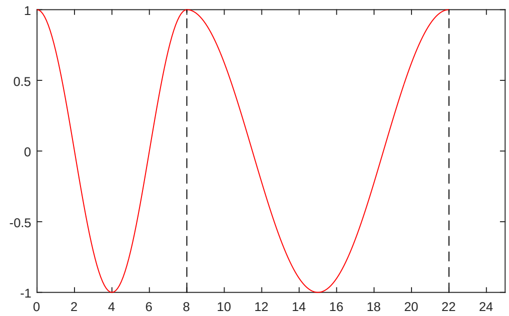


Figure 3 The graph of $\{f(t), 0 \le t \le 22\}$

In practice, $\{f(t), 0 \le t \le 22\}$ may be effected by some random disturbance, and we only obtain discrete observed values. That is, what we observed is like $\{f(t) + \varepsilon(t), t = 0, 1, \ldots, 22\}$, where the function form of f is unknown and $\{\varepsilon(t), t = 0, 1, \ldots, 22\}$ is independent identically distributed normal distribution with mean zero and variance σ^2, i.e., $\{\varepsilon(t)\} \overset{i.i.d.}{\sim} N(0, \sigma^2)$, here $\sigma^2 > 0$ is also unknown.

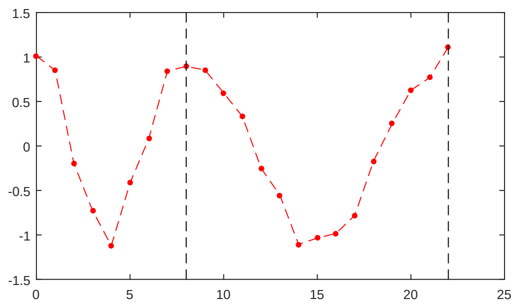


Figure 4 The sequence chart of $\{S(t) + \varepsilon(t), t = 0, 1, \ldots, 22\}$

We present a sample of $\{f(t) + \varepsilon(t), t = 0, 1, \ldots, 22\}$ in Table 1, and plot its sequence chart in Figure 4. From Figure 4 we find it is very difficult to estimate scale transformation g because we cannot establish the relation between t and $g(t)$.

t	0	1	2	3	4	5	6	7
$S(t)+\varepsilon(t)$	1.0124	0.8508	-0.1961	-0.7269	-1.1208	-0.4163	0.0825	0.8450
t	8	9	10	11	12	13	14	15
$S(t)+\varepsilon(t)$	0.8942	0.8541	0.5962	0.3324	-0.2503	-0.5533	-1.1062	-1.0354
t	16	17	18	19	20	21	22	
$S(t)+\varepsilon(t)$	-0.9833	-0.7812	-0.1717	0.2507	0.6268	0.7676	1.1127	

Table 1 Samples from $\{S(t) + \varepsilon(t), t = 0, 1, \ldots, 22\}$, where $\varepsilon \overset{i.i.d.}{\sim} N(0, 0.1^2)$

3. Necessary theories

In the previous section we have pointed out the most difficulty in estimating scale transformation g is that we hardly find the relation between t and $g(t)$. In order to estimate scale transformation g, we present some necessary theoretical results in this section.

3.1. Properties of scale transformation g

For any $t \geq 0$, denote

$$n_t = \max\{m \geq 0 : t \geq T_m\}.$$

Obviously, $t \in [T_{n_t}, T_{n_t+1})$ holds for all $t \geq 0$.

Theorem 3.1 *For any $t \geq T_1$, it follows that $g^{(n_t-k)}$ is increasing and*

$$g^{(n_t-k)}(t) \in [T_k, T_{k+1}) \tag{3.1}$$

holds for all $k = 0, 1, \ldots, n_t - 1$, where $g^{(k)}$ is the k-time composite function of g and we stipulate $g^{(1)} = g$.

Proof Noting that $g^{(n_t-k)}$ is the $(n_t - k)$-time composite function of g and g is increasing, we easily obtain that $g^{(n_t-k)}$ is increasing. In the following, we will only show (3.1) holds for all $k = 0, 1, \ldots, n_t - 1$.

When $t \in [T_1, T_2)$, it yields $n_t = 1$. Noting that g is strictly increasing, $g(T_2) = g(T_1)$ and $g(T_1) = g(T_0)$, we have

$$g^{(1)}(t) = g(t) \in [T_0, T_1)$$

holds for all $t \in [T_1, T_2)$.

Assume the statement of Theorem 3.1 holds when $t \in [T_m, T_{m+1})$, $m \geq 1$, that is,

$$g^{(m-k)}(t) \in [T_k, T_{k+1}) \tag{3.2}$$

holds for all $k = 0, 1, \ldots, m-1$. Further, for any $t \in [T_{m+1}, T_{m+2})$, it follows $n_t = m+1$. Noting that g is strictly increasing, $g(T_{m+2}) = g(T_{m+1})$ and $g(T_{m+1}) = g(T_m)$, we have

$$g^{(1)}(t) = g(t) \in [T_m, T_{m+1}) \tag{3.3}$$

holds for all $t \in [T_{m+1}, T_{m+2})$. It yields from (3.2) that

$$g^{(m-k)}(g(t)) \in [T_k, T_{k+1})$$

holds for all $k = 0, 1, \ldots, m - 1$. That is,

$$g^{(m+1-k)}(t) \in [T_k, T_{k+1}) \tag{3.4}$$

holds for all $k = 0, 1, \ldots, m - 1$. It follows from (3.3) and (3.4) that

$$g^{(n_t-k)}(t) \in [T_k, T_{k+1})$$

holds for all $k = 0, 1, \ldots, n_t - 1$.

Using the mathematical induction, we know the statement of Theorem 3.1 holds for all $t \geq T_1$. $\square$

Denote

$$L_j = g^{(j)} \quad \text{for all} \quad j = 1, 2, \ldots, \tag{3.5}$$

where $g^{(1)} = g$. That is, L_j is the j-time compound function of g. It follows from Theorem 3.1 that

$$L_j : (T_j, T_{j+1}] \to (T_0, T_1] \quad \text{for all} \quad j = 1, 2, \ldots.$$

Theorem 3.2 *$\{L_j, j = 1, 2, \ldots\}$ and g are mutually determined from each other.*

Proof It yields from (3.5) that $\{L_j, j = 1, 2, \ldots\}$ is determined by g. On the other hand, it follows from (3.5) that, for any $j = 1, 2, \ldots$, L_j is strictly increasing because g is strictly increasing. Furthermore,

$$g(t) = L_1(t) \quad \text{for all} \quad t \in (T_1, T_2] \tag{3.6}$$

and for any $j = 2, 3, \ldots$ it follows that

$$L_j(t) = g^{(j)}(t) = g^{(j-1)}(g(t)) = L_{j-1}(g(t)) \quad \text{for all} \quad t \in (T_j, T_{j+1}],$$

thus,

$$g(t) = L_{j-1}^{-1}(L_j(t)) \quad \text{for all} \quad t \in (T_j, T_{j+1}]. \tag{3.7}$$

We obtain from (3.6) and (3.7) that

$$g(t) = L_{j-1}^{-1}(L_j(t)) \quad \text{for all} \quad t \in (T_j, T_{j+1}] \tag{3.8}$$

holds for all $j = 1, 2, \ldots$, where we stipulate L_0 is the identify transformation on $(T_0, T_1]$.

It yields from (3.5) that g is determined by $\{L_j, j = 1, 2, \ldots\}$. $\square$

Remark 3.3 In the proof of Theorem 3.2, we obtain $L_{j+1}(t) = L_j(g(t))$, however, we cannot obtain $L_{j+1}(t) = g(L_j(t))$. Obviously, the domain of L_j is different from that of L_{j+1}.

Remark 3.4 When we want to estimate g, we usually estimate $\{L_1, L_2, \ldots\}$ first, then use (3.8) to work out g. Thus, in the following we will also call $\{L_1, L_2, \ldots\}$ as scale transformation and mainly consider how to estimate $\{L_1, L_2, \ldots\}$.

3.2. A shape-retention transformation with lengthwise compression

In order to estimate scale transformation $\{L_1, L_2, \ldots\}$, we need introduce a conception on shape-retention transformation with lengthwise compression.

Definition 3.5 *For any $a_0, a_1, v_0, v_1 \in R$ and $b > 0$, let $f(t)$ be a continuous function on $[a_0, a_0 + b]$, if there exist a transformation h and $\delta > 0$ such that*

$$u(t) = h(t, f(t - a_1 + a_0)), \quad t \in [a_1, a_1 + b]$$

satisfying that $u(a_1) = v_0$, $u(a_1 + b) = v_1$ and

$$u(t) - \ell^u_{(a_1,b)}(t) = \delta(f(t - a_1 + a_0) - \ell^f_{(a_0,b)}(t - a_1 + a_0))$$

hold for all $t \in [a_1, a_1 + b]$, where $\ell^f_{(a_0,b)}(t)$ denotes the value of the line through $(a_0, f(a_0))$ and $(a_0 + b, f(a_0 + b))$ at t, then h is called a shape-retention transformation with lengthwise compression of f with compression proportion δ.

Figure 5 exhibits shape-retention transformation clearly, which means the length of blue line segment always equals to that of red line segment δ for all $t \in [0, b]$.

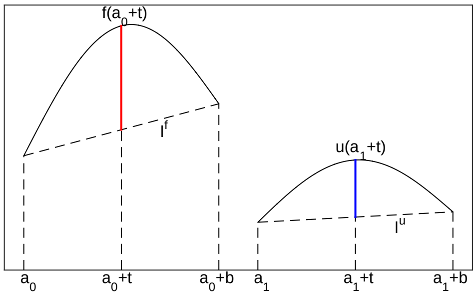


Figure 5 Shape-retention transformation with lengthwise compression

Remark 3.6 Shape-retention transformation with lengthwise compression is a very complex transformation, which blends translation process, rotation process and compression process. However, shape-retention transformation with lengthwise compression is very important for estimating scale transformation $\{L_1, L_2, \ldots\}$.

Theorem 3.7 *For any $a_0, a_1, v_0, v_1 \in R$ and $b > 0$, h is a linear shape-retention transformation with lengthwise compression proportion δ of $\{f(t), a_0 \le t \le a_0 + b\}$ into $[a_1, a_1 + b]$ satisfying $u(a_1) = v_0$ and $u(a_1 + b) = v_1$, where $u(t) \equiv h(t, f(t - a_1 + a_0))$, if and only if*

$$\begin{aligned} u(t) = & v_0 + \frac{(v_1 - v_0)(t - a_1)}{b} + \delta f(t - a_1 + a_0) - \\ & \frac{\delta}{b}[(t - a_1)f(a_0 + b) + (a_1 + b - t)f(a_0)], \quad t \in [a_1, a_1 + b]. \end{aligned} \tag{3.9}$$

Proof Sufficiency. If h is defined by (3.9), then h is obviously linear on t and f and satisfies

$$u(a_1) = v_0 \quad \text{and} \quad u(a_1 + b) = v_1. \tag{3.10}$$

Furthermore,

$$\ell^f_{(a_0,b)}(t) = f(a_0) + \frac{f(a_0 + b) - f(a_0)}{b}(t - a_0), \quad a_0 \le t \le a_0 + b \tag{3.11}$$

and

$$\ell^u_{(a_1,b)}(t) = v_0 + \frac{v_1 - v_0}{b}(t - a_1), \quad a_1 \le t \le a_1 + b. \tag{3.12}$$

It follows from (3.9), (3.12) and (3.11) that, for any $a_1 \le t \le a_1 + b$, we have

$$\begin{aligned} u(t) - \ell^u_{(a_1,b)}(t) &= \delta[f(t - a_1 + a_0) - \frac{(t - a_1)f(a_0 + b) + (a_1 + b - t)f(a_0)}{b}] \\ &= \delta[f(t - a_1 + a_0) - \ell^f_{(a_0,b)}(t - a_1 + a_0)]. \end{aligned} \tag{3.13}$$

It yields from (3.10) and (3.13) that the sufficiency of Theorem 3.7 is proved.

Necessity. If h is a linear shape-retention transformation with lengthwise compression proportion δ of $\{f(t), a_0 \le t \le a_0+b\}$ into $[a_1, a_1+b]$, then we can denote

$$u(t) \equiv h(t, f(t-a_1+a_0)) = c_0 + c_1 t + c_2 f(t-a_1+a_0), \quad a_1 \le t \le a_1+b. \tag{3.14}$$

It yields from $u(a_1)=v_0$, $u(a_1+b)=v_1$ and (3.14) that

$$c_0 + c_1 a_1 + c_2 f(a_0) = v_0, \quad c_0 + c_1(a_1+b) + c_2 f(a_0+b) = v_1. \tag{3.15}$$

Solving (3.15), we obtain

$$c_0 = v_0 - c_1 a_1 - c_2 f(a_0), \quad c_1 = \frac{v_1-v_0}{b} - c_2 \frac{f(a_0+b)-f(a_0)}{b}. \tag{3.16}$$

It follows from (3.14) and (3.16) that

$$\begin{aligned} u(t) &= v_0 + c_1(t-a_1) + c_2[f(t-a_1+a_0) - f(a_0)] \\ &= v_0 + \frac{v_1-v_0}{b}(t-a_1) + c_2[f(t-a_1+a_0) - \frac{(t-a_1)f(a_0+b)+(a_1+b-t)f(a_0)}{b}] \end{aligned} \tag{3.17}$$

for all $a_1 \le t \le a_1+b$.

Noting that

$$\ell^f_{(a_0,b)}(t) = f(a_0) + \frac{f(a_0+b)-f(a_0)}{b}(t-a_0), \quad a_0 \le t \le a_0+b$$

and

$$\ell^u_{(a_1,b)}(t) = v_0 + \frac{v_1-v_0}{b}(t-a_1), \quad a_1 \le t \le a_1+b,$$

we obtain from (3.17) that

$$u(t) - \ell^u_{(a_1,b)}(t) = c_2[f(t-a_1+a_0) - \ell^f_{(a_0,b)}(t-a_1+a_0)].$$

Owing to lengthwise compression proportion δ, we have $c_2=\delta$. Thus, it follows from (3.17) that, for all $a_1 \le t \le a_1+b$,

$$u(t) = v_0 + \frac{v_1-v_0}{b}(t-a_1) + \delta[f(t-a_1+a_0) - \frac{(t-a_1)f(a_0+b)+(a_1+b-t)f(a_0)}{b}].$$

The necessity of Theorem 3.7 is proved. □

Corollary 3.8 *For any $a, b, v_0, v_1 \in R$ satisfying $b>a$, h is a linear shape-retention transformation with lengthwise compression proportion δ of $\{f(t), a \le t \le b\}$ into $[a,b]$ satisfying $u(a)=v_0$ and $u(b)=v_1$, where $u(t) \equiv h(t, f(t))$, if and only if*

$$u(t) = v_0 + \frac{v_1-v_0}{b-a}(t-a) + \delta f(t) - \delta\frac{(t-a)f(b)+(b-t)f(a)}{b-a}, \quad t \in [a,b].$$

Theorem 3.9 *For any $a_0, a_1 \in R$ and $b>0$, h is a linear shape-retention transformation with lengthwise compression proportion δ of $\{f(t), a_0 \le t \le a_0+b\}$ into $[a_1, a_1+b]$, $u(t) \equiv h(t, f(t-a_1+a_0))$, then it follows that*

$$u(t) - \ell^u_{(a_1,s)}(t) = \delta[f(t-a_1+a_0) - \ell^f_{(a_0,s)}(t-a_1+a_0)]$$

holds for all $a_1 \le t \le a_1+b$ and $0 < s \le b$.

Proof For any $0 < s \le b$, because $\ell^f_{(a_0,s)}(z)$ is the value at z of line through $(a_0, f(a_0))$ and $(a_0+s, f(a_0+s))$, we have

$$\ell^f_{(a_0,s)}(z) = f(a_0) + \frac{f(a_0+s) - f(a_0)}{s}(z - a_0).$$

Thus, for any $a_1 \le t \le a_1 + b$ it follows that

$$\ell^f_{(a_0,s)}(t - a_1 + a_0) = f(a_0) + \frac{f(a_0+s) - f(a_0)}{s}(t - a_1). \tag{3.18}$$

Analogously, we have

$$\ell^u_{(a_1,s)}(t) = u(a_1) + \frac{u(a_1+s) - u(a_1)}{s}(t - a_1). \tag{3.19}$$

It yields from (3.19), (3.9) and (3.18) that

$$\begin{aligned}
&u(t) - \ell^u_{(a_1,s)}(t) = (u(t) - u(a_1)) - \frac{u(a_1+s) - u(a_1)}{s}(t - a_1) \\
&= \frac{u(a_1+b) - u(a_1)}{b}(t - a_1) + \delta[f(t - a_1 + a_0) - \frac{(t-a_1)f(a_0+b) + (a_1+b-t)f(a_0)}{b}] \\
&\quad \{\frac{u(a_1+b) - u(a_1)}{b} + \frac{\delta}{s}[f(a_0+s) - \frac{sf(a_0+b) + (b-s)f(a_0)}{b}]\}(t - a_1) \\
&= \delta[f(t - a_1 + a_0) - f(a_0) - \frac{f(a_0+s) - f(a_0)}{s}(t - a_1)] \\
&= \delta[f(t - a_1 + a_0) - \ell^f_{(a_0,s)}(t - a_1 + a_0)]
\end{aligned}$$

holds for all $a_1 \le t \le a_1 + b$ and $0 < s \le b$. □

Corollary 3.10 *For any $a, b \in R$ satisfying $b > a$, h is a linear shape-retention transformation with lengthwise compression proportion δ of $\{f(t), a \le t \le b\}$ into $[a, b]$, $u(t) \equiv h(t, f(t))$, then it follows that*

$$u(t) - \ell^u_{(a,s)}(t) = \delta[f(t) - \ell^f_{(a,s)}(t)]$$

holds for all $a \le t \le b$ and $0 < s \le b - a$.

Remark 3.11 When we estimate scale transformation $\{L_j, j = 1, 2, \ldots\}$, we mainly base on Corollaries 3.8 and 3.10.

4. Estimation method of scale transformation

In the previous section we have prepared some theories for estimating scale transformation. In this section, we will present a method to estimate g, where g is the scale transformation changing the j^{th} epoch into the i^{th} epoch. Thus, "g" in this section may be different from "g" in Definition 1.1. We know, if we can obtain mapping relation of $t \to g(t)$ from two different epochs, then we can obtain all mapping relation of $t \to g(t)$ from all epochs for a time series. Thus, we only present the method to estimate g from two different epochs in this section.

Assume $x_1, x_2, \ldots, x_n$ is a sample from a time series $X^{(0)} = \{x_t = f(h(t), S(t), \varepsilon(t)), t \ge 0\}$ defined by (1.2), and assume $0 = T_0 < T_1 < \cdots < T_k < \cdots$ is the dividing point series of approximate periodic function $\{S(t), t \ge 0\}$. In order to explain the method to estimate scale transformation g clearly and comprehensibly, we assume there are two groups of samples X and Y from the i^{th} and j^{th} two different epochs, where $i \neq j$, and we denote them by

$$X = (X_{T_i+1}, X_{T_i+2}, \ldots, X_{T_{i+1}}), \quad Y = (Y_{T_j+1}, Y_{T_j+2}, \ldots, Y_{T_{j+1}}).$$

In order to estimate g, we need to establish the relation between t and $g(t)$. The detail method to estimate scale transformation g includes four steps:

Step 1. Eliminate the long-term trend

Fit the data $\{(T_k, x_{T_k}),\ k = 1, 2, \ldots\}$ by a polynomial $Q(t)$, where x_{T_k} is the k-th dividing point of $X^{(0)}$, $k = 1, 2, \ldots$. Then we take

$$\tilde{X} = (X_{T_i+1} - Q(T_i + 1), X_{T_i+2} - Q(T_i + 2), \ldots, X_{T_{i+1}} - Q(T_{i+1})) \tag{4.1}$$

and

$$\tilde{Y} = (Y_{T_j+1} - Q(T_j + 1), Y_{T_j+2} - Q(T_j + 2), \ldots, Y_{T_{j+1}} - Q(T_{j+1})). \tag{4.2}$$

Step 2. Compress $\tilde{Y}$ into the size of $\tilde{X}$.

Denote

$$\delta = \frac{\max\{|X_t - Q(t)|,\ t = T_i + 1, T_i + 2, \ldots, T_{i+1}\}}{\max\{|Y_t - Q(t)|, t = T_j + 1, T_j + 2, \ldots, T_{j+1}\}}$$

and

$$Z = \delta \tilde{Y}. \tag{4.3}$$

Remark 4.1 For the convenience of reference, we denote

$$\tilde{X}_t = X_t - Q(t),$$

where $t = T_i + 1, T_i + 2, \ldots, T_{i+1}$, and

$$\tilde{X} = (\tilde{X}_{T_i+1}, \tilde{X}_{T_i+2}, \ldots, \tilde{X}_{T_{i+1}}).$$

Step 3. Generate the mapping g_0 from $\{T_j + 1, T_j + 2, \ldots, T_{j+1}\}$ to $\{T_i + 1, T_i + 2, \ldots, T_{i+1}\}$.

First, we take

$$g_0(T_j + 1) = T_i + 1 \quad \text{and} \quad g_0(T_{j+1}) = T_{i+1}. \tag{4.4}$$

Then, for any $t = T_j + 2, \ldots, T_{j+1} - 1$, we denote

$$g_0(t - 1) \in [T_i + \ell, T_i + \ell + 1],$$

where $\ell = 1, 2, \ldots, T_{i+1} - T_i - 1$. Further, we compare Z_t with $\{\tilde{X}_{T_i+\ell}, \tilde{X}_{T_i+\ell+1}, \ldots, \tilde{X}_{T_{i+1}}\}$.

If $Z_t < \min_{s=\ell,\ell+1,\ldots,T_{i+1}-T_i-1}\{\tilde{X}_{T_i+s}\}$ or $Z_t > \max_{s=\ell,\ell+1,\ldots,T_{i+1}-T_i-1}\{\tilde{X}_{T_i+s}\}$, then we take

$$g_0(t) = \min\{\hat{t} + 0.5, T_{i+1}\}, \tag{4.5}$$

where $\hat{t} = \arg\min_{s=\ell,\ell+1,\ldots,T_{i+1}-T_i-1}\{\tilde{X}_{T_i+s}\}$ or $\hat{t} = \arg\max_{s=\ell,\ell+1,\ldots,T_{i+1}-T_i-1}\{\tilde{X}_{T_i+s}\}$, respectively. Otherwise, there exists $s = \ell, \ell + 1, \ldots, T_{i+1} - T_i - 1$ satisfying

$$Z_t \in [\min\{\tilde{X}_{T_i+s}, \tilde{X}_{T_i+s+1}\},\ \max\{\tilde{X}_{T_i+s}, \tilde{X}_{T_i+s+1}\}], \tag{4.6}$$

then

$$g_0(t) \in [T_i + s, T_i + s + 1], \tag{4.7}$$

where s is the first time satisfying (4.6), and we take

$$g_0(t) = T_i + s + 0.5. \tag{4.8}$$

According to (4.4), (4.5) and (4.8) we can obtain the data set

$$\{(T_j + 1, T_i + 1);\ (t, g_0(t)), t = T_j + 2, \ldots, T_{j+1} - 1;\ (T_{j+1}, T_{i+1})\}. \tag{4.9}$$

Step 4. Estimate the scale transformation g.

After proceeding Steps 1–3, we can obtain final data pairs (4.9), then using the final data pairs of (4.9) we can estimate the scale transformation g by statistical methods and techniques.

Remark 4.2 In practice, we always estimate $\{L_j, j = 1, 2, \ldots\}$ first, then we work out g by Theorem 3.2. That is, for any $j = 1, 2, \ldots,$

$$g(t) = L_{j-1}^{-1}(L_j(t)) \quad \text{for all} \quad t \in (T_j, T_{j+1}].$$

• A method to estimate $\{L_j, j = 1, 2, \ldots\}$.

Using the final data pairs (4.9), Step 4 says we can estimate g by statistical methods and techniques. However, it does not present any concrete method to estimate g. In the following, we will present a concrete method to estimate $\{L_j, j = 1, 2, \ldots\}$, then we work out g by Theorem 3.2. First, we restate Weierstrass Approximation Theorem in algebra.

Weierstrass Approximation Theorem ([17]) *Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial p such that $|f(x) - p(x)| < \varepsilon$ holds for all $x \in [a, b]$.*

Suppose the final data pairs proceeded by Steps 1–3 as follows

$$\{(T_j + 1, T_i + 1);\ (t, g_0(t)), t = T_j + 2, \ldots, T_{j+1} - 1;\ (T_{j+1}, T_{i+1})\}.$$

In order to estimate L_j, we should ensure that

$$g(T_j + 1) = T_i + 1 \quad \text{and} \quad g(T_{j+1}) = T_{i+1}. \tag{4.10}$$

According to Weierstrass Approximation Theorem, we can use a polynomial to estimate the scale transformation L_j. Without loss of generality, assume

$$L_j(t) = c_0 + \sum_{k=1}^{q} c_k[t - (T_j + 1)]^k, \quad T_j + 1 \le t \le T_{j+1},$$

where $q \in \mathbf{N}$ is the order number of polynomial L_j. It yields from (4.10) that

$$c_0 = T_i + 1, \quad c_1 = \frac{T_{i+1} - (T_i + 1)}{T_{j+1} - (T_j + 1)} - \sum_{k=2}^{q} c_k[T_{j+1} - (T_j + 1)]^{k-1}.$$

Thus,

$$L_j(t) = (T_i + 1) + \frac{T_{i+1} - (T_i + 1)}{T_{j+1} - (T_j + 1)} \cdot [t - (T_j + 1)] +$$

$$\sum_{k=2}^{q} c_k\{[t-(T_j+1)]^{k-1}-[T_{j+1}-(T_j+1)]^{k-1}\}\cdot[t-(T_j+1)]. \tag{4.11}$$

In the following, we will estimate the unknown parameters $c_2, c_3, \ldots, c_q$ using the data pairs

$$\big\{(t, g_0(t)),\ t=T_j+2,\ldots,T_{j+1}-1\big\}.$$

According to the least square method, we can obtain the following result.

Result 4.3 Assume we use a q-order polynomial to fit L_j with the data pairs (4.9), then L_j is given by (4.11), where $C=(c_2,c_3,\ldots,c_q)^T$ is determined by

$$C=\Gamma^{-1}B,$$

where $\Gamma=(\Gamma_{uv})_{(q-1)\times(q-1)}$ and $B=(B_u)_{(q-1)\times 1}$, here

$$\begin{aligned}\Gamma_{uv}=&\sum_{s=2}^{T_{j+1}-(T_j+1)}[T_j+s-(T_j+1)]^2\cdot\{[T_j+s-(T_j+1)]^u-[T_{j+1}-(T_j+1)]^u\}\cdot\\&\{[T_j+s-(T_j+1)]^v-[T_{j+1}-(T_j+1)]^v\}\end{aligned}$$

and

$$\begin{aligned}B_u=&\sum_{s=2}^{T_{j+1}-(T_j+1)}\{[g_0(T_j+s)-(T_i+1)]-\frac{T_{i+1}-(T_i+1)}{T_{j+1}-(T_j+1)}\cdot[T_j+s-(T_j+1)]\}\cdot\\&[T_j+s-(T_j+1)]\cdot\{[T_j+s-(T_j+1)]^u-[T_{j+1}-(T_j+1)]^u\}.\end{aligned}$$

Remark 4.4 If the forms of scale transformation g and approximate periodic function S are known, then we can combine Steps 3 and 4 as a step, that is, we can estimate parameters of g and S by directly minimizing some distance of $Z_t-S(g(t))$.

5. Test for fitting effect of scale transformation

In the previous section, we mainly present estimation method of scale transformation g or $\{L_j, j=1,2,\ldots\}$. How to measure the fitting effect of scale transformation g is an unsolved problem.

Note that the effect of L_j is to change Z into $\{S(t), T_0<t\le T_1\}$, where Z is changed from Y by (4.2) and (4.3), so a "good" L_j should change Z very like $\{S(t), T_0<t\le T_1\}$. That is, $\{Z_t-S(L_j(t)), t=T_j+1, T_j+2,\ldots,T_{j+1}\}$ should be a stationary process with mean zero. Thus, we can check whether L_j is a "good" transformation by testing whether $\{Z_t-S(L_j(t)), t=T_j+1, T_j+2,\ldots,T_{j+1}\}$ is a stationary process with mean zero.

According to Theorem 3.2, $\{L_j, j=1,2,\ldots\}$ and g are mutually determined from each other, so we can test whether g is a "good" scale transformation by testing whether $L_1, L_2,\ldots$ are all "good" scale transformations. However, it is still very difficult to directly test whether $L_1, L_2,\ldots$ are all "good" scale transformations. Fortunately, we can indirectly test whether $L_1, L_2,\ldots$ are all "good" scale transformations by testing fitting effect of all $L_1, L_2,\ldots$. That is, we can check whether $L_1, L_2,\ldots$ are all "good" scale transformations by testing whether

$\{W_t, t = T_1 + 1, T_1 + 2, \ldots, T_j + 1, T_j + 2, \ldots\}$ is a stationary process with mean zero, where

$$W_t = Z_t - S(L_j(t)),\ t = T_j + 1, T_j + 2, \ldots, T_{j+1},\quad j = 1, 2, \ldots.$$

Result 5.1 For a given significance level $\alpha > 0$, if $\{W_t, t = T_1 + 1, T_1 + 2, \ldots, T_j + 1, T_j + 2, \ldots\}$ is accepted as a stationary process with mean zero, where

$$W_t = Z_t - S(L_j(t)),\ t = T_j + 1, T_j + 2, \ldots, T_{j+1},\quad j = 1, 2, \ldots, \tag{5.1}$$

then $\{g(t), t > T_1\}$ is accepted as a "good" scale transformation under the significance level α.

6. Example of estimating scale transformation

In this section, we will present an example to show the process of estimating scale transformation g with a simulated time series, which also shows that our method is very powerful.

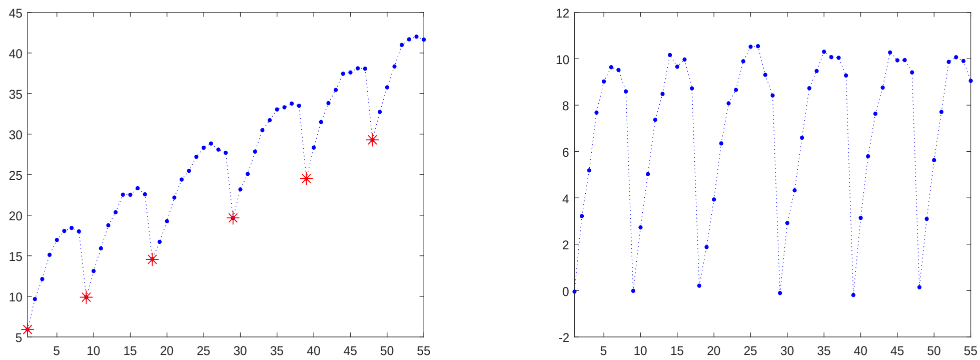
6.1. Generate an approximate periodic time series with long-term trend

Consider a time series as follows

$$x(t) = 5 + 0.5t + 10\sin\left(\frac{2(t - T_k - 1)\pi}{3(T_{k+1} - T_k - 1)}\right) + \varepsilon_t,\quad t = T_k + 1, \ldots, T_{k+1},\ k = 0, 1, 2, 3, 4, 5, \tag{6.1}$$

where $T_0 = 0, T_1 = 8, T_2 = 17, T_3 = 28, T_4 = 38, T_5 = 47, T_6 = 55$. In order to exactly repeat our computation, we set the initial state for generating random number $\{\varepsilon_t, 1 \le t \le 55\}$ as $rng(1)$ using matlab R2018b software.

We first plot the sequence chart of $\{x(t), 1 \le t \le 55\}$ in Figure 6 (a), where the starts, "$*$", are the minimum value points of each approximate periodicity.



(a) The sequence chart of $\{x(t)\}$ (b) The sequence chart of $\{\tilde{x}(t)\}$

Figure 6 The sequence charts of $\{x(t)\}$ and $\{\tilde{x}(t)\}$

6.2. Estimate the long-term trend of $\{x(t)\}$

According to Step 1, we fit the long-term trend of $\{x(t)\}$ with its minimum value points of each approximate periodicity

$$\{(1, x(1)), (9, x(9)), (18, x(18)), (29, x(29)), (39, x(39)), (48, x(48))\}$$

by a linear function h and obtain

$$h(t) = 5.46832 + 0.49336t, \quad t \geq 1. \tag{6.2}$$

Then we obtain the time series with eliminated long-term trend as follows

$$\tilde{x}(t) = x(t) - h(t), \quad t = 1, 2, \ldots, 55 \tag{6.3}$$

and we plot the sequence chart of $\{\tilde{x}(t), 1 \leq t \leq 55\}$ in Figure 6 (b).

6.3. Generate the mapping g_0 from $\{9, 10, \ldots, 55\}$ to $[1, 8]$

It follows from the sequence chart of $\{\tilde{x}(t)\}$ in Figure 6 (b) that δ in Step 2 approximately equals one, so we skip Step 2 and directly generate the mapping g_0 from $\{9, 10, \ldots, 55\}$ to $[1, 8]$ by Step 3. The results are shown in Table 2.

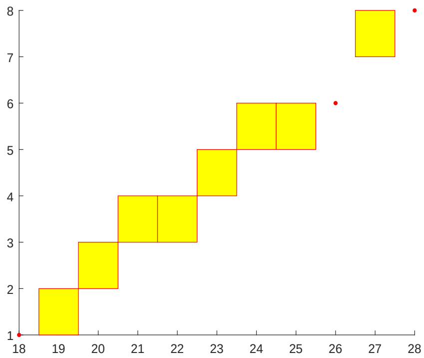
2^{nd} epoch	t	9	10	11	12	13	14	15	16	17		
	$g_{0d}(t)$	1	1	2	3	4	5	6	7	8		
	$g_{0u}(t)$	1	2	3	4	5	6	7	8	8		
3^{rd} epoch	t	18	19	20	21	22	23	24	25	26	27	28
	$g_{0d}(t)$	1	1	2	3	3	4	5	5	6	7	8
	$g_{0u}(t)$	1	2	3	4	4	5	6	6	6	8	8
4^{th} epoch	t	29	30	31	32	33	34	35	36	37	38	
	$g_{0d}(t)$	1	1	2	3	4	4	5	6	7	8	
	$g_{0u}(t)$	1	2	3	4	5	5	6	7	8	8	
5^{th} epoch	t	39	40	41	42	43	44	45	46	47		
	$g_{0d}(t)$	1	1	2	3	4	5	6	7	8		
	$g_{0u}(t)$	1	2	3	4	5	6	7	8	8		
6^{th} epoch	t	48	49	50	51	52	53	54	55			
	$g_{0d}(t)$	1	2	3	4	5	6	7	8			
	$g_{0u}(t)$	1	2	3	4	5	6	7	8			

Table 2 The mapping g_0 from $\{9, 10, \ldots, 55\}$ to $[1, 8]$

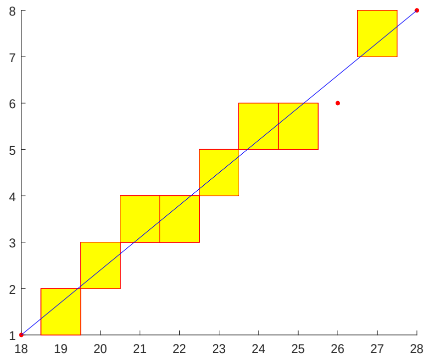
In Table 2, $g_{0d}(t)$ is the lower bound of interval and $g_{0u}(t)$ is the upper bound of interval, i.e., $g_0(t) \in [g_{0d}(t), g_{0u}(t)]$. Particularly, if $g_{0d}(t) = g_{0u}(t)$, then $g_0(t) = g_{0d}(t) = g_{0u}(t)$. For example, the mapping relation g_0 on the third approximate period $\{18, 19, \ldots, 28\}$ follows as Table 3 and we draw its chart in Figure 7 (a). It is obvious that the mapping relation g_0 on the third epoch is linear.

t	18	19	20	21	22	23	24	25	26	27	28
$g_0(t)$	1	[1,2]	[2,3]	[3,4]	[3,4]	[4,5]	[5,6]	[5,6]	6	[7,8]	8

Table 3 The mapping g_0 from $\{18, 19, \ldots, 28\}$ to $[1, 8]$



(a) The mapping relation g_0 in the 3^{rd} epoch



(b) The fitting scale transformation g in the 3^{rd} epoch

Figure 7 The mapping relation from $\{18, 19, \ldots, 28\}$ to $[1, 8]$

6.4. Estimate the scale transformation g from $\{9, 10, \ldots, 55\}$ to $[1, 8]$

The previous subsection has shown that the mapping relation g_0 on the third epoch is linear. Thus, we use a linear function to estimate the scale transformation g on the third epoch. The estimated scale transformation g on the third epoch is given as follows

$$g(t) = -11.6 + 0.7t, \quad 18 \le t \le 28$$

and its fitting effect is shown in Figure 7 (b).

Analogically, for any $i = 1, 2, 3, 4, 5$, it follows that

$$g(t) = 1 + \frac{7}{T_{i+1} - T_i - 1}(t - T_i - 1), \quad t \in \{T_i + 1, T_i + 2, \ldots, T_{i+1}\}, \tag{6.4}$$

where $T_0 = 0, T_1 = 8, T_2 = 17, T_3 = 28, T_4 = 38, T_5 = 47, T_6 = 55$. That is, the scale transformation g follows as

$$g(t) = \begin{cases} t, & t \in \{1, 2, \ldots, 8\}, \\ 1 + \frac{7}{8}(t - 9), & t \in \{9, 10, \ldots, 17\}, \\ 1 + \frac{7}{10}(t - 18), & t \in \{18, 19, \ldots, 28\}, \\ 1 + \frac{7}{9}(t - 29), & t \in \{29, 30, \ldots, 38\}, \\ 1 + \frac{7}{8}(t - 39), & t \in \{39, 40, \ldots, 47\}, \\ t - 47, & t \in \{48, 49, \ldots, 55\}. \end{cases} \tag{6.5}$$

6.5. Estimate the approximate periodic function $\{S(t), 1 \le t \le 8\}$

It yields from (6.3) and (6.5) that the adjusted data are as follows

$$D = \left\{ \big(g(t), \tilde{x}(t)\big), t = 1, 2, \ldots, 55 \right\}. \tag{6.6}$$

Basing on the adjusted data D in (6.6), we estimate the function $\{S(t), 1 \le t \le 8\}$ with a cubic polynomial and obtain

$$S(t) = -3.3544 + 3.4166t - 0.11827t^2 - 0.01556t^3, \quad 1 \le t \le 8. \tag{6.7}$$

Then, we plot the sequence chart of adjusted data D and its fitting curve of $\{S(t), 1 \le t \le 8\}$ in Figure 8.

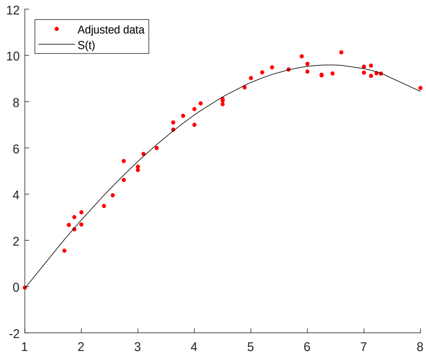


Figure 8 The sequence chart of D and the fitting curve of $\{S(t), 1 \le t \le 8\}$

6.6. Estimation effect comparison of different methods

In the subsection, we will show the estimation effect of our method. In order to distinguish it from traditional periodic method, we call it the approximate periodic method.

6.6.1. Estimation effect of approximate periodic method

It yields from (6.2), (6.7) and (6.5) that, for any $t = 1, 2, \ldots, 55$, the estimation value of $x(t)$ is given as follows

$$\hat{x}(t) = h(t) + S(g(t)),$$

i.e.,

$$\hat{x}(t) = 5.46832 + 0.49336t + (-3.3544 + 3.4166g(t) - 0.11827g^2(t) - 0.01556g^3(t)), \tag{6.8}$$

where

$$g(t) = \begin{cases} t, & t \in \{1, 2, \ldots, 8\}, \\ 1 + \frac{7}{8}(t-9), & t \in \{9, 10, \ldots, 17\}, \\ 1 + \frac{7}{10}(t-18), & t \in \{18, 19, \ldots, 28\}, \\ 1 + \frac{7}{9}(t-29), & t \in \{29, 30, \ldots, 38\}, \\ 1 + \frac{7}{8}(t-39), & t \in \{39, 40, \ldots, 47\}, \\ t - 47, & t \in \{48, 49, \ldots, 55\}. \end{cases}$$

We draw $\{x(t), t = 1, 2, \ldots, 55\}$ and its fitting values $\{\hat{x}(t), t = 1, 2, \ldots, 55\}$ by the approximate periodic method in Figure 9 (a).

6.6.2. Estimation effect of traditional periodic method

In order to compare our method (i.e., approximate periodic method) with traditional periodic methods, we will estimate $\{x(t), t = 1, 2, \ldots, 55\}$ by traditional periodic methods. Because the average period length of $\{x(t), t = 1, 2, \ldots, 55\}$ approximately equals nine, we take the period length $\tau = 8$, 9 and 10 for the traditional periodic methods, respectively.

Step 1. Compute the τ-step difference of $x(t)$ by

$$\Delta_\tau x(t) = x(t) - x(t - \tau), \quad t = \tau + 1, \tau + 2, \ldots, 55,$$

where Δ_τ is the τ-step difference operator.

Step 2. Stationarity test and pure randomness test for $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots, 55\}$

Using the augmented Dickey-Fuller test (i.e., "adftest" function in Matlab R2018b) and the Ljung-Box Q-test (i.e., "lbqtest" function in Matlab R2018b) to test stationarity and pure randomness, respectively. The test results are presented in Table 4, which indicates that the residual $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots, 55\}$ is stationary and not pure random for $\tau = 8$ and 9, and stationary and pure random for $\tau = 10$.

τ	Augmented Dickey-Fuller test			Ljung-Box Q-test			Test result
	Statistic	p-Value	h	Statistic	p-Value	h	
8	-2.8149	0.0062	1	101.08	$8.0524 \times e^{-13}$	1	stationary, not pure random
9	-2.1476	0.0319	1	33.451	0.030089	1	stationary, not pure random
10	-2.3986	0.0176	1	24.124	0.23701	0	stationary, pure random

Table 4 Stationarity test and pure randomness test for $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots, 55\}$

Step 3. Establish ARMA models of $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots\}$

It yields from Table 4 that $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots\}$ is a white noise while $\tau = 10$, and a stationary & not pure random series while $\tau = 8$ or 9. Thus, we will establish ARMA models of $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots\}$ while $\tau = 8$ or 9, and white noise model while $\tau = 10$ as follows

$$\Phi(B)\Delta_\tau x(t) = \Theta(B)\varepsilon_t, \tag{6.9}$$

where B is the delay operator that $B^m x(t) = x(t - m)$ holds for all $m = 1, 2, \ldots$, and $\{\varepsilon_t, t = \tau + 1, \tau + 2, \ldots\}$ is a white noise series.

τ	Model	$\Phi(B)$	$\Theta(B)$	σ_ε^2
8	ARMA(2,1)	$1 - 1.491B + 0.4851B^2$	$1 - 0.9613B$	3.7412
9	ARMA(2,1)	$1 - 1.291B + 0.299B^2$	$1 - B$	2.4488
10	white noise	0	1	2.7322

Table 5 ARMA models of $\{\Delta_\tau x(t),\ t = \tau + 1, \tau + 2, \ldots\}$

Step 4. Estimation effect of traditional periodic methods

It yields from Steps 1–3 that, for any $\ell > 0$,

$$x(t) = x(t - \tau) + \Delta_\tau x(t),$$

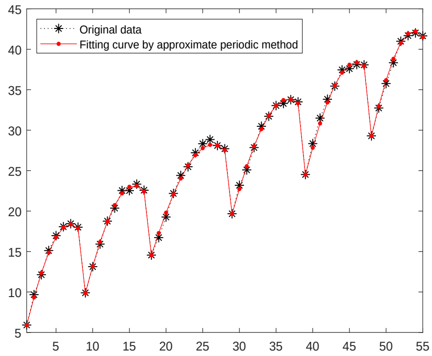
so

$$\hat{x}_t(\ell) = \tilde{x}_t(\ell - \tau) + \tilde{Y}_t(\ell), \quad t = \tau + 1, \tau + 2, \ldots, 55, \tag{6.10}$$

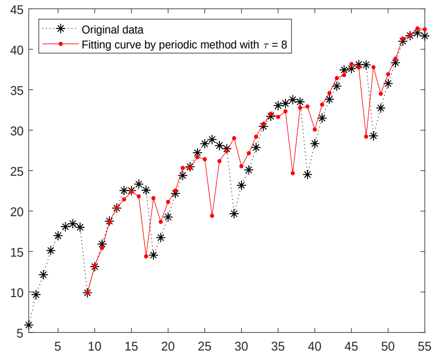
where $\hat{x}_t(\ell)$ denotes the ℓ-step estimation about $x(t+\ell)$ at time t, $Y(t) = \Delta_\tau x(t)$,

$$\tilde{x}_t(k) = \begin{cases} x(t+k), & k \le 0 \\ \hat{x}_t(k), & k > 0 \end{cases} \quad \text{and} \quad \tilde{Y}_t(k) = \begin{cases} Y(t+k), & k \le 0 \\ \hat{Y}_t(k), & k > 0 \end{cases}.$$

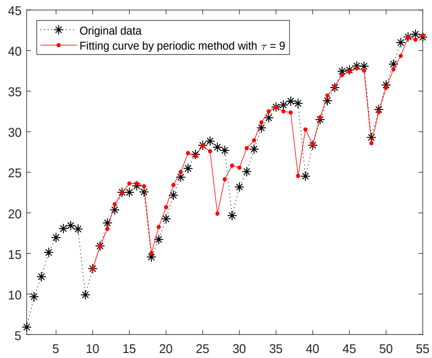
We draw $\{x(t), t = 1, 2, \ldots, 55\}$ and its fitting values $\{\hat{x}(t), t = 1, 2, \ldots, 55\}$ by traditional periodic methods with $\tau = 8$, 9 and 10 in Figures 9 (b), 9 (c) and 9(d), respectively. It follows from comparing Figure 9 (a) with Figures 9 (b)–(d) that approximate periodic method is far better than traditional periodic methods in the sense of fitting effect.



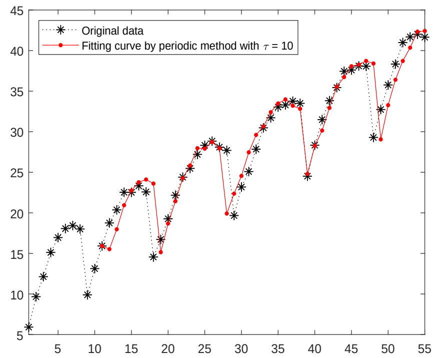
(a) Fitting effect by approximate periodic method



(b) Fitting effect by traditional periodic method ($\tau = 8$)



(c) Fitting effect by traditional periodic method ($\tau = 9$)



(d) Fitting effect by traditional periodic method ($\tau = 10$)

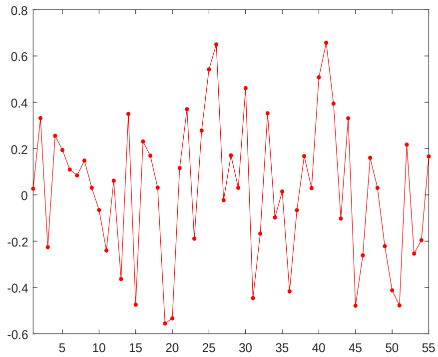
Figure 9 Fitting effect of $\{x(t), t = 1, 2, \ldots, 55\}$ by different methods

6.6.3. Residual comparison of different methods

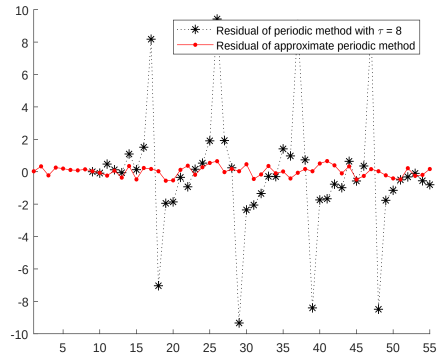
According to the previous calculation, we obtain the residual by approximate periodic method and the residuals by traditional periodic methods with $\tau = 8$, 9 and 10 in Table 6 and draw their charts in Figure 10, which shows approximate periodic method is far more powerful than traditional periodic methods in fitting effect.

Method type		Mean	STD	Maximum absolute deviation
Approximate periodic method		0.0254	0.3102	0.6571
Traditional periodic method	$\tau = 8$	-0.1708	3.7412	9.4254
	$\tau = 9$	$-5.5625 \times e^{-8}$	2.4488	8.9452
	$\tau = 10$	0.0044	2.6363	9.1144

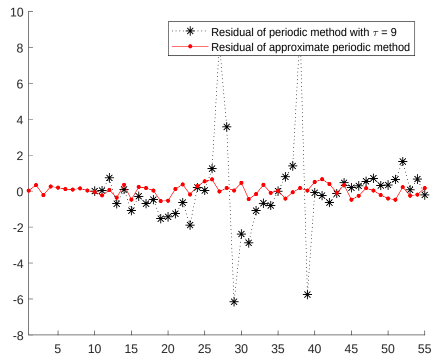
Table 6 Residual comparison of different methods



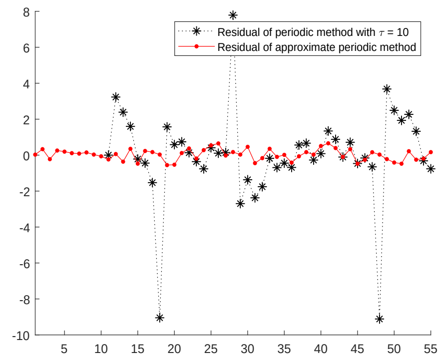
(a) Residual by approximate periodic method



(b) Residual comparison of traditional periodic method ($\tau = 8$) and approximate periodic method



(c) Residual comparison of traditional periodic method ($\tau = 9$) and approximate periodic method



(d) Residual comparison of traditional periodic method ($\tau = 10$) and approximate periodic method

Figure 10 Residual comparison by different methods

7. Conclusions

In the paper, shape-retention transformation with lengthwise compression is first brought forward, and some necessary and sufficient conditions for the transformations are obtained. Then, basing on the linear shape-retention transformation with lengthwise compression we bring forward a method to estimate the scale transformation of approximate periodic time series with long-term trend. At last, a simulated example is analyzed by our method and traditional periodic methods. The results show that our method is far effective and very powerful.

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