Journal of Mathematical Research with Applications
May, 2021, Vol. 41, No. 3, pp. 259–264
DOI:10.3770/j.issn:2095-2651.2021.03.003
Http://jmre.dlut.edu.cn

Boundedness of an Integral Operator on Bloch-Type Spaces

Xiaoyang HOU[1,*], **Chao LIU**[2]
1. *Department of Basic, Wenzhou Business College, Zhejiang* 325035*, P. R. China;*
2. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China*

Abstract In this paper, we study the boundedness of an integral operator K over the unit disk $\mathbb{D}$, defined as $Kf(z)=\int_{\mathbb{D}}\frac{f(w)}{1-z\bar{w}}\mathrm{d}A(w)$, which can be viewed as a cousin of the classical Bergman projection, and we establish satisfactory boundedness results between Bloch-type spaces, H^∞ and L^p spaces.

Keywords Boundedness; integral operator; Bloch-type space; Hardy space

MR(2020) Subject Classification 30H30; 31A10; 34C11

1. Introduction

Let H^∞ be the space of all bounded holomorphic functions on the unit disk $\mathbb{D}$ with the norm $\|f\|_\infty=\sup_{z\in\mathbb{D}}|f(z)|$. For $0<\alpha<\infty$, the Bloch-type space $\mathcal{B}_\alpha$ consists of all analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty. \tag{1.1}$$

$\mathcal{B}_\alpha$ is a Banach space [1] with the corresponding norm

$$\|f\|_\alpha=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|. \tag{1.2}$$

Note that $\mathcal{B}_1=\mathcal{B}$ is the classical Bloch space.

This paper is devoted to studying an integral operator K on the Bloch-type space $\mathcal{B}_\alpha$, defined as

$$Kf(z)=\int_{\mathbb{D}}\frac{f(w)}{1-z\bar{w}}\mathrm{d}A(w), \tag{1.3}$$

where $\mathrm{d}A$ denotes normalized Lebesgue measure over the unit disk $\mathbb{D}$, i.e., $\mathrm{d}A=\frac{1}{\pi}\mathrm{d}x\mathrm{d}y$.

This K operator, which can be viewed as a cousin of the classical Bergman projection, is important and ubiquitous in complex analysis and operator theory, but the study of K as an integral operator has just started [2, 3]. In the paper [2], the authors completely determined all pairs (p,q) such that $K: L^p(\mathbb{D})\to L^q(\mathbb{D})$ is bounded, and recently, a generalization of this result on the unit ball $\mathbb{B}$ was done by Cheng, Hou and Liu [3]. Motivated by these results, we investigate boundedness of the operator K from $\mathcal{B}_\alpha$ to H^∞; from $L^p(\mathbb{D})$ to $\mathcal{B}_\alpha$; and from

Received December 2, 2019; Accepted October 24, 2020
Supported by the Natural Science Foundation of Zhejiang Province (Grant No. LY14A010021).
* Corresponding author
E-mail address: xyhou@wzbc.edu.cn (Xiaoyang HOU); chaoliumath1@gmail.com (Chao LIU)

$H^p(\mathbb{D})$ to $\mathcal{B}_\alpha$. Namely, we seek to characterize the boundedness of this Bergman-type projection operator. Another reason for our studying the Bloch-type space is that the norm in (1.2) is defined without using an integral, and we are interested in the situation of the integral operator K relating to 'non-integral' spaces.

2. Boundedness of $K:\mathcal{B}_\alpha\to H^\infty$

Our first main result is the following

Theorem 2.1 *Let $0<\alpha<\infty$. Then $K:\mathcal{B}_\alpha\to H^\infty$ is bounded if and only if $0<\alpha<2$.*

Before proving the theorem, we present two lemmas.

Lemma 2.2 ([4]) *For $s>-1$ and t real, let*

$$I_{s,t}(z)=\int_{\mathbb{D}}\frac{(1-|w|^2)^s}{|1-z\bar{w}|^{2+s+t}}\mathrm{d}A(w),\quad z\in\mathbb{D}.$$

Then

(1) $I_{s,t}(z)$ is bounded in z if $t<0$;

(2) $I_{s,t}(z)\sim-\log(1-|z|^2)$ as $|z|\to1^-$ if $t=0$;

(3) $I_{s,t}(z)\sim(1-|z|^2)^{-t}$ as $|z|\to1^-$ if $t>0$.

To prove Theorem 2.1, we need to introduce an auxiliary operator. For any $\alpha>0$, let Q_α denote the operator defined by

$$Q_\alpha f(z)=\alpha\int_{\mathbb{D}}\frac{f(w)}{(1-z\bar{w})^{\alpha+1}}\mathrm{d}A(w),\quad z\in\mathbb{D}.$$

Lemma 2.3 ([1]) *For each $\alpha>0$, the operator Q_α maps $L^\infty(\mathbb{D})$ boundedly onto $\mathcal{B}_\alpha$, and there exists a constant $C>0$, such that*

$$C^{-1}\|f\|_\alpha\le\inf\{\|g\|_\infty: f=Q_\alpha g, g\in L^\infty(\mathbb{D})\}\le C\|f\|_\alpha.$$

Based on Lemma 2.3, we can use Q_α to transform the estimation of the $\mathcal{B}_\alpha$-norm into the integral form.

Proof of Theorem 2.1 For any $f\in\mathcal{B}_\alpha$, there exists $g\in L^\infty(\mathbb{D})$ such that $f=Q_\alpha(g)$, by Lemma 2.3. So we have

$$Kf(z)=\int_{\mathbb{D}}\frac{f(w)}{1-z\bar{w}}\mathrm{d}A(w)=\alpha\int_{\mathbb{D}}\frac{1}{1-z\bar{w}}\int_{\mathbb{D}}\frac{g(u)}{(1-w\bar{u})^{1+\alpha}}\mathrm{d}A(u)\mathrm{d}A(w),\tag{2.1}$$

thus,

$$|Kf(z)|\le\alpha\|g\|_\infty\int_{\mathbb{D}}\frac{1}{|1-z\bar{w}|}\mathrm{d}A(w)\int_{\mathbb{D}}\frac{1}{|1-w\bar{u}|^{1+\alpha}}\mathrm{d}A(u).\tag{2.2}$$

In the following proof, it is enough to show

$$|Kf(z)|\le C\|g\|_\infty,$$

for some constant C, since by taking the infimum over $g\in L^\infty(\mathbb{D})$, then Lemma 2.3 implies that $\|Kf\|_\infty\le C\|f\|_\alpha$, i.e., $K:\mathcal{B}_\alpha\to H^\infty$ is bounded.

We divide the proof into 3 cases.

Case 1. $0<\alpha<1$. Since $\int_{\mathbb{D}}\frac{1}{|1-w\bar{u}|^{1+\alpha}}\mathrm{d}A(u)$ and $\int_{\mathbb{D}}\frac{1}{|1-z\bar{w}|}\mathrm{d}A(w)$ are bounded in w and in z, respectively, by Lemma 2.2, from (2.2), we get

$$|Kf(z)|\le C_1\|g\|_\infty\int_{\mathbb{D}}\frac{1}{|1-z\bar{w}|}\mathrm{d}A(w)\le C_2\|g\|_\infty.$$

Case 2. $1<\alpha<2$. $\int_{\mathbb{D}}\frac{1}{|1-w\bar{u}|^{1+\alpha}}\mathrm{d}A(u)\sim(1-|w|^2)^{1-\alpha}$ by Lemma 2.2, so from (2.2), we have

$$|Kf(z)|\le C\|g\|_\infty\int_{\mathbb{D}}\frac{(1-|w|^2)^{1-\alpha}}{|1-z\bar{w}|}\mathrm{d}A(w),$$

and the above integration is bounded by Lemma 2.2 again.

Case 3. $\alpha=1$. Notice that $\mathcal{B}\subset L_a^1(\mathbb{D})$, the Bergman 1-space, so by Fubini's theorem and the reproducibility of the Bergman kernel, we obtain

$$\begin{aligned}Kf(z)&=\int_{\mathbb{D}}g(u)\int_{\mathbb{D}}\frac{1}{1-z\bar{w}}\cdot\frac{1}{(1-w\bar{u})^2}\mathrm{d}A(w)\mathrm{d}A(u)\\&=\int_{\mathbb{D}}g(u)\overline{\int_{\mathbb{D}}\frac{1}{1-w\bar{z}}\cdot\frac{1}{(1-u\bar{w})^2}\mathrm{d}A(w)}\mathrm{d}A(u)\\&=\int_{\mathbb{D}}g(u)\overline{(\frac{1}{1-u\bar{z}})}\mathrm{d}A(u)=\int_{\mathbb{D}}\frac{g(u)}{1-z\bar{u}}\mathrm{d}A(u).\end{aligned}$$

It follows that

$$|Kf(z)|\le\|g\|_\infty\int_{\mathbb{D}}\frac{1}{|1-z\bar{u}|}\mathrm{d}A(u)\le C\|g\|_\infty.$$

So we have proved that K maps $\mathcal{B}_\alpha$ boundedly into H^∞ when $0<\alpha<2$. Conversely, when $\alpha\ge 2$, $K:\mathcal{B}_\alpha\to H^\infty$ is unbounded, in fact, we only need to give an example when $\alpha=2$ since $\mathcal{B}_2\subset\mathcal{B}_\alpha$ when $\alpha>2$. The following Lemma is required, which will show that $\mathcal{B}_\alpha$ contains lots of interesting functions.

Lemma 2.4 ([5]) *Let $\alpha>0$ and λ_n be a sequence of positive integers satisfying*

$$1<\lambda\le\frac{\lambda_{n+1}}{\lambda_n}\le M<+\infty,\quad n\ge 1.$$

Then for $f(z)=\sum_{n=1}^\infty a_nz^{\lambda_n}$, $f(z)\in\mathcal{B}_\alpha$ if and only if $\limsup_{n\to\infty}|a_n|\lambda_n^{1-\alpha}<\infty$.

Now, let λ be a positive integer, $\lambda>1$, $a_n=\lambda^{n+1},\lambda_n=\lambda^n$, and $\alpha=2$. Then

$$\lim_{n\to\infty}a_n\lambda_n^{1-\alpha}=\lim_{n\to\infty}\frac{\lambda^{n+1}}{\lambda^n}=\lambda<\infty,$$

so $f(z)=\sum_{n=1}^\infty\lambda^{n+1}z^{\lambda^n}\in\mathcal{B}_2$ by Lemma 2.4, but

$$\begin{aligned}Kf(z)&=\int_{\mathbb{D}}\frac{f(w)}{1-z\bar{w}}\mathrm{d}A(w)=\int_{\mathbb{D}}\sum_{n=1}^\infty\lambda^{n+1}w^{\lambda^n}\sum_{m=0}^\infty z^m\bar{w}^m\mathrm{d}A(w)\\&=\sum_{n=1}^\infty\lambda^{n+1}z^{\lambda^n}\int_{\mathbb{D}}|w|^{2\lambda^n}\mathrm{d}A(w)=\sum_{n=1}^\infty\frac{\lambda^{n+1}}{\lambda^n+1}z^{\lambda^n}.\end{aligned}$$

It is easy to verify that $\sum_{n=1}^\infty\frac{\lambda^{n+1}}{\lambda^n+1}$ is divergent, so $Kf(z)\notin H^\infty$. This completes the proof of Theorem 2.1. □

3. Boundedness of $K : L^p(\mathbb{D}) \to \mathcal{B}_\alpha$

In this section, we characterize the boundedness of the operator $K : L^p(\mathbb{D}) \to \mathcal{B}_\alpha$.

Theorem 3.1 *Let $p \ge 1$. Then $K : L^p(\mathbb{D}) \to \mathcal{B}_\alpha$ is bounded if and only if $\alpha \ge \frac{2}{p}$.*

Proof For any $f \in L^p(\mathbb{D})$, we have

$$|Kf(0)| = \Big|\int_{\mathbb{D}} f(w)\mathrm{d}A(w)\Big| \le \|f\|_p,$$

so we only need to prove

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(Kf)'(z)| \le C\|f\|_p. \tag{3.1}$$

If $p > 1$, using Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}(1-|z|^2)^\alpha|(Kf)'(z)| =&(1-|z|^2)^\alpha\Big|\int_{\mathbb{D}}\frac{\bar{w}f(w)}{(1-z\bar{w})^2}\mathrm{d}A(w)\Big|\\ \le&(1-|z|^2)^\alpha\Big(\int_{\mathbb{D}}|\bar{w}f(w)|^p\mathrm{d}A(w)\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{D}}\frac{1}{|1-z\bar{w}|^{2q}}\mathrm{d}A(w)\Big)^{\frac{1}{q}}\\ \le&||f||_p(1-|z|^2)^\alpha C(1-|z|^2)^{\frac{-2q+2}{q}}\\ =&C||f||_p(1-|z|^2)^{\alpha-2+\frac{2}{q}} = C||f||_p(1-|z|^2)^{\alpha-\frac{2}{p}},\end{aligned} \tag{3.2}$$

where q is a real number satisfying $\frac{1}{p}+\frac{1}{q}=1$. Notice that, when $p=\infty$, the above inequalities still hold.

If $p = 1$,

$$\begin{aligned}(1-|z|^2)^\alpha|(Kf)'(z)| &= (1-|z|^2)^\alpha\Big|\int_{\mathbb{D}}\frac{\bar{w}f(w)}{(1-z\bar{w})^2}\mathrm{d}A(w)\Big|\\ &\le C(1-|z|^2)^\alpha\int_{\mathbb{D}}\frac{|f(w)|}{(1-|z|^2)^2}\mathrm{d}A(w)\\ &= C||f||_1(1-|z|^2)^{\alpha-2}.\end{aligned} \tag{3.3}$$

It is then easy to see that $K : L^p(\mathbb{D}) \to \mathcal{B}_\alpha$ is bounded when $\alpha \ge \frac{2}{p}$ from (3.1)–(3.3), completing the proof of sufficiency.

Now, we give an example to show that $K : L^p(\mathbb{D}) \to \mathcal{B}_\alpha$ is unbounded for $\alpha < \frac{2}{p}$.

Take a real number a in $(\alpha, \frac{2}{p})$, and let $f_r(z) = \frac{1}{(1-rz)^a}, 0 < r < 1$. Since $a < \frac{2}{p}$, a simple application of Lemma 2.2 shows that $f_r \in L^p(\mathbb{D})$, and

$$\sup_{0<r<1}\|f_r\|_p < \infty.$$

Now, by Taylor expansion and Stirling's formula, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(Kf_r)'(z)| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\Big|\int_{\mathbb{D}}\frac{\bar{w}f_r(w)}{(1-z\bar{w})^2}\mathrm{d}A(w)\Big|\\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\Big|\sum_{n=0}^{\infty}\frac{\Gamma(n+1+a)}{n!\Gamma(a)}r^{n+1}z^n\int_{\mathbb{D}}|w|^{2(n+1)}\mathrm{d}A(w)\Big|\\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\Big|\sum_{n=0}^{\infty}\frac{\Gamma(n+1+a)}{n!\Gamma(a)}\frac{1}{n+2}r^{n+1}z^n\Big|\end{aligned}$$

$$
\begin{aligned}
&\geq (1-r^2)^{\alpha}\Big|\sum_{n=0}^{\infty}\frac{\Gamma(n+1+a)}{n!\Gamma(a)}\frac{1}{n+2}r^{2n}\Big|r\\
&\sim (1-r^2)^{\alpha}\frac{r}{(1-r^2)^{a}}\\
&=\frac{r}{(1-r^2)^{a-\alpha}}\to\infty,\ \text{as } r\to 1^-.
\end{aligned}\tag{3.4}
$$

Moreover,

$$
|Kf_r(0)|=\Big|\int_{\mathbb{D}}f_r(w)\mathrm{d}A(w)\Big|=\Big|\int_{\mathbb{D}}\frac{1}{(1-rw)^{a}}\mathrm{d}A(w)\Big|\geq\frac{1}{(1+r)^{a}}.
$$

Thus $\|Kf_r\|_{\alpha}\to\infty$, as $r\to 1^-$, and the proof of Theorem 3.1 is completed. □

Corollary 3.2 *For any $\alpha>0$, $K: L^{\infty}\to\mathcal{B}_{\alpha}$ is bounded.*

Corollary 3.3 *$K: L^2(\mathbb{D})\to\mathcal{B}_{\alpha}$ is bounded if and only if $\alpha\geq 1$.*

4. Boundedness of $K: H^p(\mathbb{D})\to\mathcal{B}_{\alpha}$

In this section, we characterize boundedness of the operator K between the Hardy space $H^p(\mathbb{D})$ and the Bloch type space $\mathcal{B}_{\alpha}$.

Theorem 4.1 *Let $p\geq 1$. Then $K: H^p(\mathbb{D})\to\mathcal{B}_{\alpha}$ is bounded if and only if $\alpha\geq\frac{1}{p}$.*

Proof We divide the proof into 2 cases.

Case 1. When $p>1$, the sufficiency proof is similar to the above section by applying the following Lemma 4.2 and Lemma 2.2. We leave it to interested readers.

Lemma 4.2 ([4]) *Suppose $0<p<\infty$ and $f\in H^p$. Then*

$$
|f(z)|\leq\frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}}}.
$$

Case 2. When $p=1$,

$$
\begin{aligned}
(1-|z|^2)^{\alpha}|(Kf)'(z)|=&(1-|z|^2)^{\alpha}\Big|\int_{\mathbb{D}}\frac{\bar{w}f(w)}{(1-z\bar{w})^2}\mathrm{d}A(w)\Big|\\
&\leq (1-|z|^2)^{\alpha}\int_0^1 2r\sup_{\theta}\frac{r}{|1-zre^{-i\theta}|^2}\int_0^{2\pi}|f(re^{i\theta})|\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}r\\
&\leq (1-|z|^2)^{\alpha}\|f\|_{H^1}\int_0^1 2r^2\sup_{\theta}\frac{1}{|1-zre^{-i\theta}|^2}\mathrm{d}r\\
&=(1-|z|^2)^{\alpha}\|f\|_{H^1}\int_0^1\frac{2r^2}{(1-|z|r)^2}\mathrm{d}r\\
&=(1-|z|^2)^{\alpha}\|f\|_{H^1}\int_0^1 2r^2\Big(\sum_{n=0}^{\infty}(n+1)|z|^n r^n\Big)\mathrm{d}r\\
&=2(1-|z|^2)^{\alpha}\|f\|_{H^1}\sum_{n=0}^{\infty}\frac{n+1}{n+3}|z|^n
\end{aligned}
$$

$$\leq 2(1-|z|^2)^{\alpha}\|f\|_{H^1}\frac{1}{1-|z|^2}=2\|f\|_{H^1}(1-|z|^2)^{\alpha-1}.$$

So when $\alpha \geq 1$, $K: H^1(\mathbb{D}) \to \mathcal{B}_\alpha$ is bounded, and the sufficiency proof is completed.

Next, we give an example to show that this operator is unbounded when $\alpha < \frac{1}{p}$, and we omit the corresponding proof.

Example 4.3 For any $\alpha < \frac{1}{p}$ and for all $a \in (\alpha, \frac{1}{p})$, let $f_r(z) = \frac{1}{(1-rz)^a}$. Since $ap < 1$, it is easy to show that $f_r \in H^p(\mathbb{D}), 0 < r < 1$ and

$$\sup_{0<r<1} \|f_r\|_{H^p} < \infty,$$

and $\|Kf_r\|_\alpha \to \infty$ as $r \to 1^-$. □

Acknowledgements We thank the referees for their time and comments.

References

[1] Kehe ZHU. *Bloch type spaces of analytic functions*. Rocky Mountain J. Math., 1993, **23**(3): 1143–1177.

[2] Guozheng CHENG, Xiang FANG, Zipeng WANG, et al. *The hyper-singular cousin of the Bergman projection*. Trans. Amer. Math. Soc., 2017, **369**(12): 8643–8662.

[3] Guozheng CHENG, Xiaoyang HOU, Chao LIU. *The singular integral operator induced by Drury Arveson kernel*. Complex Anal. Oper. Theory, 2018, **12**(4): 917–929.

[4] Kehe ZHU. *Operator Theory in Function Spaces*. 2nd edition, American Mathematical Society, Providence, 2007.

[5] S. YAMASHITA. *Gap series and α-Bloch functions*. Yokohama Math. J., 1980, **28**(1): 31–36.