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Robustness and Iterative Reconstruction of g-Fusion Frames in Hilbert Spaces

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Abstract Regarding the application of the fusion frames and generalization of them in data proceeding, their iterative is of particular importance when one of their members is deleted. In this note, a method for reconstruction of generalized fusion frames and error operator with its upper bound are presented. Also, the approximation operator for these frames will be introduced and we study some results about them.

Keywords Parseval frame; g-frame; g-fusion frame

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1. Introduction and preliminaries

Frames which are a general case of basis in vector spaces, have a significant role in both pure and applied mathematics, and are among the fundamental research area in mathematics, computer science and quantum information and several new applications have been developed, e.g., signal processing, image processing, data compression and sampling theory. This topic was introduced by Duffin and Scheaffer [1], and nowadays frames are studied in seven branches:

(1) Continuous frames (or c-frames): they have been introduced in measure spaces [2];

(2) Generalized frames (or g-frames): they are for bounded operators on Hilbert spaces [3];

(3) Frame of subspaces (or fusion frames): they have been discussed for closed subspaces of a Hilbert space [4];

(4) Frame of operators (or K-frames): they were presented in [5] while studying about the atomic systems with respect to a bounded operator K;

(5) Controlled frames: which were introduced to improve the numerical output in relation to algorithms for inverting the frame operator [6];

(6) Weaving frames (or woven): they were motivated by a question in distributed signal processing [7];

(7) The combination of each two frames: e.g., c-fusion frames [8], g-fusion frames [9], K-fusion frames [10], cK-fusion frames [11], continuous weaving fusion frames [12], and etc.

Robustness of Parseval fusion frames under erasure have been employed by Bodmann and et. al. [13] for optimal transmission of quantum states and packet encoding. Kutyniok et. al. [14]

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presented fusion frames which are optimally resilient against noise and erasure for random signals and further, Casazza and Kutyniok [15] have studied this topic and they presented sufficient conditions on the robustness of a fusion frame with respect to erasures of subspaces. Afterwards, Ahmadi [16] continued it by introducing an operator and yielded an iterative reconstruction by the operator.

In this paper, we focus on the study of those topics on generalized fusion frames and we will show some new results about these frames and provide some conditions to be deleted yet still leave a g-fusion frame. This results can be used in signal processing and image processing, when some data are deleted during the transfer. In Theorem 1.4, we present a new result for the eigenvalues of the g-fusion frame operator and then, in Theorem 1.6, we show that each g-fusion frame can produce a different ordinary frame. And we focus on the conditions that we get a g-fusion frame after deleting some subspaces. Finally, in Theorem 2.7, a useful result about Parseval g-fusion frames is presented.

Throughout this paper, H and K are separable Hilbert spaces and $\mathcal{B}(H, K)$ is the collection of all the bounded linear operators of H into K. If $K = H$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$. Also, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces where $\mathbb{J}$ is a subset of $\mathbb{Z}$.

Definition 1.1 *Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

The constants A and B are called frame bounds. If $A = B$, we say that $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval frame.

Definition 1.2 *A family $\{\Lambda_j \in \mathcal{B}(H, H_j)\}_{j\in\mathbb{J}}$ is called a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, if there exist $0 < A \leq B < \infty$ such that for every $f \in H$,*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leq B\|f\|^2. \tag{1.2}$$

Definition 1.3 ([9]) *Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e., $v_j > 0$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$. We say $\Lambda := (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is a generalized fusion frame (or g-fusion frame) for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{1.3}$$

Throughout this paper, Λ denotes a triple (W_j, Λ_j, v_j) with $j \in \mathbb{J}$ unless otherwise noted. If $A = B$, then we say Λ is a tight g-fusion frame and we say Λ is a Parseval g-fusion frame whenever $A = B = 1$. When the right hand side of (1.3) holds, Λ is called a g-fusion Bessel sequence for H with bound B. If Λ is a g-fusion Bessel sequence, then the synthesis and the *analysis operators of the g-fusion frames are defined by (for more details, we refer to [9])*

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H, \quad T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j,$$
$$T_\Lambda^* : H \longrightarrow \mathscr{H}_2, \quad T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}},$$

where

$$\mathscr{H}_2 = \Big\{\{f_j\}_{j\in\mathbb{J}} : f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\Big\},$$

with the inner product defined by $\langle\{f_j\},\{g_j\}\rangle = \sum_{j\in\mathbb{J}}\langle f_j, g_j\rangle$*. It is clear that* $\mathscr{H}_2$ *is a* Hilbert *space with pointwise operations. Thus, the g-fusion frame operator is given by*

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

Therefore,

$$A\,\mathrm{Id}_H \le S_\Lambda \le B\,\mathrm{Id}_H.$$

This means that S_Λ *is a bounded, positive and invertible operator (with adjoint inverse). The next result gives a new identity about the eigenvalues of the operator* S_Λ *which is general case in [17, Theorem 1.1.12].*

Theorem 1.4 *Let* Λ *be a g-fusion frame for* H *where* $\dim H = n$ *and* $|\mathbb{J}| < \infty$*, and also* $\{\lambda_j\}_{j=1}^n$ *be the eigenvalues of the operator* S_Λ*. Then, there exists an orthonormal basis* $\{e_k\}_{k=1}^n$ *for* H *such that*

$$\sum_{k=1}^n \lambda_k = \sum_{k=1}^n \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} e_k\|^2.$$

Proof Since S_Λ is self-adjoint, there exists an orthonormal basis $\{e_k\}_{k=1}^n$ for H such that $S_\Lambda e_k = \lambda_k e_k$ for each $1 \le k \le n$. Hence,

$$\sum_{k=1}^n \lambda_k = \sum_{k=1}^n \lambda_k \langle e_k, e_k\rangle = \sum_{k=1}^n \langle S_\Lambda e_k, e_k\rangle = \sum_{k=1}^n \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} e_k\|^2.$$

Now, if Λ is a Parseval g-fusion frame, then $S_\Lambda = Id_H$ and therefore,

$$\sum_{k=1}^n \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} e_k\|^2 = n.$$

The following shows an interesting property between T_Λ and T_Θ^*. For this, we need the concept of a trace-class operator: an operator U on H is called a trace-class operator if

$$\mathrm{tr}(|U|) := \sum_{j\in\mathbb{J}} \langle |U| e_j, e_j\rangle < \infty,$$

where $\{e_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for E and $|U| = (U^*U)^{\frac{1}{2}}$. $\square$

Theorem 1.5 *Let* $|\mathbb{J}| < \infty$*, also* $\Lambda = (W_j, \Lambda_j, v_j)$ *and* $\Theta = (W_j, \Theta_j, w_j)$ *be two g-fusion Bessel sequence for* H*, where* $\Lambda_j, \Theta_j \in \mathcal{B}(H, H_j)$*. Let* $\phi := T_\Lambda T_\Theta^*$*. Then* ϕ *is a trace class operator.*

Proof Suppose that $\phi = u|\phi|$ is a polar decomposition of the operator ϕ, where $u \in \mathcal{B}(H)$ is a partial isometry, therefore, $|\phi| = u^* T_\Lambda T_\Theta^*$. Assume that $\{e_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for H, then

$$\begin{aligned}
\operatorname{tr}(|\phi|) &= \sum_{j\in\mathbb{J}} \langle |\phi| e_j, e_j\rangle = \sum_{j\in\mathbb{J}} \langle T_\Theta^* e_j, T_\Lambda^* u e_j\rangle \\
&= \sum_{j\in\mathbb{J}} \big\langle \{w_k \Theta_k \pi_{W_k} e_j\}_{k\in\mathbb{J}}, \{v_k \Lambda_k \pi_{W_k} u e_j\}_{k\in\mathbb{J}} \big\rangle \\
&= \sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{J}} \langle w_k \Theta_k \pi_{W_k} e_j, v_k \Lambda_k \pi_{W_k} u e_j\rangle \\
&\leq \sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{J}} \|w_k \Theta_k \pi_{W_k} e_j\| . \|v_k \Lambda_k \pi_{W_k} u e_j\| \\
&\leq \sum_{j\in\mathbb{J}} \Big(\sum_{k\in\mathbb{J}} \|w_k \Theta_k \pi_{W_k} e_j\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{J}} \|v_k \Lambda_k \pi_{W_k} u e_j\|^2 \Big)^{\frac{1}{2}} \\
&\leq \sum_{j\in\mathbb{J}} \sqrt{B_\Lambda B_\Theta} \|u e_j\| = \sqrt{B_\Lambda B_\Theta}\ |\mathbb{J}| < \infty. \quad \square
\end{aligned}$$

In the next theorem, we show a relation between ordinary frames and g-fusion frames.

Theorem 1.6 *For each $j \in \mathbb{J}$ let $\Lambda_j \in \mathcal{B}(H, H_j)$ and $v_j > 0$. Let $\{f_{ij}\}_{i\in\mathbb{I}_j}$ be a frame for H_j with bounds A_j and B_j. Define a sequence of subspaces $W_j = \overline{\operatorname{span}}\{\Lambda_j^* f_{ij}\}_{i\in\mathbb{I}_j}$ for each $j \in \mathbb{J}$ and suppose that*

$$0 < A := \inf_{j\in\mathbb{J}} A_j \leq B := \sup_{j\in\mathbb{J}} B_j < \infty.$$

The following assertions are equivalent:

(1) $\{v_j \Lambda_j^* f_{ij}\}_{j\in\mathbb{J}, i\in\mathbb{I}_j}$ is a frame for H.

(2) $\Lambda_j(W_j)$ are closed subspaces of H_j for every $j \in \mathbb{J}$ and $\{e_{ij}\}_{j\in\mathbb{J}, i\in\mathbb{I}_j}$ are orthonormal bases for them such that $\{v_j \pi_{W_j} \Lambda_j^* e_{ij}\}_{j\in\mathbb{J}, i\in\mathbb{I}_j}$ is a frame for H.

(3) $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is a g-fusion frame for H.

Proof First, we prove that (1) and (3) are equivalent. Suppose that $\{v_j \Lambda_j^* f_{ij}\}_{j\in\mathbb{J}, i\in\mathbb{I}_j}$ is a frame for H with frame bounds C and D. For each $f \in H$, we have

$$\begin{aligned}
A \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 &\leq \sum_{j\in\mathbb{J}} A_j v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq \sum_{j\in\mathbb{J}} \sum_{i\in\mathbb{I}_j} |\langle v_j \Lambda_j \pi_{W_j} f, f_{ij}\rangle|^2 \\
&= \sum_{j\in\mathbb{J}} \sum_{i\in\mathbb{I}_j} |\langle \pi_{W_j} f, v_j \Lambda_j^* f_{ij}\rangle|^2 \\
&= \sum_{j\in\mathbb{J}} \sum_{i\in\mathbb{I}_j} |\langle f, v_j \Lambda_j^* f_{ij}\rangle|^2 \leq D\|f\|^2.
\end{aligned}$$

This means that Λ is a g-fusion Bessel sequence for H with a bound $\frac{D}{A}$. With the same method, we can show that $\frac{C}{B}$ is a lower frame bound for Λ. For the opposite case, assume that Λ is a

g-fusion frame with bounds C and D. For each $f \in H$ we have

$$\begin{aligned}\sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle f, v_j\Lambda_j^* f_{ij}\rangle|^2 &= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle \pi_{W_j} f, v_j\Lambda_j^* f_{ij}\rangle|^2\\ &= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j} v_j^2|\langle \Lambda_j\pi_{W_j} f, f_{ij}\rangle|^2\\ &\leq \sum_{j\in\mathbb{J}} B_j v_j^2\|\Lambda_j\pi_{W_j} f\|^2 \leq BD\|f\|^2,\end{aligned}$$

and it is easy to check that AC is a lower frame bound.

Now, according to the following:

$$v_j^2\|\Lambda_j\pi_{W_j} f\|^2 = v_j^2\Big\|\sum_{i\in\mathbb{I}_j}\langle \Lambda_j\pi_{W_j} f, e_{ij}\rangle e_{ij}\Big\|^2 = \sum_{i\in\mathbb{I}_j}|\langle f, v_j\pi_{W_j}\Lambda_j^* e_{ij}\rangle|^2,$$

we aim that (2) and (3) are equivalent. □

2. Main results

Suppose that $\{W_j\}_{j\in\mathbb{J}}$ and $\{Z_j\}_{j\in\mathbb{J}}$ are two closed subspaces of H and $\{v_j\}_{j\in\mathbb{J}}$ is a set of weights. Also, Λ_j and Θ_j are bounded operators in $\mathcal{B}(H, H_j)$. We define the approximation operator with respect to Λ and $\Theta := (Z_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ as follows:

$$\Psi : H \longrightarrow H, \quad \Psi f = \sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^*(v_j\Lambda_j\pi_{W_j} f).$$

The following can be found in the text of Banach spaces:

Lemma 2.1 *Let $(X, \|.\|)$ be a Banach space and $U : X \to X$ be a bounded operator such that $\|I - U\| < 1$. Then U is invertible and $U^{-1} = \sum_{k=0}^{n}(I-U)^k$. Moreover,*

$$\|U^{-1}\| \leq \frac{1}{1 - \|I - U\|}.$$

Theorem 2.2 *Let $C_1, C_2 > 0$ and $0 \leq \gamma < 1$ be real numbers such that for each $f \in H$ and $\{f_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ the following assertions hold:*

(1) $\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j} f\|^2 \leq C_1\|f\|^2$;

(2) $\|\sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^* f_j\|^2 \leq C_2\|\{f_j\}\|_2^2$;

(3) $\|f - \Psi f\|^2 \leq \gamma\|f\|^2$.

The Λ is a g-fusion frame for H with bounds $C_2^{-1}(1-\gamma)^2$ and C_1. Also, Θ is a g-fusion frame for H with bounds $C_1^{-1}(1-\gamma)^2$ and C_2.

Proof Assume that $f \in H$, with items (1) and (2) we get

$$\|\Psi f\|^2 \leq C_2\|\{v_j\Lambda_j\pi_{W_j} f\}\|_2^2 = C_2\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j} f\|^2 \leq C_2C_1\|f\|^2.$$

Hence, Ψ is a bounded operator. Via item (3) we have $\|I - \Psi\| \leq \sqrt{\gamma} < 1$. So, by Lemma 2.1, Ψ is invertible and $\|\Psi^{-1}\| \leq (1-\gamma)^{-1}$. Thus,

$$\|f\|^2 = \|\Psi^{-1}\Psi f\|^2 \leq (1-\gamma)^{-2}\|\Psi f\|$$

$$\le C_2(1-\gamma)^{-2}\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2$$
$$\le C_2C_1(1-\gamma)^{-2}\|f\|^2.$$

So, we conclude that

$$C_2^{-1}(1-\gamma)^2\|f\|^2 \le \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 \le C_1\|f\|^2,$$

and the first part is proved. Next, we verify two inequalities which are dual to (1) and (2) for Θ. Let $f \in H$ and we have

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^2 &= \big(\langle\sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^*\Theta_j\pi_{Z_j}f, f\rangle\big)^2\\ &\le \|\sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^*\Theta_j\pi_{Z_j}f\|^2\|f\|^2\\ &\le C_2\|f\|^2\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2.\end{aligned}$$

Therefore,

$$\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2 \le C_2\|f\|^2.$$

For second inequality, for $\{f_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$, we can write

$$\begin{aligned}\|\sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*f_j\|^2 &= \Big(\sup_{\|f\|=1}\big|\big\langle\sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*f_j, f\big\rangle\big|\Big)^2\\ &\le \Big(\sup_{\|f\|=1}\big|\sum_{j\in\mathbb{J}}\big\langle f_j, v_j\Lambda_j\pi_{W_j}f\big\rangle\big|\Big)^2\\ &\le \|\{f_j\}\|_2^2\Big(\sup_{\|f\|=1}\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)\\ &\le C_1\|\{f_j\}\|_2^2.\end{aligned}$$

Now by similar argument and applying an approximation operator of the form

$$\Psi^*f = \sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*(v_j\Theta_j\pi_{Z_j}f),$$

we can establish Θ has required properties. □

The next result is a generalization of [15, Theorem 3.2] for g-fusion frames.

Theorem 2.3 *Let Λ be a g-fusion frame for H with bounds A and B, and also $\mathbb{I} \subset \mathbb{J}$. Then the following statements hold.*

(1) If $\{\Lambda_j\}_{j\in\mathbb{I}}$ is a g-frame for H with the lower frame bound B and we have $v_j > 1$ for each $j \in \mathbb{I}$, then $\bigcap_{j\in\mathbb{I}} W_j = \{0\}$.

(2) If $\{\Lambda_j\}_{j\in\mathbb{I}}$ is a tight g-frame for H with the lower frame bound B and we have $v_j = 1$ for each $j \in \mathbb{I}$, then $\bigcap_{j\in\mathbb{I}} W_j \perp \operatorname{span}\{W_j\}_{j\in\mathbb{J}\setminus\mathbb{I}}$.

(3) If $C := \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\|^2 < A$, then $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}\setminus\mathbb{I}}$ is a g-fusion frame for H with bounds $A - C$ and B.

Proof (1) For every $f \in \bigcap_{j\in\mathbb{I}} W_j$ and $j \in \mathbb{I}$ we have $\pi_{W_j} f = f$. So,

$$B\|f\|^2 < \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j f\|^2 = \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \le \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \le B\|f\|^2.$$

Thus, $f = 0$.

(2) For each $f \in \bigcap_{j\in\mathbb{I}} W_j$, we have

$$B\|f\|^2 = \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \le \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \le B\|f\|^2.$$

Therefore, $\sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 = 0$ and it shows that $f \perp \mathrm{span}\{W_j\}_{j\in\mathbb{J}\setminus\mathbb{I}}$.

(3) The upper bound is evident. For the lower bound, if $f \in H$ we get

$$\begin{aligned}\sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 - \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\ &\ge A\|f\|^2 - \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\|^2\|f\|^2 \\ &= (A - C)\|f\|^2.\end{aligned}$$

When the set $\mathbb{I}$ which is introduced in Theorem 2.3 is singleton, then we can get the following result. □

Corollary 2.4 *Let Λ be a g-fusion frame for H with frame bounds A and B. If there exists $j_0 \in \mathbb{J}$ such that $v_{j_0}^2\|\Lambda_{j_0}\|^2 < A$, then $(W_j, \Lambda_j, v_j)_{j\ne j_0}$ is a g-fusion frame for H with bounds $A - v_{j_0}^2\|\Lambda_{j_0}\|^2$ and B.*

The following is a generalization in [15, Corollary 3.4].

Corollary 2.5 *Let Λ be a tight g-fusion frame for H with bound A and $j_0 \in \mathbb{J}$. Then the following assertions are equivalent.*

(1) $v_{j_0}^2\|\Lambda_{j_0}\pi_{W_{j_0}}\|^2 < A$.

(2) $(W_j, \Lambda_j, v_j)_{j\ne j_0}$ is a g-fusion frame for H.

Proof The proof of (1)⇒ (2) is clear from Corollary 2.4. For the opposite, assume that C is a lower frame bound of $(W_j, \Lambda_j, v_j)_{j\ne j_0}$. For each $0 \ne f \in H$ we have

$$\begin{aligned}C\|f\|^2 &\le \sum_{j\ne j_0} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\ &= \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 - v_{j_0}^2\|\Lambda_{j_0}\pi_{W_{j_0}} f\|^2 \\ &= (A\|f\|^2 - v_{j_0}^2\|\Lambda_{j_0}\pi_{W_{j_0}} f\|^2).\end{aligned}$$

Hence,

$$0 < C \le A - v_{j_0}^2 \frac{\|\Lambda_{j_0}\pi_{W_{j_0}} f\|^2}{\|f\|^2}.$$

Therefore, $A - v_{j_0}^2\|\Lambda_{j_0}\pi_{W_{j_0}}\|^2 > 0$. □

In next result, we provide a new g-fusion frame for the space H by deleting a number of members of a Parseval frame for H_j.

Theorem 2.6 *Let Λ be a g-fusion frame for H with bounds A and B. For each $j \in \mathbb{J}$, let $\{f_{ij}\}_{i\in\mathbb{I}_j} \in \Lambda_j(W_j)$ be a Parseval frame for H_j which $\{f_{ij}\}_{i\in\mathbb{I}_j\setminus\mathbb{L}_j}$ is a frame for H_j with the lower frame bound C_j for each finite subset $\mathbb{L}_j \subset \mathbb{I}_j$ and all $j \in \mathbb{J}$. If $\widetilde{W}_j := \overline{\text{span}}\{\Lambda_j^* f_{ij}\}_{i\in\mathbb{I}_j\setminus\mathbb{L}_j}$, then $(\widetilde{W}_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is a g-fusion frame for H with bounds $(\min_{j\in\mathbb{J}} C_j)A$ and B.*

Proof It is clear that $\widetilde{W}_j$ are closed subspaces of H for each $j \in \mathbb{J}$ and B is an upper frame bound of $(\widetilde{W}_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$. For each $f \in H$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{\widetilde{W}_j} f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j} |\langle \Lambda_j \pi_{\widetilde{W}_j} f, f_{ij}\rangle|^2 \\
&\geq \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j\setminus\mathbb{L}_j} |\langle \pi_{\widetilde{W}_j} f, \Lambda_j^* f_{ij}\rangle|^2 \\
&= \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j\setminus\mathbb{L}_j} |\langle \Lambda_j \pi_{W_j} f, f_{ij}\rangle|^2 \\
&\geq \sum_{j\in\mathbb{J}} v_j^2 C_j \|\Lambda_j \pi_{W_j} f\|^2 \\
&\geq (\min_{j\in\mathbb{J}} C_j) \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \\
&\geq (\min_{j\in\mathbb{J}} C_j) A \|f\|^2.
\end{aligned}$$

Now, we aim to study the approximation Ψ in finite case similar to the view presented in [15]. Suppose that $\mathbb{J} = \{1, 2, \ldots, m\}$ is finite and Λ is a Parseval g-fusion frame for H. For every $j_0 \in \mathbb{J}$, we consider the following operator: $D_{j_0} : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$, $D_{j_0}\{f_j\}_{j\in\mathbb{J}} = \delta_{j,j_0} f_{j_0}$. We define the associated 1-erasure reconstruction error $\mathcal{E}_1(\Lambda)$ to be

$$\mathcal{E}_1(\Lambda) = \max_{j\in\mathbb{J}} \|T_\Lambda D_j T_\Lambda^*\|.$$

Since

$$\|T_\Lambda D_j T_\Lambda^*\| = \sup_{\|f\|=1} \|T_\Lambda D_j T_\Lambda^* f\| = v_j^2 \sup_{\|f\|=1} \|\pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\| \leq v_j^2 \|\Lambda_j\|^2,$$

therefore, $\mathcal{E}_1(\Lambda) = \max_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\|^2$. □

Theorem 2.7 *Let $\Lambda_j(W_j)$ be closed subspaces, $\mathbb{J} = \{1, 2, \ldots, m\}$ and Λ be a Parseval g-fusion frame for finite dimensional H and also $|H_j| < \infty$ for each $j \in \mathbb{J}$. Then the following conditions are equivalent.*

(1) Λ satisfies $\mathcal{E}_1(\Lambda) = \min_{j\in\mathbb{J}} \mathcal{E}_1(\widetilde{W}_j, \Lambda_j, \widetilde{v}_j)_{j\in\mathbb{J}}$, where $(\widetilde{W}_j, \Lambda_j, \widetilde{v}_j)_{j\in\mathbb{J}}$ is a Parseval g-fusion frame for H with $\dim \widetilde{W}_j = \dim W_j$ for each $j \in \mathbb{J}$.

(2) For each $j \in \mathbb{J}$ we have $v_j^2 \|\Lambda_j\|^2 = \frac{\dim H}{m.\dim W_j}$.

Proof Assume that $\{e_{ij}\}_{i\in\mathbb{I}_j}$ is an orthonormal basis for $\Lambda_j(W_j)$ for each $j \in \mathbb{J}$. Via Theorem 1.6, the sequence $\{v_j \pi_{W_j} \Lambda_j^* e_{ij}\}_{j=1,i=1}^{m,\dim \Lambda_j(W_j)}$ is a Parseval frame for H. In [18, Eq. (17)], we can get

$$\dim H = \sum_{j=1}^{m} \sum_{i=1}^{\dim \Lambda_j(W_j)} v_j^2 \|\pi_{W_j} \Lambda_j^* e_{ij}\|^2 \leq \sum_{j=1}^{m} \dim \Lambda_j(W_j) v_j^2 \|\Lambda_j\|^2.$$

So, there exists j such that $\dim H \leq m. \dim \Lambda_j(W_j) v_j^2 \|\Lambda_j\|^2$. Since the dimensions as well as the number of subspaces are fixed, we conclude that $\mathcal{E}_1(\Lambda)$ is minimal if and only if

$$\dim H = m. \dim \Lambda_j(W_j) v_j^2 \|\Lambda_j\|^2, \quad \forall j \in \mathbb{J}. \quad \square$$

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