Journal of Mathematical Research with Applications
May, 2021, Vol. 41, No. 3, pp. 279–290
DOI:10.3770/j.issn:2095-2651.2021.03.006
Http://jmre.dlut.edu.cn

Almost Cosymplectic p-Spheres and Almost Cosymplectic Metric Bi-Structures on Three-Manifolds

Jin LI*, **Ximin LIU**
School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024*, P. R. China*

Abstract The purpose of this paper is to study almost cosymplectic p-spheres and almost cosymplectic metric bi-structures. Firstly, we show some properties of almost cosymplectic p-spheres. Then we introduce the notion of almost cosymplectic metric bi-structures and give some results on three dimensional manifolds admitting almost cosymplectic metric bi-structures. Moreover, we investigate three dimensional manifolds with almost cosymplectic metric bi-structures when the $(1,1)$-type tensor fields h_1 and h_2 are being of codazzi type and cyclic parallel.

Keywords almost cosymplectic circles; almost cosymplectic spheres; almost cosymplectic metric bi-structures

MR(2020) Subject Classification 53D15; 53C25

1. Introduction

In recent years, after Goldberg and Yano [1] introduced the notion of almost cosymplectic manifolds, almost cosymplectic manifolds were studied by many authors. In [2], Perrone classified all simply connected homogeneous almost cosymplectic three-manifolds. In [3–6], the authors considered three dimensional almost cosymplectic manifolds satisfying certain conditions. Geiges and Gonzalo [7] introduced the notion of contact circles on three-manifolds in 1995. In 2005, Zessin [8] studied contact p-spheres, and proved that a contact circle (resp., a contact sphere) is taut if and only if it is round on a three-manifold. Montano, Nicola and Yudin [9] introduced almost cosymplectic circles and almost cosymplectic spheres. Moreover, they showed that any 3-Sasakian manifold admits a sphere of Sasakian structures which is both taut and round. In 2017, Perrone [10] introduced a Riemannian approach to the study of taut contact circles on three-manifolds. The author gave a complete classification of simply complete three-manifolds which admit a bi-H-contact metric structure.

In this paper, we investigate almost cosymplectic p-spheres and almost cosymplectic metric bi-structures. In Section 2, we give some properties of almost cosymplectic p-spheres. According to the notation of bi-contact metric structures in [10], we introduce the definition of almost cosymplectic metric bi-structures in Section 3. According to the structures of almost cosymplectic metric bi-structures, we construct a global orthonormal basis on M^3. We conclude that if there

Received February 25, 2020; Accepted May 24, 2020
Supported by the National Natural Science Foundation of China (Grant No. 11431009).
* Corresponding author
E-mail address: janelee0907@163.com (Jin LI); ximinliu@dlut.edu.cn (Ximin LIU)

exists a cosymplectic metric bi-structure on M^3, then the scalar curvature is zero. Finally, we study some special conditions on three-manifolds with almost cosymplectic metric bi-structures. We show that for three-manifolds with almost cosymplectic metric bi-structures, we have the following results: if tensors $\{h_i\}_{i=1,2}$ are of Codazzi type, then any almost cosymplectic manifold $(M,\varphi_i,\xi_i,\eta_i)$ is cosymplectic and locally isometric to the flat Euclidean space $\mathbb{R}^3$; if the tensors $\{h_i\}_{i=1,2}$ are cyclic parallel, then the scalar curvature of M^3 is 0, $-4\beta^2$ or $-4\alpha^2$.

2. Preliminaries

Let M be a manifold of dimension $2n+1$, φ a $(1,1)$-type tensor field, ξ a global vector field, called the *Reeb vector fileld* or the *characteristic vector field*, η a 1-form dual to ξ. The triplet (φ,ξ,η) is called an *almost contact structure* if the following relations hold:

$$\varphi^2=-I+\eta\otimes\xi,\ \eta\circ\varphi=0,\ \varphi\circ\xi=0.$$

An almost contact structure endowed with an associated metric g such that

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for any $X,Y\in\mathfrak{X}(M)$ is called an almost contact metric structure. The fundamental 2-form Φ is defined by $\Phi(X,Y)=g(X,\varphi Y)$ for any $X,Y\in\mathfrak{X}(M)$. An almost contact metric structure is called contact metric structure if $d\eta=\Phi$ and called almost cosymplectic structure if Φ and η are closed. As a consequence, any almost contact manifold is orientable, and the $\eta\wedge\Phi^n$ does not vanish everywhere on M.

An almost cosymplectic metric structure is said to be normal when the Nijenhuis tensor $[\varphi,\varphi]=0$, where $[\varphi,\varphi]=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]$ for any $X,Y\in\mathfrak{X}(M)$. It should be noted that an almost contact metric structure (φ,ξ,η,g) is cosymplectic if and only if φ is parallel, i.e., $\nabla\varphi=0$ (see [11, p.95]). Any three dimensional almost contact metric manifold fulfils $|\nabla\varphi|^2=2|\nabla\xi|^2$, as the consequence of this, we obtain that any three dimensional almost contact metric manifold is cosymplectic if and only if $\nabla\xi=0$ (see [12, p.248]). The (1, 1)-type tensor field h on almost contact metric manifolds is defined by $h=\frac{1}{2}\mathfrak{L}_\xi\varphi$. We also have the following properties for almost cosymplectic manifolds [2]:

$$\nabla_\xi\varphi=0,\ \nabla\xi=h\varphi,\ h\varphi=-\varphi h,\ h\xi=0. \tag{2.1}$$

Let $\{(\eta_\lambda,\Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be a pair of the linear combination about the 1-form $(\eta_1,\eta_2,\ldots,\eta_{p+1})$ and the 2-form $(\Phi_1,\Phi_2,\ldots,\Phi_{p+1})$, where $\eta_\lambda=\lambda_1\eta_1+\cdots+\lambda_{p+1}\eta_{p+1}$, $\Phi_\lambda=\lambda_1\Phi_1+\cdots+\lambda_{p+1}\Phi_{p+1}$ for any $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{p+1})\in\mathbb{S}^p$. If the corresponding almost contact structure of pair $(\eta_\lambda,\Phi_\lambda)$ for any $\lambda\in\mathbb{S}^p$ is almost cosymplectic structure, then the $\{(\eta_\lambda,\Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ is called almost cosymplectic p-sphere. Especially, an almost cosymplectic p-sphere is called an almost cosymplectic circle or an almost cosymplectic sphere if $p=1$ or $p=2$, respectively. We also use $\{(\eta_1,\Omega_1),(\eta_2,\Omega_2)\}$ to indicate $\{(\eta_\lambda,\Phi_\lambda)\}_{\lambda\in\mathbb{S}^1}$.

An almost cosymplectic p-sphere is said to be taut if the volume forms are equal to every $\lambda,\lambda'\in\mathbb{S}^p$, i.e., $(\sum_{i=1}^{p+1}\lambda_i\eta_i)\wedge(\sum_{i=1}^{p+1}\lambda_i\Phi_i)^n=(\sum_{i=1}^{p+1}\lambda_i'\eta_i)\wedge(\sum_{i=1}^{p+1}\lambda_i'\Phi_i)^n$. An almost cosymplectic p-sphere is said to be round if the vector field $\xi_\lambda = \lambda_1\xi_1 + \cdots + \lambda_{p+1}\xi_{p+1}$ is the Reeb vector field of the corresponding almost contact structure, i.e., $i_{\xi_\lambda}\eta_\lambda = 1, i_{\xi_\lambda}\Phi_\lambda = 0$. We have the following properties for almost cosymplectic p-spheres.

Lemma 2.1 ([9]) *On $(4n+1)$-dimensional manifolds, almost cosymplectic p-spheres do not exist for $p \geq 1$.*

Proof We now prove that there is not an almost cosymplectic p-sphere on M of dimension 5. Assume that $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ is an almost cosymplectic p-sphere on M^5, then we have

$$\eta_\lambda \wedge \Phi_\lambda^2 = \sum_{i,j,k=1}^{p+1} \lambda_i\lambda_j\lambda_k(\eta_i \wedge \Phi_j \wedge \Phi_k),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) \in \mathbb{S}^p$. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be a basis of T_pM, where p is a point of M. Then we consider the function from $\mathbb{R}^{p+1}$ to $\mathbb{R}$ defined by

$$f(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) = \sum_{i,j,k=1}^{p+1} \lambda_i\lambda_j\lambda_k(\eta_i \wedge \Phi_j \wedge \Phi_k)(e_1, e_2, e_3, e_4, e_5).$$

It is a homogeneous polynomial function of degree 3. We have $f(-\lambda_1, -\lambda_2, \ldots, -\lambda_{p+1}) = -f(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$ for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) \in \mathbb{S}^p$. If it is positive at some point of $\mathbb{R}^{p+1}$, it is negative at its antipode. Therefore, f should have zero in $\mathbb{S}^p$, $\eta_\lambda \wedge \Phi_\lambda^2$ is not a volume form in this condition. So $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ is not an almost cosymplectic p-sphere on M^5.

Generally, when the dimension of M is $4n+1$, the degree of polynomial function is $2n+1$, which is odd, so the polynomial function has zero on $\mathbb{S}^{p+1}$. Thus there is not an almost cosymplectic p-sphere in dimension $4n+1$. □

Note that the degree of polynomial function is $2n$ when the dimension of M is $4n-1$, so there is no restriction to the existence of almost cosymplectic p-spheres in these dimensions. There is an example about cosymplectic circles on 7 dimensional manifolds in [9].

Lemma 2.2 *Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be an almost cosymplectic p-sphere. Then for every fixed i, there must be a j, such that $i_{\xi_i}\Phi_j \neq 0$ for $i, j \in 1, 2, \ldots, p+1$, $i \neq j$.*

Proof Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be an almost cosymplectic p-sphere and fix i. Suppose $i_{\xi_i}\Phi_j(p) = 0$ for all $j = 1, \ldots, p+1$ and $p \in M$, then we have

$$i_{\xi_i}\Phi_\lambda(p) = \lambda_1 i_{\xi_i}\Phi_1(p) + \cdots + \lambda_{p+1} i_{\xi_i}\Phi_{p+1}(p) = 0.$$

If $\eta_\lambda(\xi_i)(p) = 0$, then $(\eta_\lambda \wedge \Phi_\lambda^n)(\xi_i, \cdots)$ vanished at p, and the $\{(\eta_\lambda, \Phi_\lambda)\}$ cannot be an almost cosymplectic p-sphere. We put $a = \eta_\lambda(\xi_i)(p) \neq 0$, $\xi_i' = \frac{\xi_i}{a}$, then we have $i_{\xi_i'}\Phi_\lambda(p) = 0$, $\eta_\lambda(\xi_i')(p) = 1$ at p. Thus the Reeb vector field of structure $\{(\eta_\lambda, \Phi_\lambda)\}$ is ξ_i'. The structure $(-\eta_i, -\Phi_i)$ is also an almost cosymplectic p-sphere and the Reeb vector field is $-\xi_i$. We define a function f from $\mathbb{S}^p$ to $\mathbb{R}$ by $\xi_\lambda(p) = f(\lambda)\xi_i(p)$, f is continuous and $f(\lambda_1) = 1$ for $\lambda_1 = (0, \ldots, 0, 1(i\text{th}), 0, \ldots, 0)$, $f(\lambda_2) = -1$ for $\lambda_2 = (0, \ldots, 0, -1(i\text{th}), 0, \ldots, 0)$. There exists some $\lambda_0 \in \mathbb{S}^p$, such that $\xi_{\lambda_0}(p) = f(\lambda_0)\xi_i(p) = 0$. Since $\xi_\lambda(p) = \xi_i' \neq 0$, we get a contradiction. So there exists a number j, s.t. $i_{\xi_i}\Phi_j(p) \neq 0$. □

Corollary 2.3 *Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be an almost cosymplectic p-sphere. If $i_{\xi_i}\Phi_j \neq 0$ for every $i, j \in 1, \dots, p+1$, $i \neq j$, then the Reeb vector fields ξ_i of (η_i, Φ_i) are everywhere linearly independent.*

Proof Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be an almost cosymplectic p-sphere. $\xi_1, \xi_2, \dots, \xi_{p+1}$ are the corresponding Reeb vector fields, respectively. If there is a set of number $a_1, a_2, \dots, a_{p+1}$ on $\mathbb{R}$, s.t. $a_1\xi_1+a_2\xi_2+\cdots+a_{p+1}\xi_{p+1} = 0$. Then we have $i_{a_1\xi_1+a_2\xi_2+\cdots+a_{p+1}\xi_{p+1}}\Phi_i = 0$ for $i = 1, 2, \dots, p+1$, i.e.,

$$\begin{cases} a_1 i_{\xi_1}\Phi_1 + a_2 i_{\xi_2}\Phi_1 + \cdots + a_{p+1} i_{\xi_{p+1}}\Phi_1 = 0, \\ a_1 i_{\xi_1}\Phi_2 + a_2 i_{\xi_2}\Phi_2 + \cdots + a_{p+1} i_{\xi_{p+1}}\Phi_2 = 0, \\ \qquad\qquad \cdots \\ a_1 i_{\xi_1}\Phi_{p+1} + a_2 i_{\xi_2}\Phi_{p+1} + \cdots + a_{p+1} i_{\xi_{p+1}}\Phi_{p+1} = 0. \end{cases}$$

The coefficient matrix is

$$\begin{vmatrix} i_{\xi_1}\Phi_1 & i_{\xi_2}\Phi_1 & \cdots & i_{\xi_{p+1}}\Phi_1 \\ i_{\xi_1}\Phi_2 & i_{\xi_2}\Phi_2 & \cdots & i_{\xi_{p+1}}\Phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ i_{\xi_1}\Phi_{p+1} & i_{\xi_2}\Phi_{p+1} & \cdots & i_{\xi_{p+1}}\Phi_{p+1} \end{vmatrix}.$$

The rank of coefficient matrix must be $p+1$ due to $i_{\xi_i}\Phi_i = 0$ and $i_{\xi_i}\Phi_j \neq 0$. So the equation set must have zero solution, we get $(a_1, a_2, \dots, a_{p+1}) = (0, 0, \dots, 0)$. Thus $\xi_1, \xi_2, \dots, \xi_{p+1}$ are linearly independent. □

Especially, for almost cosymplectic circle, i.e., $p = 1$, we have the following conclusions:

Corollary 2.4 ([9]) *Let $\{(\eta_1\Phi_1), (\eta_2, \Phi_2)\}$ be an almost cosymplectic circle. Then $i_{\xi_1}\Phi_2$ and $i_{\xi_2}\Phi_1$ never vanish, ξ_1 and ξ_2 are everywhere linearly independent.*

Proposition 2.5 *An almost cosymplectic p-sphere is round if and only if the following conditions are satisfied:*

(i) $\eta_i(\xi_j) + \eta_j(\xi_i) = 0$ *for* $i, j = 1, 2, \dots, p+1, i \neq j$;

(ii) $i_{\xi_i}\Phi_j + i_{\xi_j}\Phi_i = 0$ *for* $i, j = 1, 2, \dots, p+1$.

Proof Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^p}$ be an almost cosymplectic p-sphere. If it is round, then we have $\eta_\lambda(\xi_\lambda) = 1$, $i_{\xi_\lambda}\Phi_\lambda = 0$, and they are equivalent to

$$\eta_\lambda(\xi_\lambda) = \sum_{i,j=1}^{p+1} \lambda_i\lambda_j\eta_i(\xi_j) = \sum_{i=1}^{p+1} \lambda_i^2\eta_i(\xi_i) + \sum_{i\neq j} \lambda_i\lambda_j\eta_i(\xi_j) = 1,$$

$$i_{\xi_\lambda}\Phi_\lambda = \sum_{i,j=1}^{p+1} \lambda_i\lambda_j i_{\xi_i}\Phi_j = \sum_{i=1}^{p+1} \lambda_i^2 i_{\xi_i}\Phi_i + \sum_{i\neq j} \lambda_i\lambda_j i_{\xi_i}\Phi_j = 0.$$

Then we get

$$\sum_{i\neq j} \lambda_i\lambda_j\eta_i(\xi_j) = 0, \quad \sum_{i\neq j} \lambda_i\lambda_j i_{\xi_i}\Phi_j = 0.$$

By substituting $\lambda_i = \lambda_j = \frac{1}{\sqrt{2}}$, $\lambda_k = 0$, where $i, j = 1, \dots, p+1$, $k \neq i, j$, we obtain

$$\eta_i(\xi_j) + \eta_j(\xi_i) = 0, \quad i_{\xi_i}\Phi_j + i_{\xi_j}\Phi_i = 0. \ \square \tag{2.2}$$

Proposition 2.6 *Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda \in \mathbb{S}^p}$ be an almost cosymplectic p-sphere on M^3. Then M is taut if and only if the following conditions are satisfied:*

(i) $\eta_i \wedge \Phi_i = \eta_j \wedge \Phi_j$ for $i, j = 1, 2, \dots, p+1$;

(ii) $\eta_i \wedge \Phi_j = -\eta_j \wedge \Phi_i$ for $i, j = 1, 2, \dots, p+1, i \neq j$.

Proof If the almost cosymplectic p-sphere is taut, then we have

$$\Big(\sum_{i=1}^{p+1} \lambda_i \eta_i\Big) \wedge \Big(\sum_{j=1}^{p+1} \lambda_j \Phi_j\Big) = \Big(\sum_{i=1}^{p+1} \lambda'_i \eta_i\Big) \wedge \Big(\sum_{j=1}^{p+1} \lambda'_j \Phi_j\Big)$$

for any $\lambda = (\lambda_1, \dots, \lambda_{p+1}), \lambda' = (\lambda'_1, \dots, \lambda'_{p+1}) \in \mathbb{S}^p$. By taking $(\lambda_1, \dots, \lambda_{p+1}) = (0, \dots, 1(j\text{th}), \dots, 0)$ and $(\lambda_1, \dots, \lambda_{p+1}) = (0, \dots, 1(i\text{th}), \dots, 0)$, we get

$$\eta_i \wedge \Phi_i = \eta_j \wedge \Phi_j, \quad i, j = 1, 2, \dots, p+1. \tag{2.3}$$

By taking $(\lambda_1, \dots, \lambda_{p+1}) = (0, \dots, \frac{1}{\sqrt{2}}(i\text{th}), \dots, \frac{1}{\sqrt{2}}(j\text{th}), \dots, 0)$ and $(\lambda'_1, \dots, \lambda'_{p+1}) = (0, \dots, 1$ $(i\text{th}$ or $j\text{th}), \dots, 0)$, according to (2.3), we obtain

$$\eta_i \wedge \Phi_j = -\eta_j \wedge \Phi_i, \quad i, j = 1, 2, \dots, p+1, i \neq j. \tag{2.4}$$

On the other hand, if (2.3) and (2.4) are fulfilled, we have

$$\begin{aligned} &(\lambda_1\eta_1 + \cdots \lambda_{p+1}\eta_{p+1}) \wedge (\lambda_1\Phi_1 + \cdots \lambda_{p+1}\Phi_{p+1}) \\ &= \sum_{i=1}^{p+1} \lambda_i^2 \eta_i \wedge \Phi_i + \sum_{i<j} \lambda_i\lambda_j(\eta_i \wedge \Phi_j + \eta_j \wedge \Phi_i) \\ &= \eta_i \wedge \Phi_i. \end{aligned}$$

Then the almost cosymplectic p-sphere is taut. $\square$

Proposition 2.7 *Let $\{(\eta_1, \Phi_1), (\eta_2, \Phi_2), (\eta_3, \Phi_3)\}$ be an almost cosymplectic sphere on M^3. If $i_{\xi_i}\Phi_j \neq 0$ for every $i, j = 1, 2, 3, i \neq j$, then it is taut if and only if it is round.*

Proof Let $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda \in \mathbb{S}^2}$ be a taut almost cosymplectic sphere on M^3. Then (2.3) and (2.4) hold. Let ξ_i be the Reeb vector field of (η_i, Φ_i), $i = 1, 2, 3$. We have $i_{\xi_i}\eta_i = 1$, $i_{\xi_i}\Phi_i = 0$. Then according to (2.3), we have

$$i_{\xi_j}\Phi_i = i_{\xi_j} i_{\xi_i}(\eta_i \wedge \Phi_i) = i_{\xi_j} i_{\xi_i}(\eta_j \wedge \Phi_j) = -i_{\xi_i} i_{\xi_j}(\eta_j \wedge \Phi_j) = -i_{\xi_i}\Phi_j. \tag{2.5}$$

By applying the equation (2.4) on the vector field ξ_i, we obtain

$$\Phi_j - \eta_i \wedge i_{\xi_i}\Phi_j = -i_{\xi_i}\eta_j \wedge \Phi_i. \tag{2.6}$$

By applying the equation (2.6) on the vector field ξ_j, we obtain $-i_{\xi_j}\eta_i \wedge i_{\xi_i}\Phi_j = -i_{\xi_i}\eta_j \wedge i_{\xi_j}\Phi_i$. Then by using (2.5), we get $i_{\xi_i}\Phi_j(\eta_i(\xi_j) + \eta_j(\xi_i)) = 0$. Since $i_{\xi_i}\Phi_j \neq 0$, we have

$$\eta_i(\xi_j) + \eta_j(\xi_i) = 0. \tag{2.7}$$

From (2.5), (2.7) and Proposition 2.5, we conclude that the taut almost cosymplectic sphere is round.

Next we suppose that $\{(\eta_\lambda, \Phi_\lambda)\}_{\lambda\in\mathbb{S}^2}$ is a round almost cosymplectic sphere on M^3 and $i_{\xi_i}\Phi_j \neq 0$ for every $i, j = 1, 2, 3, i \neq j$. According to Proposition 2.5, we have (2.2), and according to Corollary 2.3, we have that ξ_1, ξ_2, ξ_3 are linearly independent. By straightforward computation, we prove that

$$\eta_\lambda \wedge \Phi_\lambda(\xi_1, \xi_2, \xi_3) = \Phi_1(\xi_2, \xi_3) = \eta_1 \wedge \Phi_1(\xi_1, \xi_2, \xi_3).$$

So the round almost cosymplectic sphere is taut. □

Remark 2.8 According to Corollary 2.3, the condition of $i_{\xi_i}\Phi_j \neq 0$ is unnecessary for almost cosymplectic circle, so we say that roundness is equivalent to tautness for almost cosymplectic circles on dimension 3.

3. Almost cosymplectic metric bi-structures

In [10], D. Perrone introduced and studied the notion of bi-contact metric structures on three-manifolds. According to the Lemma 3.2 in [10], we have known that for a pair of almost contact metric structures $\{(\varphi_i, \xi_i, \eta_i, g)\}_{i=1,2}$, the condition $g(\xi_1, \xi_2) = 0$ is equivalent to

$$\varphi_1\varphi_2 + \varepsilon\eta_1 \otimes \xi_2 = -(\varphi_2\varphi_1 + \varepsilon\eta_2 \otimes \xi_1), \tag{3.1}$$

where $\varphi_2\xi_1 = \varepsilon\varphi_1\xi_2, \varepsilon = \pm1$. Now, using this property, we consider the notion of almost cosymplectic metric bi-structures on M^3.

Let M be a three-manifold. A pair of almost cosymplectic metric structures $\{(\varphi_i, \xi_i, \eta_i, g)\}_{i=1,2}$ is said to be almost cosymplectic metric bi-structure if the two almost cosymplectic metric structures satisfy (3.1). Moreover, when $\varepsilon = +1$ (resp., $\varepsilon = -1$), the almost cosymplectic metric bi-structure is called negative (resp., positive).

After introducing the notions of almost cosymplectic metric bi-structures, we show some results between almost cosymplectic circles and almost cosymplectic metric bi-structures.

Proposition 3.1 *Let $\{(\eta_1, \Phi_1), (\eta_2, \Phi_2)\}$ be an almost cosymplectic circle on M^3. Then it is taut if and only if the corresponding almost cosymplectic metric structures are positive almost cosymplectic metric bi-structure.*

Proof If $\{(\eta_1, \Phi_1), (\eta_2, \Phi_2)\}$ is a taut almost cosymplectic circle, the corresponding two almost cosymplectic metric structures are $(\varphi_1, \xi_1, \eta_1, g)$, $(\varphi_2, \xi_2, \eta_2, g)$. From Remark 2.8 and Proposition 2.5, we have $\eta_1(\xi_2) + \eta_2(\xi_1) = 0$ which means $g(\xi_1, \xi_2) = 0$, the almost cosymplectic metric structures satisfy (3.1), therefore $\{(\varphi_i, \xi_i, \eta_i, g)\}_{i=1,2}$ is an almost cosymplectic metric bi-structure. Put $\xi_3 = \varphi_1\xi_2 = \varepsilon\varphi_2\xi_1$, then (ξ_1, ξ_2, ξ_3) is a global orthonormal basis on M^3. By computation, we get $(\eta_1 \wedge \Phi_1)(\xi_1, \xi_2, \xi_3) = -1$ and $(\eta_2 \wedge \Phi_2)(\xi_1, \xi_2, \xi_3) = \varepsilon$. Using Proposition 2.6, we get $\varepsilon = -1$, i.e., the almost cosymplectic metric bi-structure is positive.

Conversely, if $\{(\varphi_i, \xi_i, \eta_i, g)\}_{i=1,2}$ is a positive almost cosymplectic metric bi-structure, we have $g(\xi_1,\xi_2) = \eta_1(\xi_2) = \eta_2(\xi_1) = 0$ and $\varphi_2\xi_1 = -\varphi_1\xi_2$. Then we obtain $i_{\xi_2}\Phi_1 + i_{\xi_1}\Phi_2 = 0$. Thus we conclude that $\{(\eta_i,\Phi_i)\}_{i=1,2}$ is a taut almost cosymplectic circle by applying Proposition 2.5. □

Let $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ be an almost cosymplectic metric bi-structure on M^3. Put

$$\xi_3 = \varphi_1\xi_2,\ \eta_3 = -\eta_2\circ\varphi_1,\ \varphi_3 = \varphi_1\varphi_2 + \varepsilon\eta_2\otimes\xi_1. \tag{3.2}$$

Then we have that (ξ_1,ξ_2,ξ_3) is a global orthonormal basis and $\xi_3 = \varphi_1\xi_2 = \varepsilon\varphi_2\xi_1$. We have the following properties:

$$\begin{aligned}
&\varphi_3\xi_1 = (\varphi_1\varphi_2 + \varepsilon\eta_2\otimes\xi_1)\xi_1 = \varphi_1\varphi_2\xi_1 = \varepsilon\varphi_1^2\xi_2 = -\varepsilon\xi_2,\\
&\varphi_3\xi_2 = (\varphi_1\varphi_2 + \varepsilon\eta_2\otimes\xi_1)\xi_2 = \varepsilon\xi_1,\\
&\varphi_3\xi_3 = (\varphi_1\varphi_2 + \varepsilon\eta_2\otimes\xi_1)\xi_3 = \varphi_1\varphi_2\xi_3 = \varepsilon\varphi_1\varphi_2^2\xi_1 = 0,\\
&\varphi_3^2\xi_1 = \varphi_3(-\varepsilon\xi_2) = -\xi_1,\ \varphi_3^2\xi_2 = \varphi_3(\varepsilon\xi_1) = -\xi_2,\ \varphi_3^2\xi_3 = 0,\\
&\eta_3(\xi_1) = \eta_3(\xi_2) = 0,\ \ \eta_3(\xi_3) = 1,\\
&g(\varphi_3\xi_i,\varphi_3\xi_j) = g(\xi_i,\xi_j) - \eta_3(\xi_i)\eta_3(\xi_j),\ \ i,j = 1,2,3.
\end{aligned}$$

Therefore, we obtain that $(\varphi_3,\xi_3,\eta_3,g)$ is an almost contact metric structure. Moreover, we have the following properties for this structure.

Theorem 3.2 *If $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ is an almost cosymplectic metric bi-structure on a three-manifold M, then there exists a global basis of vector fields $\{\xi_1,\xi_2,\xi_3\}$ such that*

$$[\xi_2,\xi_3] = \alpha\xi_3,\ [\xi_3,\xi_1] = \beta\xi_3,\ [\xi_1,\xi_2] = \gamma\xi_3, \tag{3.3}$$

where ξ_1,ξ_2 are the corresponding Reeb vector fields of the two almost cosymplectic structures; $\xi_3\in\ker\eta_1\cap\ker\eta_2$; $\alpha,\ \beta$ and γ are smooth functions satisfying $\xi_3(\gamma)+\xi_1(\alpha)+\xi_2(\beta) = 0$. In particular, the third almost contact metric structure $(\varphi_3,\xi_3,\eta_3,g)$ is almost cosymplectic if and only if $\alpha=\beta=\gamma=0$.

Proof If $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ is an almost cosymplectic metric bi-structure on M^3 and

$$\xi_3 = \varphi_1\xi_2,\ \eta_3 = -\eta_2\circ\varphi_1,\ \varphi_3 = \varphi_1\varphi_2 + \varepsilon\eta_2\otimes\xi_1.$$

We have known that (ξ_1,ξ_2,ξ_3) is a global orthonormal basis. If we suppose

$$[\xi_1,\xi_2] = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma\xi_3,$$

$$[\xi_2,\xi_3] = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha\xi_3,$$

$$[\xi_3,\xi_1] = \beta_1\xi_1 + \beta_2\xi_2 + \beta\xi_3,$$

it is known that $(\varphi_1,\xi_1,\eta_1,g)$ and $(\varphi_2,\xi_2,\eta_2,g)$ are almost cosmplectic metric structures, so we have $\nabla\xi_1 = h_1\varphi_1$ and $\nabla\xi_2 = h_2\varphi_2$. Thus $\nabla_{\xi_1}\xi_1 = 0$, $\nabla_{\xi_2}\xi_2 = 0$. Therefore, we get $\alpha_2=\beta_1=\gamma_1=\gamma_2=0$. Due to $d\eta_1(\xi_2,\xi_3)=0$ and $d\eta_2(\xi_1,\xi_3)=0$, we obtain $\alpha_1=\beta_2=0$. Then

$$[\xi_2,\xi_3] = \alpha\xi_3,\ [\xi_3,\xi_1] = \beta\xi_3,\ [\xi_1,\xi_2] = \gamma\xi_3.$$

Thus we have

$$d\Phi_3(\xi_1,\xi_2,\xi_3) = -\Phi_3([\xi_1,\xi_2],\xi_3) + \Phi_3([\xi_1,\xi_3],\xi_2) - \Phi_3([\xi_2,\xi_3],\xi_1) = 0. \tag{3.4}$$

Furthermore, from Jacobi identity, we get

$$[[\xi_1,\xi_2],\xi_3] + [[\xi_2,\xi_3],\xi_1] + [[\xi_3,\xi_1],\xi_2] = -[\xi_3(\gamma) + \xi_1(\alpha) + \xi_2(\beta)]\xi_3 = 0,$$

which means that $\xi_3(\gamma) + \xi_1(\alpha) + \xi_2(\beta) = 0$.

We also obtain

$$2d\eta_3(\xi_1,\xi_2) = -\eta_3([\xi_1,\xi_2]) = -\gamma,$$

$$2d\eta_3(\xi_2,\xi_3) = -\eta_3([\xi_2,\xi_3]) = -\alpha,$$

$$2d\eta_3(\xi_3,\xi_1) = -\eta_3([\xi_3,\xi_1]) = -\beta.$$

According to (3.4), if $(\varphi_3,\xi_3,\eta_3,g)$ is almost cosymplectic metric structure, we must have $\gamma = \alpha = \beta = 0$. □

Corollary 3.3 *If $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ is a cosymplectic metric bi-structure on M^3, then there exists a third almost contact metric structure $(\varphi_3,\xi_3,\eta_3,g)$ which satisfies (3.2) and is cosymplectic metric structure.*

Proof According to Theorem 3.2, we find the tensor $h_1 = \frac{1}{2}\mathfrak{L}_{\xi_1}\varphi_1$ and $h_2 = \frac{1}{2}\mathfrak{L}_{\xi_2}\varphi_2$ satisfying:

$$\begin{aligned}
2h_1\xi_2 &= (\mathfrak{L}_{\xi_1}\varphi_1)\xi_2 = [\xi_1,\varphi_1\xi_2] - \varphi_1[\xi_1,\xi_2] = \gamma\xi_2 - \beta\xi_3,\\
2h_1\xi_3 &= (\mathfrak{L}_{\xi_1}\varphi_1)\xi_3 = [\xi_1,\varphi_1\xi_3] - \varphi_1[\xi_1,\xi_3] = -\beta\xi_2 - \gamma\xi_3,\\
2h_2\xi_1 &= (\mathfrak{L}_{\xi_2}\varphi_2)\xi_1 = [\xi_2,\varphi_2\xi_1] - \varphi_2[\xi_2,\xi_1] = -\varepsilon\gamma\xi_1 + \varepsilon\alpha\xi_3,\\
2h_2\xi_3 &= (\mathfrak{L}_{\xi_2}\varphi_2)\xi_3 = [\xi_2,\varphi_2\xi_3] - \varphi_2[\xi_2,\xi_3] = \varepsilon\alpha\xi_1 + \varepsilon\gamma\xi_3.
\end{aligned} \tag{3.5}$$

Then we get

$$2h_1 = \begin{bmatrix} 0 & 0 & 0\\ 0 & \gamma & -\beta\\ 0 & -\beta & -\gamma \end{bmatrix},\quad 2h_2 = \begin{bmatrix} -\varepsilon\gamma & 0 & \varepsilon\alpha\\ 0 & 0 & 0\\ \varepsilon\alpha & 0 & \varepsilon\gamma \end{bmatrix}.$$

So if $(\varphi_i,\xi_i,\eta_i,g)_{i=1,2}$ are cosymplectic manifolds, we have $\nabla\xi_i = h_i\varphi_i = 0$, which means $\varphi_i h_i X = 0$ for any vector field $X \in \mathfrak{X}(M)$. By applying φ_i on $\varphi_i h_i$, we get $h_i = 0$ which means $\alpha = \beta = \gamma = 0$. About the tensor $h_3 = \frac{1}{2}\mathfrak{L}_{\xi_3}\varphi_3$, we have

$$\begin{aligned}
2h_3\xi_1 &= (\mathfrak{L}_{\xi_3}\varphi_3)\xi_1 = [\xi_3,\varphi_3\xi_1] - \varphi_3[\xi_3,\xi_1] = \varepsilon\alpha\xi_3,\\
2h_3\xi_2 &= (\mathfrak{L}_{\xi_3}\varphi_3)\xi_2 = [\xi_3,\varphi_3\xi_2] - \varphi_3[\xi_3,\xi_2] = \varepsilon\beta\xi_3.
\end{aligned}$$

Therefore, according to Theorem 3.2, we get if these two almost cosymplectic metric structures are cosymplectic, the third almost contact metric structure is almost cosymplectic. Moreover, due to $h_3 = 0$ it is cosymplectic. □

According to the above results, on a three dimensional Riemannian manifold M^3 admitting almost cosymplectic metric bi-structures, there exists a global basis of vector fields $\{\xi_1,\xi_2,\xi_3\}$

satisfying (3.3). Using (3.3) and the Levi-Civita equation, we get

$$(\nabla_{\xi_i}\xi_j) = \begin{pmatrix} 0 & \frac{\gamma}{2}\xi_3 & -\frac{\gamma}{2}\xi_2 \\ -\frac{\gamma}{2}\xi_3 & 0 & \frac{\gamma}{2}\xi_1 \\ -\frac{\gamma}{2}\xi_2+\beta\xi_3 & \frac{\gamma}{2}\xi_1-\alpha\xi_3 & -\beta\xi_1+\alpha\xi_2 \end{pmatrix}. \tag{3.6}$$

Then according to (3.3) and (3.6), we calculate the following Riemannian curvature tensor which is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for any $X,\ Y,\ Z \in \mathfrak{X}(M)$:

$$\begin{aligned} &R(\xi_2,\xi_1)\xi_1 = -\frac{3}{4}\gamma^2\xi_2 + (\frac{1}{2}\xi_1(\gamma)+\beta\gamma)\xi_3, \\ &R(\xi_2,\xi_1)\xi_3 = (-\frac{1}{2}\xi_1(\gamma)-\beta\gamma)\xi_1 + (-\frac{1}{2}\xi_2(\gamma)+\alpha\gamma)\xi_2, \\ &R(\xi_2,\xi_3)\xi_1 = (-\frac{1}{2}\xi_2(\gamma)+\alpha\gamma)\xi_2 + (\frac{1}{2}\xi_3(\gamma)+\xi_2(\beta)-\alpha\beta)\xi_3, \\ &R(\xi_3,\xi_1)\xi_1 = (\frac{1}{2}\xi_1(\gamma)+\beta\gamma)\xi_2 + (-\xi_1(\beta)+\frac{1}{4}\gamma^2-\beta^2)\xi_3, \\ &R(\xi_3,\xi_1)\xi_2 = (-\frac{1}{2}\xi_1(\gamma)-\beta\gamma)\xi_1 + (\xi_1(\alpha)+\frac{1}{2}\xi_3(\gamma)+\alpha\beta)\xi_3, \\ &R(\xi_3,\xi_2)\xi_2 = (-\frac{1}{2}\xi_2(\gamma)+\alpha\gamma)\xi_1 + (\xi_2(\alpha)+\frac{1}{4}\gamma^2-\alpha^2)\xi_3. \end{aligned} \tag{3.7}$$

Then we obtain

$$\begin{aligned} &Q\xi_1 = (-\xi_1(\beta)-\frac{\gamma^2}{2}-\beta^2)\xi_1 + (\frac{\xi_1(\alpha)+\xi_3(\gamma)}{2}+\alpha\beta)\xi_2 + (-\frac{\xi_2(\gamma)}{2}+\alpha\gamma)\xi_3, \\ &Q\xi_2 = (\xi_1(\alpha)+\frac{\xi_3(\gamma)}{2}+\alpha\beta)\xi_1 + (\xi_2(\alpha)-\frac{\gamma^2}{2}-\alpha^2)\xi_2 + (\frac{\xi_1(\gamma)}{2}+\gamma\beta)\xi_3, \\ &Q\xi_3 = (-\frac{\xi_2(\gamma)}{2}+\alpha\gamma)\xi_1 + (\frac{\xi_1(\gamma)}{2}+\beta\gamma)\xi_2 + (-\xi_1(\beta)+\xi_2(\alpha)+\frac{\gamma^2}{2}-\alpha^2-\beta^2)\xi_3. \end{aligned} \tag{3.8}$$

If $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ is a cosymplectic metric bi-structure, we have $\alpha=\beta=\gamma=0$. Therefore, we have the following results:

Theorem 3.4 *If $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$ is a cosymplectic metric bi-structure on M^3, then the scalar curvature $scal=0$.*

4. Some results of h_i satisfying certain conditions

In this section, we discuss some special properties of Riemannian manifold M^3 with almost cosymplectic metric bi-structures. Let $\{\xi_1,\xi_2,\xi_3=\varphi_1\xi_2=\varepsilon\varphi_2\xi_1\}$ be the global orthonormal basis satisfying (3.2). First of all, using Eqs. (3.3) and (3.6), we compute the following formulas, for the tensor h_1, we have

$$\begin{aligned} &(\nabla_{\xi_1}h_1)\xi_2 = (\frac{1}{2}\xi_1(\gamma)+\frac{1}{2}\beta\gamma)\xi_2 + (-\frac{1}{2}\xi_1(\beta)+\frac{1}{2}\gamma^2)\xi_3, \\ &(\nabla_{\xi_1}h_1)\xi_3 = (-\frac{1}{2}\xi_1(\beta)+\frac{1}{2}\gamma^2)\xi_2 + (-\frac{1}{2}\xi_1(\gamma)-\frac{1}{2}\beta\gamma)\xi_3, \\ &(\nabla_{\xi_2}h_1)\xi_1 = (-\frac{1}{4}\beta\gamma)\xi_2 + (-\frac{1}{4}\gamma^2)\xi_3, \\ &(\nabla_{\xi_2}h_1)\xi_2 = -\frac{1}{4}\beta\gamma\xi_1 + \frac{1}{2}\xi_2(\gamma)\xi_2 - \frac{1}{2}\xi_2(\beta)\xi_3, \end{aligned} \tag{4.1}$$

$$(\nabla_{\xi_2}h_1)\xi_3=-\frac{1}{4}\gamma^2\xi_1-\frac{1}{2}\xi_2(\beta)\xi_2-\frac{1}{2}\xi_2(\gamma)\xi_3,$$
$$(\nabla_{\xi_3}h_1)\xi_1=(\frac{1}{4}\gamma^2+\frac{1}{2}\beta^2)\xi_2+\frac{1}{4}\beta\gamma\xi_3,$$
$$(\nabla_{\xi_3}h_1)\xi_2=(\frac{1}{4}\gamma^2+\frac{1}{2}\beta^2)\xi_1+(\frac{1}{2}\xi_3(\gamma)-\alpha\beta)\xi_2+(-\frac{1}{2}\xi_3(\beta)-\alpha\gamma)\xi_3,$$
$$(\nabla_{\xi_3}h_1)\xi_3=\frac{1}{4}\beta\gamma\xi_1+(-\frac{1}{2}\xi_3(\beta)-\alpha\gamma)\xi_2+(-\frac{1}{2}\xi_3(\gamma)+\alpha\beta)\xi_3;$$

for the tensor h_2, we have

$$\begin{aligned}
&(\nabla_{\xi_1}h_2)\xi_1=-\frac{\varepsilon}{2}\xi_1(\gamma)\xi_1-\frac{\varepsilon}{4}\alpha\gamma\xi_2+\frac{\varepsilon}{2}\xi_1(\alpha)\xi_3,\\
&(\nabla_{\xi_1}h_2)\xi_2=-\frac{\varepsilon}{4}\alpha\gamma\xi_1-\frac{\varepsilon}{4}\gamma^2\xi_3,\\
&(\nabla_{\xi_1}h_2)\xi_3=\frac{\varepsilon}{2}\xi_1(\alpha)\xi_1-\frac{\varepsilon}{4}\gamma^2\xi_2+\frac{\varepsilon}{2}\xi_1(\gamma)\xi_3,\\
&(\nabla_{\xi_2}h_2)\xi_1=(-\frac{\varepsilon}{2}\xi_2(\gamma)+\frac{\varepsilon}{2}\alpha\gamma)\xi_1+(\frac{\varepsilon}{2}\xi_2(\alpha)+\frac{\varepsilon}{2}\gamma^2)\xi_3,\\
&(\nabla_{\xi_2}h_2)\xi_3=(\frac{\varepsilon}{2}\xi_2(\alpha)+\frac{\varepsilon}{2}\gamma^2)\xi_1+(\frac{\varepsilon}{2}\xi_2(\gamma)-\frac{\varepsilon}{2}\alpha\gamma)\xi_3,\\
&(\nabla_{\xi_3}h_2)\xi_1=(-\frac{\varepsilon}{2}\xi_3(\gamma)-\varepsilon\alpha\beta)\xi_1+(\frac{\varepsilon}{4}\gamma^2+\frac{\varepsilon}{2}\alpha^2)\xi_2+(\frac{\varepsilon}{2}\xi_3(\alpha)-\varepsilon\beta\gamma)\xi_3,\\
&(\nabla_{\xi_3}h_2)\xi_2=(\frac{\varepsilon}{4}\gamma^2+\frac{\varepsilon}{2}\alpha^2)\xi_1+\frac{\varepsilon}{4}\alpha\gamma\xi_3,\\
&t(\nabla_{\xi_3}h_2)\xi_3=(\frac{\varepsilon}{2}\xi_3(\alpha)-\varepsilon\beta\gamma)\xi_1+\frac{\varepsilon}{4}\alpha\gamma\xi_2+(\frac{\varepsilon}{2}\xi_3(\gamma)+\varepsilon\alpha\beta)\xi_3.
\end{aligned}\tag{4.2}$$

On a Riemannian manifold, a $(1,1)$-type tensor T is said to be of Codazzi type if it satisfies $(\nabla_X T)Y=(\nabla_Y T)X$ for any vector field X, Y. By the previous calculations, we obtain:

Theorem 4.1 *Let M be a three-manifold with almost cosymplectic metric bi-structures. If the tensor $\{h_i\}_{i=1,2}$ are being of Codazzi type, then the any almost cosymplectic manifold $(M,\varphi_i,\xi_i,\eta_i)$ is cosymplectic and locally isometric to the flat Euclidean space $\mathbb{R}^3$.*

Proof Let M be a three-manifold with almost cosymplectic metric bi-structures $\{(\varphi_i,\xi_i,\eta_i,g)\}_{i=1,2}$. If h_1 is being of codazzi type, i.e., $(\nabla_X h_1)Y=(\nabla_Y h_1)X$ for any vector field X, Y, replace X and Y by ξ_1 and ξ_2, respectively. Then using (4.1), we get

$$\begin{cases}\frac{1}{2}\xi_1(\gamma)+\frac{3}{4}\beta\gamma=0,\\ \frac{1}{2}\xi_1(\beta)-\frac{3}{4}\gamma^2=0.\end{cases}\tag{4.3}$$

Then replacing X and Y by ξ_1 and ξ_3, respectively, we obtain

$$\begin{cases}-\frac{1}{2}\xi_1(\beta)-\frac{1}{2}\beta^2+\frac{1}{4}\gamma^2=0,\\ -\frac{1}{2}\xi_1(\gamma)-\frac{3}{4}\beta\gamma=0.\end{cases}\tag{4.4}$$

Applying the formula $\frac{1}{2}\xi_1(\beta)=\frac{3}{4}\gamma^2$ in the first term of (4.4), we obtain $\gamma^2+\beta^2=0$, which means $\gamma=0$ and $\beta=0$.

Similarly, using the same way for h_2, we get

$$\begin{cases} \frac{\varepsilon}{2}\xi_2(\gamma) - \frac{3\varepsilon}{4}\alpha\gamma = 0, \\ \frac{\varepsilon}{2}\xi_2(\alpha) + \frac{3\varepsilon}{4}\gamma^2 = 0. \end{cases} \tag{4.5}$$

$$\begin{cases} \frac{\varepsilon}{2}\xi_2(\alpha) - \frac{\varepsilon}{2}\alpha^2 + \frac{\varepsilon}{4}\gamma^2 = 0, \\ \frac{\varepsilon}{2}\xi_2(\gamma) - \frac{3\varepsilon}{4}\alpha\gamma = 0. \end{cases} \tag{4.6}$$

Using $\frac{\varepsilon}{2}\xi_2(\alpha) = -\frac{3\varepsilon}{4}\gamma^2$ in the first term of (4.6), we get $\gamma^2 + \alpha^2 = 0$, which means $\alpha = 0$ and $\gamma = 0$. So we have $\alpha = \beta = \gamma = 0$. Therefore, we have that $(M, \varphi_i, \xi_i, \eta_i)_{i=1,2,3}$ are cosymplectic. According to [2], any of them is locally isometric to the flat Euclidean space $\mathbb{R}^3$. □

On a Riemannian manifold, we say a symmetric $(1,1)$-type tensor field T is cyclic parallel if it satisfies

$$g((\nabla_X T)Y, Z) + g((\nabla_Y T)Z, X) + g((\nabla_Z T)X, Y) = 0 \tag{4.7}$$

for any vector field X, Y, Z. We have the following result:

Theorem 4.2 *Let M be a three-manifold with almost cosymplectic metric bi-structure. Suppose that the tensors $\{h_i\}_{i=1,2}$ are cyclic parallel, then the scalar curvature of M^3 is 0, $-4\beta^2$ or $-4\alpha^2$.*

Proof According to (4.1), (4.2) and (4.7), we obtain

$$\begin{cases} g((\nabla_{\xi_3} h_1)\xi_3, \xi_3) = -\frac{1}{2}\xi_3(\gamma) + \alpha\beta = 0, \\ g((\nabla_{\xi_3} h_2)\xi_3, \xi_3) = \frac{\varepsilon}{2}\xi_3(\gamma) + \varepsilon\alpha\beta = 0. \end{cases}$$

From this, we have $\xi_3(\gamma) = \alpha\beta = 0$. Replacing $X = \xi_1$, $Y = Z = \xi_2$ and $X = Y = Z = \xi_2$ in (4.7) for h_1 respectively, we get $\xi_1(\gamma) = \xi_2(\gamma) = 0$. Then we conclude that γ is a constant.

By replacing $(X, Y, Z) = (\xi_1, \xi_2, \xi_3)$, (ξ_2, ξ_2, ξ_3), (ξ_2, ξ_3, ξ_3) in (4.7) for h_1 respectively, we have the following relationships:

$$\begin{cases} -\xi_1(\beta) + \gamma^2 + \beta^2 = 0, \\ \xi_3(\gamma) - 2\xi_2(\beta) - 2\alpha\beta = 0, \\ \frac{1}{2}\xi_2(\gamma) + \xi_3(\beta) + 2\alpha\gamma = 0. \end{cases} \tag{4.8}$$

By replacing $(X, Y, Z) = (\xi_1, \xi_1, \xi_3)$, (ξ_1, ξ_2, ξ_3), (ξ_1, ξ_3, ξ_3) in (4.7) for h_2 respectively, we have the following relationships:

$$\begin{cases} \xi_1(\alpha) - \frac{1}{2}\xi_3(\gamma) - \alpha\beta = 0, \\ \xi_2(\alpha) + \gamma^2 + \alpha^2 = 0, \\ \frac{1}{2}\xi_1(\gamma) + \xi_3(\alpha) - 2\beta\gamma = 0. \end{cases} \tag{4.9}$$

According to $\alpha\beta = 0$, we have three conditions.

If $\alpha = 0$ and $\beta = 0$, from the first term of (4.8) and the second term of (4.9), we get that $\gamma = 0$. In this case, the scalar curvature is 0.

If $\alpha = 0$ and $\beta \neq 0$, by the second term of (4.9), we conclude that $\gamma = 0$ and (4.8) is equivalent to $\xi_1(\beta) = \gamma^2$, $\xi_2(\beta) = 0$, $\xi_3(\beta) = 0$. Applying these in (3.8), we obtain $Q\xi_1 = -2\beta^2\xi_1$, $Q\xi_2 = 0$, $Q\xi_3 = -2\beta^2\xi_3$. It follows that the scalar curvature is $-4\beta^2$.

If $\alpha \neq 0$ and $\beta = 0$, according to the first term of (4.8), we get that $\gamma = 0$ and (4.9) is equivalent to $\xi_1(\alpha) = 0$, $\xi_2(\alpha) = -\alpha^2$, $\xi_3(\alpha) = 0$. In this context, we obtain

$$Q\xi_1 = 0,\ Q\xi_2 = -2\alpha^2\xi_2,\ Q\xi_3 = -2\alpha^2\xi_3.$$

It follows that the scalar curvature is $-4\alpha^2$. This completes the proof. □

Acknowledgements We thank the referees for their time and comments.

References

[1] S. GOLDBERG, K. YANO. *Integrability of almost cosymplectic structures*. Pacific J. Math., 1969, **31**: 373–382.

[2] D. PERRONE. *Classification of homogeneous almost cosymplectic three-manifolds*. Differ. Geom. Appl., 2012, **30**(1): 49–58.

[3] Wenjie WANG. *A class of three dimensional almost cokähler manifolds*. Palest. J. Math., 2017, **6**(1): 111–118.

[4] Yaning WANG. *Almost contact metric manifolds with local Riemannian and Ricci symmetries*. Mediterr. J. Math., 2019, **16**(4): 81.

[5] Yaning WANG. *Almost co-kähler manifolds satisfying some symmetry conditions*. Turkish J. Math., 2016, **40**(4): 740–752.

[6] Yaning WANG. *Curvature homogeneity and ball-homogeneity on almost cokähler 3-manifolds*. Bull. Korean Math. Soc., 2019, **56**(1): 253–263.

[7] H. GEIGES, J. GONZALO. *Contact geometry and complex surfaces*. Invent. Math., 1995, **121**(1): 147–209.

[8] M. ZESSIN. *On contact p-spheres*. Ann. Inst. Fourier., 2005, **55**(4): 1167–1194.

[9] B. C. MONTANO, A. D. NICOLA, I. YUDIN. *Cosymplectic p-spheres*. J. Geom. Phys., 2016, **100**: 68–79.

[10] D. PERRONE. *Taut contact circles and bi-contact metric structures on three-manifolds*. Ann. Global Anal. Geom., 2017, **52**(2): 213–235.

[11] D. E. BLAIR. *Riemannian Geometry of Contact and Symplectic Manifolds*. Birkhäuser Boston, Ltd., Boston, MA, 2010.

[12] Z. OLSZAK. *On almost cosymplectic manifolds*. Kodai Math. J., 1981, **4**(2): 239–250.

[13] D. PERRONE. *Taut contact circles on H-contact 3-manifolds*. Int. Math. Forum, 2006, **1**(25-28): 1285–1296.

[14] D. PERRONE. *Homogeneous contact Riemannian three-manifolds*. Illinois J. Math., 1998, **42**(2): 243–256.