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A New Proof of the Stronger Second Mean Value Theorem for Integrals

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Abstract R. Witula et al obtained a stronger version of the second mean value theorem for integral with some restrictions. In this paper, the stronger version theorem is proved without any restriction. The result is first restricted to the Riemann integrable functions and can be easily generalized to L^p integrable functions by using the well-known result that continuous functions are dense in the Banach space $L^p[a,b]$ for any $p \geq 1$.

Keywords second mean value theorem for integrals; Riemann integrable; Lebesgue integrable

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1. Introduction

The classical form of the second mean value theorem for integral is as follows.

Theorem 1.1 (Second mean value theorem for integrals) *Let f be Riemann integrable on $[a,b]$ and g be monotone on $[a,b]$. Then, there exists c, being an inner point of $[a,b]$, such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a+)\int_a^c f(x)\mathrm{d}x + g(b-)\int_b^c f(x)\mathrm{d}x,$$

where $g(a+) = \lim_{x\to a+} g(x)$ and $g(b-) = \lim_{x\to b-} g(x)$.

In [1], the following result is obtained.

Theorem 1.2 *Let $g \geq 0$ with domain $[a,b]$ be a monotonic function and f with domain $[a,b]$ be a real Lebesgue integrable function.*

(1) (standard version) If for every $c \in (a,b)$, we have

$$0 \neq \int_a^c f(x)\mathrm{d}x \neq \int_a^b f(x)\mathrm{d}x,$$

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then for any $A \le \min\{g(a+), g(b-)\}$ *and* $B \ge \max\{g(a+), g(b-)\}$ *there exists* $\xi = \xi(A,B) \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = A\int_a^\xi f(x)\mathrm{d}x + B\int_\xi^b f(x)\mathrm{d}x \quad \text{if } g(a+) < g(b-),$$

$$\int_a^b f(x)g(x)\mathrm{d}x = B\int_a^\xi f(x)\mathrm{d}x + A\int_\xi^b f(x)\mathrm{d}x \quad \text{if } g(a+) > g(b-).$$

(2) (generalization) Let A_0,B_0 *be real numbers,* $\xi_0 \in (a,b]$ *and*

$$\int_a^b f(x)g(x)\mathrm{d}x = A_0\int_a^{\xi_0} f(x)\mathrm{d}x + B_0\int_{\xi_0}^b f(x)\mathrm{d}x.$$

If $A_0 < B_0$ *and* $\int_a^{\xi_0} f(x)\mathrm{d}x \ne 0$, *then for any* $A < A_0$, *there exists* $\xi = \xi(A) \in (a, \xi_0)$ *such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = A\int_a^\xi f(x)\mathrm{d}x + B_0\int_\xi^b f(x)\mathrm{d}x.$$

If $A_0 > B_0$ *and* $\int_{\xi_0}^b f(x)\mathrm{d}x \ne 0$, *then for any* $B > B_0$ *there exists* $\eta = \eta(B) \in (\xi_0, b]$ *such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = A_0\int_a^\eta f(x)\mathrm{d}x + B\int_\eta^b f(x)\mathrm{d}x.$$

2. Main result

The main result of this paper is to prove Theorem 1.2 without any restriction, restated as follows.

Theorem 2.1 *Let* g *with domain* $[a,b]$ *be a monotonic function and* f *with domain* $[a,b]$ *be a real Riemann integrable function.*

(1) If $g(a+) < g(b-)$, *then for any* $A \le g(a+)$ *and* $B \ge g(b-)$ *there exists* $c = c(A,B) \in [a,b]$, *such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = A\int_a^c f(x)\mathrm{d}x + B\int_c^b f(x)\mathrm{d}x. \tag{2.1}$$

(2) If $g(a+) > g(b-)$, *then for any* $A \ge g(a+)$ *and* $B \le g(b-)$ *there exists* $d = d(A,B) \in [a,b]$, *such that*

$$\int_a^b f(x)g(x)\mathrm{d}x = A\int_a^d f(x)\mathrm{d}x + B\int_d^b f(x)\mathrm{d}x. \tag{2.2}$$

Proof We only prove (2.1), since (2.2) can be proved similarly. Let

$$h(a) = A,\ h(b) = B\ ,\ h(x) = g(x) \ \text{ for } \ a < x < b.$$

Then, h is also an increasing function and we only need to prove

$$\int_a^b f(x)h(x)\mathrm{d}x = \int_a^b f(x)g(x)\mathrm{d}x = h(a)\int_a^c f(x)\mathrm{d}x + h(b)\int_c^b f(x)\mathrm{d}x. \tag{2.3}$$

It is easy to check that (2.3) is equivalent to

$$\int_a^b f(x)u(x)\mathrm{d}x = u(b)\int_c^b f(x)\mathrm{d}x, \tag{2.4}$$

where $u(x) = h(x) - h(a) \geq 0$. For any natural number n, denote $x_i = a + \frac{i(b-a)}{n}$, $0 \leq i \leq n$, and

$$F(x) = \int_a^x f(s)\mathrm{d}s, \quad a \leq x \leq b.$$

Let $t_1 = a$, $t_n = b$, $t_i \in [x_{i-1}, x_i]$ for $2 \leq i \leq n-1$. We also denote $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ being the infimum and supremum of f on the interval $[x_{i-1}, x_i]$, respectively. Since $u(x) \geq 0$, we have

$$\sum_{i=1}^n m_i u(t_i)\Delta x_i \leq \sum_{i=1}^n f(t_i)u(t_i)\Delta x_i \leq \sum_{i=1}^n M_i u(t_i)\Delta x_i, \tag{2.5}$$

$$\sum_{i=1}^n m_i u(t_i)\Delta x_i \leq \sum_{i=1}^n u(t_i)\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \leq \sum_{i=1}^n M_i u(t_i)\Delta x_i, \tag{2.6}$$

where $\Delta x_i = x_i - x_{i-1}$. Denote

$$E_n = \sum_{i=1}^n f(t_i)u(t_i)\Delta x_i - \sum_{i=1}^n u(t_i)\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x.$$

Since $u(x) \geq 0$ is increasing, (2.5) and (2.6) show that

$$|E_n| \leq \sum_{i=1}^n [M_i - m_i]u(t_i)\Delta x_i \leq u(b)\sum_{i=1}^n [M_i - m_i]\Delta x_i \to 0, \quad \text{as} \quad n \to \infty, \tag{2.7}$$

since f is Riemann integrable and $\Delta x_i = \frac{b-a}{n} \to 0$, as $n \to \infty$. Thus, using Abel's summation formula, we obtain

$$\begin{aligned}\sum_{i=1}^n f(t_i)u(t_i)\Delta x_i &= \sum_{i=1}^n u(t_i)\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x + E_n \\ &= \sum_{i=1}^n u(t_i)[F(x_i) - F(x_{i-1})] + E_n \\ &= u(t_n)F(x_n) - \sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)]F(x_i) - u(t_1)F(x_0) + E_n. \end{aligned} \tag{2.8}$$

Let m and M be the minimum and maximum values of F, respectively. Noting that $u(t_{i+1}) - u(t_i) \geq 0$, it holds that

$$m\sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)] \leq \sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)]F(x_i) \leq M\sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)].$$

That is,

$$m[u(t_n) - u(t_1)] \leq \sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)]F(x_i) \leq M[u(t_n) - u(t_1)]. \tag{2.9}$$

Noting that $u(t_n) = u(b)$, $u(t_1) = u(a) = 0$ and the assumption $g(b-) > g(a+)$ implies $u(b) \geq g(b-) - g(a+) > 0$. According to (2.9), we obtain

$$m \leq \frac{\sum_{i=1}^{n-1}[u(t_{i+1}) - u(t_i)]F(x_i)}{u(b)} \leq M.$$

According to the intermediate value theorem for continuous function, there exists $c_n \in [a,b]$ such that

$$\sum_{i=1}^{n-1}[u(t_{i+1})-u(t_i)]F(x_i) = u(b)F(c_n) = u(b)\int_a^{c_n} f(s)\mathrm{d}s. \tag{2.10}$$

According to (2.8) and (2.10), and noting that $u(t_n) = u(b)$, $F(x_0) = 0$ and $F(x_n) = \int_a^b f(s)\mathrm{d}s$, it holds that

$$\sum_{i=1}^{n-1} f(t_i)u(t_i)\Delta x_i = u(b)\int_{c_n}^{b} f(s)\mathrm{d}s + E_n. \tag{2.11}$$

Since $c_n \in [a,b]$, according to Bolzano-Weierstrass theorem, the sequence (c_n) has a convergent subsequence, say $\lim_{k\to\infty} c_{n_k} = c$. Replacing n by n_k in (2.11) and asking $k \to \infty$, we have

$$\lim_{k\to\infty}\sum_{i=1}^{n_k} f(t_i)u(t_i)\Delta x_i = \int_a^b f(x)u(x)\mathrm{d}x, \tag{2.12}$$

$$\lim_{k\to\infty}[u(b)\int_{c_{n_k}}^{b} f(s)\mathrm{d}s + E_{n_k}] = u(b)\int_c^b f(s)\mathrm{d}s. \tag{2.13}$$

(2.4) can be obtained from Eqs. (2.11)–(2.13). Theorem 2.1 is thus proved. □

Theorem 2.1 can be easily generalized to the Banach space $L^p[a,b]$ for any $p \geq 1$ as follows.

Theorem 2.2 *Theorem 2.1 holds also for a function $f \in L^p[a,b]$ for any $p \geq 1$.*

The proof of Theorem 2.2 depends on the following Lusin's Theorem and Tietze's Extension Theorem.

Lusin's Theorem *Let f be a real-valued measurable function with domain $[a,b]$. Then, for any $\varepsilon > 0$, there is a compact set $K \subset [a,b]$ with the measure $m([a,b]\setminus K) < \varepsilon$ such that the restriction of f to K is continuous.*

Tietze's Extension Theorem *Let $K \subset [a,b]$ be a compact set and f be continuous on K. Then f can be extended to a continuous function g defined on $[a,b]$ such that $g|_K = f|_K$ with* $\max\{|g(x)|; a \leq x \leq b\} = \max\{|f(x)|; x \in K\}$.

Using the Lusin's Theorem and Tietze's Extension Theorem, we can prove that continuous functions are dense in the Banach space $L^p[a,b]$ $(p \geq 1)$. That is, if $f \in L^p[a,b]$ $(p \geq 1)$, then, for any $\varepsilon > 0$ there exists a continuous function f_ε such that

$$\|f_\varepsilon - f\|_p < \varepsilon. \tag{2.14}$$

To prove (2.14), we define

$$f_N(x) = \begin{cases} f(x), & |f(x)| < N, \\ N, & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$, there exists an N, such that

$$\|f_N - f\|_p < \frac{\varepsilon}{2}. \tag{2.15}$$

According to the Lusin's Theorem, for $2^{-p}\frac{\varepsilon^p}{(2N)^p}$, there is a compact set $K \subset [a,b]$ with the measure $m([a,b]\setminus K) < 2^{-p}\frac{\varepsilon^p}{(2N)^p}$ such that the restriction of f_N to K is continuous.

According to the Tietze's Extension Theorem, the restriction of f_N to K can be extended to a continuous function defined on $[a,b]$, say f_ε, such that $f_\varepsilon|_K = f_N|_K$ with $\max\{|f_\varepsilon(x)|; a \le x \le b\} = \max\{|f_N(x)|; x \in K\} \le N$. Therefore, we obtain

$$\begin{aligned} &\int_a^b |f_N(x) - f_\varepsilon(x)|^p \mathrm{d}x = \int_{[a,b]\backslash K} |f_N(x) - f_\varepsilon(x)|^p \mathrm{d}x \\ &\le \int_{[a,b]\backslash K} 2^{p-1}(|f_N(x)|^p + |f_\varepsilon(x)|^p)\mathrm{d}x \le (2N)^p m([a,b]\backslash K) < 2^{-p}\varepsilon^p, \end{aligned} \tag{2.16}$$

where the following inequality is used.

For any real numbers c, d and $p \ge 1$, it holds that

$$|c+d|^p \le 2^{p-1}(|c|^p + |d|^p).$$

According to (2.15) and (2.16), we obtain

$$\|f_\varepsilon - f\|_p < \varepsilon.$$

Therefore, there exists a sequence of continuous functions (f_n) such that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Since f_n is continuous and therefore is Riemann integrable. If g is monotone, according to Theorem 2.1, there exists $c_n \in [a,b]$ such that

$$\int_a^b f_n(x)g(x)\mathrm{d}x = A\int_a^{c_n} f_n(x)\mathrm{d}x + B\int_{c_n}^b f_n(x)\mathrm{d}x. \tag{2.17}$$

Without loss of generality, assume that $\lim_{n\to\infty} c_n = c$. Asking $n \to \infty$, (2.17) shows that Theorem 2.1 still works for a function $f \in L^p[a,b]$. Theorem 2.2 is proved.

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