Journal of Mathematical Research with Applications
May, 2021, Vol. 41, No. 3, pp. 291–302
DOI:10.3770/j.issn:2095-2651.2021.03.007
Http://jmre.dlut.edu.cn

Infimum of Topological Entropies of Homotopy Classes of Maps on Infra-solvmanifolds of Type (R)

Baojun HUANG[1,2]

1. *Department of Electronic and Information Engineering, Bozhou University, Anhui* 236800*, P. R. China;*

2. *School of Mathematical Science, Huaibei Normal University, Anhui* 235000*, P. R. China*

Abstract Let f be a continuous map on infra-solvmanifold M of type (R) and $N^{\infty}(f)$ be the asymptotic Nielsen number of f. In this paper, the sufficient conditions to assure that $\log N^{\infty}(f)$ is the infimum of topological entropies of the homotopy class of the map f are given by using Nielsen fixed point theory. These conclusions will generalize the similar results on infra-nilmanifolds.

Keywords infimum; topological entropy; homotopy; infra-solvmanifolds of type (R)

MR(2020) Subject Classification 55M20; 37B40

1. Introduction

Let $f : X \to X$ be a continuous map on a compact metric space X. The pair (f, X) is called a topological dynamical system. The complexity of a dynamical system is one of the main topics in the study of the dynamical system. There are several ways to measure complexity of a topological dynamical system. Topological entropy is a topological invariant and a topological dynamical system with positive entropy means that the complexity of the system is "big". This invariant measures the complexity of the topological dynamical system in the following way: let $f : X \to X$ be a continuous map on a compact metric space and $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$ be an orbit segment of length n . Then the topological entropy $h(f)$ of f measures the exponential growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution. Therefore, it is meaningful to estimate or calculate topological entropy.

In [1], Ivanov has proved that if f is a continuous map on the compact connected polyhedron X, then the topological entropy $h(f)$ and the asymptotic Nielsen number $N^{\infty}(f)$ of f satisfy

$$h(f) \geq \log N^{\infty}(f).$$

And then the above conclusion was also proved by Jiang with simpler method in [2]. Since $N^{\infty}(f)$ is a homotopy invariant, we have

$$\inf\{h(g) | g \simeq f : X \to X\} \geq \log N^{\infty}(f). \tag{1.1}$$

Received April 29, 2020; Accepted November 13, 2020
Supported by Research Foundation of Bozhou for the Introduction of Talents, Natural Science Foundation of Anhui Province (Grant No. KJ2019A1300).
E-mail address: huangbaojun006-6@163.com

In [2], Jiang raised the following open problem: what are the conditions for $\log N^\infty(f)$ to be the best lower bound for $h(f)$ of all maps homotopic to f? In other words, when does the equality hold in the inequality (1.1) (see [2, Question 5.3])? For this question, the author first obtains the following results in [3]:

Theorem 1.1 ([3, Theorem 1]) *Let $f: T^m \to T^m$ be a continuous map on m dimensional torus. Then the equality*

$$\inf\{h(g)|g \simeq f : T^m \to T^m\} = \log N^\infty(f)$$

holds if and only if one of the following conditions is satisfied

(i) 1 is not in the spectrum of $f_*: H_1(T^m, \mathcal{R}) \to H_1(T^m, \mathcal{R})$;

(ii) 1 is in the spectrum of f_*, but the norms of its all other eigenvalues are not more than 1.

Theorem 1.2 ([3, Theorem 2]) *Let $f : M := G/\Gamma \to M$ be a continuous map on compact nilmanifold, F be the unique extension of $f_\sharp : \Gamma \to \Gamma(\Gamma \approx \pi_1(M))$ on Lie group G and F_* be the homomorphism of Lie algebra of G induced by F. Then the equality*

$$\inf\{h(g)|g \simeq f : M \to M\} = \log N^\infty(f)$$

holds if and only if one of the following conditions is satisfied

(i) 1 is not in the spectrum of F_*;

(ii) 1 is in the spectrum of F_*, but the norms of its all other eigenvalues of are not more than 1.

Recently, as a generalization of the above results, the author obtained the following results in [4]:

Theorem 1.3 ([4, Theorem 4.1]) *Let $f : M := \pi\backslash G \to M$ be a continuous map on the infra-nilmanifold M with holonomy group H, and $\Phi = (b, B)$ be a homotopy lift of f. Then the equality*

$$\inf\{h(g)|g \simeq f : M = \pi\backslash G \to M\} = \log N^\infty(f)$$

holds if and only if one of the following conditions is satisfied

(i) $1 \notin \sigma(B_*)$;

(ii) $1 \in \sigma(B_*)$ but $sp(B_*) \leq 1$;

(iii) $1 \in \sigma(B_*)$, $sp(B_*) > 1$, and $N^\infty = \Pi_{|\lambda_i|>1}|\lambda_i|$; where $\sigma(B_*)$ and $sp(B_*)$ denote the spectrum and the spectral radius of $B_*, \lambda_i \in \sigma(B_*)$.

These results give a part of answer of Jiang's question above. However, there are still much works to be done before the question can be completely solved. In this paper, we continue to consider the above equality for other continuous maps. We will give the sufficient conditions to assure that $\log N^\infty(f)$ is the infimum of topological entropies of homotopy classes of the maps on infra-solvmanifolds of type (R). Since the class of solvable Lie groups of type (R) contains all nilpotent Lie groups, these results will essentially generalize some conclusions in [4].

2. Preliminaries

In this section we will recall some necessary preliminaries on the infra-solvmanifolds of type (R), and list some basic concepts of topological entropy and asymptotic Nielsen number which are related to the discussion of this paper.

Definition 2.1 ([5]) *Let G be a connected Lie group. The semi-direct product* $\mathrm{aff}(G) := G \rtimes \mathrm{End}(G)$ *with the binary operation*

$$(a, A)(b, B) = (a \cdot Ab, AB)$$

is called the semigroup of affine endomorphisms of G, where $\mathrm{End}(G)$ *is the set of all endomorphisms of G.*

In particular, $\mathrm{Aff}(G) := G \rtimes \mathrm{Aut}(G)$ is called the affine group of G, where $\mathrm{Aut}(G)$ is the set of all automorphisms of G . The elements of $\mathrm{aff}(G)$ and $|rmAff(G)$ are called affine endomorphisms and affine automorphisms of G:

Definition 2.2 ([5]) *Let G be a connected Lie group. The action of the semigroup* $\mathrm{aff}(G)$ *on G is defined by*

$$(a, A)z := a \cdot Az, \quad z \in G$$

and

$$(a, A)(b, B)(z) := (a, A)((b, B)(z)), \quad z \in G.$$

Definition 2.3 ([5]) *Let G be a connected and simply connected solvable Lie group. A discrete subgroup Γ of G is called a lattice of G if the orbit space G/Γ is compact, and in this case G/Γ is called a special solvmanifold.*

Definition 2.4 ([5]) *Let $\pi \subset Aff(G)$ be a torsion-free finite extension of the lattice Γ. The orbit space $\pi\backslash G$ is called an infra-solvmanifold on model G. $H := \pi/\Gamma(\subset \mathrm{Aut}(G))$ is called the holonomy group of π or $\pi\backslash G$.*

Clearly, the infra-solvmanifold $\pi\backslash G$ is finitely covered by the special solvmanifold G/Γ with the covering transformation group equal to H. When $H = 1$, $\pi \subset G$ and the orbit space $\pi\backslash G$ is exactly a special solvmanifold.

Definition 2.5 ([5]) *An infra-solvmanifold $\pi\backslash G$ is called of type (R) if G is of type (R), that is, every adjoint representation $ad(X)$ of the Lie algebra of G has only real eigenvalues.*

Let $f : X \to X$ be a continuous map on the compact metric space X. For given $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be an (n, ε)-separated subset under f if for each pair $x \neq y$ in E there exists i $(0 \leq i < n)$ such that $d(f^i(x), f^i(y)) \geq \varepsilon$. Let $s_n(\varepsilon, f)$ denote the largest cardinality of any (n, ε)-separated subset E under f. Then we have the following definition:

Definition 2.6 ([2]) *Write $h(f, \varepsilon) := \limsup_n \frac{1}{n} \log s_n(\varepsilon, f)$. Then*

$$h(f) := \lim_{\varepsilon \to 0} h(f, \varepsilon)$$

is called the topological entropy of f.

Remark 2.7 $s_n(\varepsilon, f)$ is the greatest number of orbit segments $\{x, f(x), f^2(x), \dots, f^{n-1}(x)\}$ of length n that can be distinguished one from another provided we can only distinguish between points of X that are at least ε apart. If $h(f, \varepsilon) > 0$, then, up to resolution $\varepsilon > 0$, the number $s_n(\varepsilon, f)$ of distinguishable orbit segments grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution.

Let f be a continuous map on the compact connected polyhedron X. To define the asymptotic Nielsen number of f, we define an equivalence relation on fixed point set $\mathrm{Fix}(f)$ of f as follows: For $x_0, x_1 \in \mathrm{Fix}(f)$, $x_0 \sim x_1$ if and only if there exists a path c from x_0 to x_1 such that $c \simeq f \circ c\{x_0, x_1\}$. An equivalence class of this relation is called a fixed point class of f. To each fixed point class F, one can assign an integer $\mathrm{ind}(f, F)$. A fixed point class F is called essential if $\mathrm{ind}(f, F) \neq 0$. Now, the Nielsen number of f is defined by

$$N(f) := \text{the number of essential fixed point classes of } f.$$

This leads to the following definition:

Definition 2.8 ([2]) *Let f be a continuous map on the compact connected polyhedron X. Then*

$$N^{\infty}(f) := \max\{1, \limsup_{n} [N(f^n)]^{\frac{1}{n}}\}$$

is called the asymptotic Nielsen number of f.

3. Infimum of topological entropies of homotopy classes of maps on infra-solvmanifolds of type (R)

Let $f : M = \pi \setminus G \to M$ be a continuous map on the infra-solvmanifold M of type (R). Then it induces an endomorphism $f_\sharp$ on $\pi_1(M)$. By [5, Theorem 2.2], there exists affine endomorphism $\Phi = (b, B)$ on G such that for all $\alpha \in \pi_1(M) \approx \pi$ we have

$$f_\sharp(\alpha) \circ \Phi = \Phi \circ \alpha, \tag{3.1}$$

B is unique up to $\mathrm{Inn}(G)$, the inner automorphism group of G. This implies that the affine endomorphism $\Phi = (b, B)$ induces a continuous map $\Phi_{(b,B)} : M \to M$ on the infra-solvmanifold M, which is homotopic to f. That is, f has an affine homotopy lift $\Phi = (b, B)$.

To estimate or calculate the asymptotic Nielsen number of f, we first give the following two lemmas.

Lemma 3.1 *Suppose $f : M = \pi \setminus G \to M$ is a continuous map on the infra-solvmanifold M of type (R) with holonomy group H, and $\Phi = (b, B)$ is a homotopy lift of f. Then for any $A_1 \in H$, there exists a sequence $A_1, A_2, \dots$ of elements of H such that*

(i) $A_{i+1}B = BA_i$ for $i \in \mathbb{N}$;

(ii) The sequence is a periodic sequence from some A_j.

Proof We write the natural projection from π to $H = \pi/\Gamma$ as ρ, then for any $A_1 \in H$, there is $x \in \pi$ such that $\rho(x) = A_1$. Write $A_i := \rho(f_\sharp^{i-1}(x))$, $x := (a_1, A_1)$, $f_\sharp^{i-1}(x) := (a_i, A_i)$. Then we have

$$f_\sharp^i(x)(b,B)=(b,B)f_\sharp^{i-1}(x),$$

by (3.1), i.e.,

$$(a_{i+1},A_{i+1})(b,B)=(b,B)(a_i,A_i).$$

So

$$A_{i+1}B=BA_i.$$

Also H is finite, thus there exist $j\geq 1$ and $k\geq 1$ such that$A_{j+k}=A_j$, and so, we have $A_{j+k+1}=A_{j+1},A_{j+k+2}=A_{j+2},\dots$ (in general $A_{j+nk+l}=A_{j+l}$ for all $l,n\in\mathbb{N}$). That is, the sequence is a periodic sequence from some A_j. □

We will refer to the sequence $A_1,A_2,\dots,A_j,\dots,A_{j+k-1},A_{j+k}=A_j,\dots$ in Lemma 3.1 above as a periodic sequence for A_1, associated to f, with period k starting from position j. The following lemma is a generalization of [6, Lemma 3.1], the proof is also similar.

Lemma 3.2 *Suppose $f:M=\pi\setminus G\to M$ is a continuous map on the infra-solvmanifold M of type (R) with holonomy group H, and $\Phi=(b,B)$ is a homotopy lift of f. Let $A_1,A_2,\dots,A_j,\dots,A_{j+k-1},A_{j+k}=A_j,\dots$ be a periodic sequence for $A_1\in H$, associated to f with period k starting from position j. Then*

(i) For any $i\in\mathbb{N},\det(I-(A_1)_*(B)_*)=\det(I-(A_i)_*(B)_*)$;

(ii) $B_*^k(A_j)_*=(A_j)_*B_*^k$;

(iii) There is $l\in\mathbb{N}$, such that $((A_j)_*B_*)^l=B_*^l$,

where $()_*$ denotes the differential of $()$.

Because averaging formula for the Nielsen numbers of maps on infra-nilmanifolds can be generalized to infra-solvmanifolds of type (R) (see [5, 7]), we have the following.

Theorem 3.3 *Suppose $f:M=\pi\setminus G\to M$ is a continuous map on the infra-solvmanifold M of type (R) with holonomy group H, and $\Phi=(b,B)$ is a homotopy lift of f. Then the asymptotic Nielsen number $N^\infty(f)$ of f satisfies*

$$1\leq N^\infty(f)\leq\prod_{|\lambda_i|>1}|\lambda_i|.$$

In particular, $N^\infty(f)=\prod_{|\lambda_i|>1}|\lambda_i|$ when $1\notin\sigma(B_*)$ and $sp(B_*)>1$; $N^\infty(f)=1$ when $sp(B_*)\leq 1$, where $\sigma(B_*)$ and $sp(B_*)$ denote the spectrum and the spectral radius of B_*, $\lambda_i\in\sigma(B_*)$.

Proof Let $\mu_1,\mu_2,\dots,\mu_m$ be the eigenvalues of $(A_jB)_*$ with each eigenvalue listed as many times as its algebraic multiplicity. By Lemma 3.2 (iii), $((A_jB)_*)^l=B_*^l$, so

$$\{\mu_1^l,\mu_2^l,\dots,\mu_m^l\}=\{\lambda_1^l,\lambda_2^l,\dots,\lambda_m^l\}.$$

We may assume that $\mu_i^l=\lambda_i^l$, $i=1,2,\dots,m$. Thus $|\lambda_i|=|\mu_i|$. We have

$$|\det(I-(A_1)_*(B)_*)|=|\det(I-(A_j)_*(B)_*)|=\prod_{i=1}^m|1-\mu_i|\leq\prod_{i=1}^m(1+|\mu_i|)=\prod_{i=1}^m(1+|\lambda_i|).$$

So

$$|\det(I-(A_1)_*B_*^n)| \le \prod_{i=1}^{m}(1+|\lambda_i^n|),$$

and hence

$$N(f^n) = \frac{1}{|H|}\sum_{A\in H}|\det(I-A_*B_*^n)| \le \prod_{i=1}^{m}(1+|\lambda_i^n|) \tag{3.2}$$

by [5, Theorem 4.3]. Therefore,

$$N(f^n)^{1/n} = (\frac{1}{|H|}\sum_{A\in H}|\det(I-A_*B_*^n)|)^{1/n} \le \Big(\prod_{i=1}^{m}(1+|\lambda_i^n|)\Big)^{1/n},$$

and so

$$\begin{aligned}\limsup N(f^n)^{1/n} &\le \limsup \Big(\prod_{i=1}^{m}(1+|\lambda_i^n|)\Big)^{1/n}\\ &= \prod_{|\lambda_i|>1}|\lambda_i| \limsup \Big(\prod_{|\lambda_i|>1}(1+|\lambda_i^{-n}|)\Big)^{\frac{1}{n}}\Big(\prod_{|\lambda_i|\le 1}(1+|\lambda_i^n|)\Big)^{\frac{1}{n}}\end{aligned}$$

i.e.,

$$N^\infty(f) \le \prod_{|\lambda_i|>1}|\lambda_i|. \tag{3.3}$$

When $1\notin\sigma(B_*)$ and $sp(B_*)>1$,

$$N(f^n) = \frac{1}{|H|}\sum_{A\in H}|\det(I-A_*B_*^n)| \ge \frac{1}{|H|}|\det(I-B_*^n)|.$$

So

$$N(f^n)^{1/n} \ge |\det(I-B_*^n)|^{1/n}/|H|^{1/n}.$$

Also note that

$$\begin{aligned}|\det(I-B_*^n)| &= \prod_{i=1}^{m}|1-|\lambda_i^n|\\ &= \prod_{|\lambda_i|>1}|\lambda_i^n|\prod_{|\lambda_i|>1}|1-\lambda_i^{-n}|\prod_{|\lambda_i|<1}|1-\lambda_i^n|\prod_{|\lambda_i|=1\,(\lambda_i\text{ are roots of unity})}|1-\lambda_i^n|\\ &\quad \prod_{|\lambda_i|=1\,(\lambda_i\text{ are not roots of unity})}|1-\lambda_i^n|\end{aligned}$$

But $\prod_{|\lambda_i|=1\,(\lambda_i\text{ are roots of unity})}|1-\lambda_i^n|$ takes only limited numbers of values for any natural number n. Therefore, if

$$\Delta_n := \prod_{|\lambda_k|>1}|1-\lambda_k^{-n}|\prod_{|\lambda_k|<1}|1-\lambda_k^n|\prod_{|\lambda_k|=1\,(\lambda_k\text{ are roots of unity})}|1-\lambda_k^n|,$$

then $(\Delta_{n_s})^{\frac{1}{n_s}} \to 1$ for the subsequence $\{\Delta_{n_s}\}$ of all nonzero terms of Δ_n. Also if

$$\sigma_{n_s} := \prod_{|\lambda_i|=1\,(\lambda_i\text{ are not roots of unity})}|1-\lambda_i^{n_s}| = \prod_{j=1}^{t_0}|1-\lambda_{i_j}^{n_s}||1-\bar\lambda_{i_j}^{n_s}|, \lambda_{i_j} = e^{\theta_j \mathrm{i}},$$

then

$$\sigma_{n_s} = 4^{t_0}|\sin n_s\frac{\theta_1}{2}\sin n_s\frac{\theta_2}{2}\cdots\sin n_s\frac{\theta_{t_0}}{2}|^2.$$

Then, by [4, Lemma 3.1]

$$\limsup_{n_s}(\sigma_{n_s})^{\frac{1}{n_s}} = \limsup_{n_s}(4^{t_0})^{\frac{1}{n_s}}|\sin n_s\frac{\theta_1}{2}\sin n_s\frac{\theta_2}{2}\cdots\sin n_s\frac{\theta_{t_0}}{2}|^{\frac{2}{n_s}} = 1.$$

i.e.,

$$\limsup_{n_s}|\det(I-B^{n_s})|^{\frac{1}{n_s}} = \prod_{|\lambda_i|>1}|\lambda_i|.$$

Hence,

$$\limsup N(f^n)^{1/n} \geq \limsup|\det(I-B_*^n)|^{1/n} = \prod_{|\lambda_i|>1}|\lambda_i|.$$

So

$$N^\infty(f) \geq \prod_{|\lambda_i|>1}|\lambda_i|. \tag{3.4}$$

(3.3) and (3.4) imply

$$N^\infty(f) = \prod_{|\lambda_i|>1}|\lambda_i|.$$

When $\mathrm{sp}(B_*) \leq 1$, we have $|\lambda_i| \leq 1$ for any $\lambda_i \in \sigma(B_*)$. By (3.2), it follows that $N(f^n) \leq 2^m$ and $N^\infty(f) = 1$. □

Let $M = \pi \setminus G$ be an infra-solvmanifold of type (R) with holonomy group H. By [8, Lemma 3.1], we can choose a fully invariant subgroup $\Lambda \subset \Gamma$ of π which is of finite index. Therefore, $f_\sharp(\Lambda) \subset \Lambda$ and so $f_\sharp$ induces the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda & \longrightarrow & \pi & \longrightarrow & J & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle f'_\sharp} & & \big\downarrow{\scriptstyle f_\sharp} & & \big\downarrow{\scriptstyle \bar{f}_\sharp} & & \\
1 & \longrightarrow & \Lambda & \longrightarrow & \pi & \longrightarrow & J & \longrightarrow & 1,
\end{array}$$

Diagram 1 Commutative diagram of the fundamental groups

where $J := \pi/\Lambda$ is finite. Applying (3.1) for $\lambda \in \Lambda \subset \pi$, we see that

$$f_\sharp(\lambda) = bB(\lambda)b^{-1} = \tau_b B(\lambda),$$

where τ_b is the conjugation by b. The homomorphism $f'_\sharp : \Lambda \to \Lambda$ induces a unique Lie group homomorphism $F := \tau_b B : G \to G$, and hence a homomorphism F_* on the Lie algebra of G. On the other hand, since $f_\sharp(\Lambda) \subset \Lambda$, we have the following two facts: (i) f and $\Phi_{(b,B)}$ can be lifted to $\bar{f}$ and ϕ_f on the special solvmanifold $N := G/\Lambda$ which finitely and regularly covers M and has J as its group of covering transformations; (ii) F induces a map ϕ_F on the special solvmanifold N.

Theorem 3.4 *Let $f : M = \pi \setminus G \to M$ be a continuous map on the infra-solvmanifold M of type (R) with a holonomy group H, and $\Phi = (b, B)$ be a homotopy lift of f. If $1 \notin \sigma(B_*)$, then*

$$h(f) \geq h(\Phi_{(b,B)}) = \log N^\infty(f).$$

Hence $\Phi_{(b,B)}$ minimizes the entropy in the homotopy class of f.

Proof The Lie group G has a right invariant metric d (see [9, p.24]). Similarly, there is left invariant metric d' on G. Then we have

$$\begin{aligned} h_{d'}(\Phi) =& h_{d'}(L_b \circ B) = h_{d'}(B) = h_{d'}(B^*) \\ =& \begin{cases} \sum_{|\lambda_i|>1} \log|\lambda_i|, & \mathrm{sp}(B_*) > 1; \\ 0, & \mathrm{sp}(B_*) \le 1 \end{cases} \end{aligned} \tag{3.5}$$

similar to [10, Corollary11,16]. In the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{\phi_f} & N \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{p}} \\ M & \xrightarrow{\Phi_{(b,B)}} & M, \end{array}$$

Diagram 2 Commutative diagram of the Lie group G and its orbit spaces

we take any metric d'' on M. By compactness the lift d''_Λ of d'' to N is equivalent to the projection d'_Λ of d' to N (see [11, Lemma 3.1]).

So,

$$h(\phi_f) = h_{d''_\Lambda}(\phi_f) = h_{d'_\Lambda}(\phi_f) = h_{d'}(\Phi) \tag{3.6}$$

where the last equal sign comes from the first proposition in [12, p.184]. And from [11, Proposition 2.1], we have

$$h(\Phi_{b,B}) = h(\phi_f). \tag{3.7}$$

Hence

$$h(f) \ge \log N^\infty(f) = \log \prod_{|\lambda_j|>1} |\lambda_j| = \log \prod_{|\mu_j|>1} |\mu_j| = h(\Phi_{(b,B)})$$

when $\mathrm{sp}(B_*) > 1$ by Theorem 3.3 and the equalities (3.5) to (3.7). In addition, $N^\infty(f) = 1$ and $h(\Phi_{(b,B)}) = 0 = \log N^\infty(f)$ when $\mathrm{sp}(B_*) \le 1$. Therefore,

$$h(f) \ge \log N^\infty(f) = h(\Phi_{(b,B)}). \quad \square$$

Theorem 3.5 *Let $f : M = \pi \setminus G \to M$ be a continuous map on the infra-solvmanifold M of type (R) with a holonomy group H, and $\Phi = (b, B)$ be a homotopy lift of f. Then we have*

$$\inf\{h(g) | g \simeq f : \pi \setminus G \to \pi \setminus G\} = \log N^\infty(f), \tag{3.8}$$

provided that one of the following conditions holds:

(i) $1 \notin \sigma(B_*)$;

(ii) $\mathrm{sp}(B_*) \le 1$;

(iii) $\mathrm{sp}(B_*) > 1$, *and* $N^\infty(f) = \prod_{|\lambda_i|>1} |\lambda_i|$, *where* $\lambda_i \in \sigma(B_*)$.

Proof Suppose (i) holds. Since $\Phi_{(b,B)} \simeq f$ and

$$h(g) \geq \log N^{\infty}(g) = \log N^{\infty}(f) = h(\Phi_{(b,B)})$$

for any $g \simeq f$ by Theorem 3.4, the equality (3.5) holds.

Suppose (ii) holds. Then

$$h(\Phi_{(b,B)}) = h(\phi_f) = 0 = \log N^{\infty}(f).$$

And since $f \simeq \Phi_{(b,B)}$, the equality (3.5) holds.

Suppose (iii) holds. Then

$$h(\Phi_{(b,B)}) = h(\phi_f) = \sum_{|\lambda_i|>1} \log|\lambda_i| = \log N^{\infty}(f).$$

And since $\Phi_{(b,B)} \simeq f$ and $h(g) \geq \log N^{\infty}(g) = \log N^{\infty}(f)$ for any $g \simeq f$, the equality (3.8) holds. □

Corollary 3.6 *Let $f : N = G/\pi \to N$ be a continuous map on the special solvmanifold N of type (R), and $\Phi = (b, B)$ be a homotopy lift of f. Then we have*

$$\inf\{h(g) | g \simeq f : N = G/\pi \to N\} = \log N^{\infty}(f), \tag{3.9}$$

provided that one of the following conditions holds:

(i) $1 \notin \sigma(B_*)$;

(ii) $1 \in \sigma(B_*)$, *but* $\mathrm{sp}(B_*) \leq 1$.

Proof When condition (i) holds, the conclusion is derived directly from Theorem 3.5 (i). When condition (ii) holds we have

$$N(f^n) = |\det(I - B_*^n)| = 0,$$

so $N^{\infty}(f) = 1$. Also $\mathrm{sp}(B_*) \leq 1$, hence

$$h(\phi_f) = 0 = \log N^{\infty}(f).$$

And since $f = \phi_f$, the equality (3.9) holds. □

Remark 3.7 When G is nilpotent Lie group $\sigma(F_*) = \sigma(B_*)$ by [8, Lemma 3.2]. Hence Theorem 3.5 and Corollary 3.6 generalize the sufficiency of Theorem 4.1 in [4] and Theorem 2 in [3].

Example 3.8 The solvable Lie group Sol is one of the eight geometries that one considers in the study of 3-manifolds [12]. One can describe Sol as a semi-direct product $\mathbb{R}^2 \rtimes_{\varphi} \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the map

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

i.e.,

$$(x, y, t) \cdot (x', y', t') := ((x, y) + (x', y')\varphi(t), t + t') = (x + e^t x', y + e^{-t} y', t + t').$$

Clearly, the Lie group Sol can be embedded into $\mathrm{Aff}(\mathbb{R}^2) \subset M_3(\mathbb{R})$ as

$$\begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y and t are real numbers. Consider the equality

$$\exp Xs = \begin{pmatrix} e^{t(s)} & 0 & x(s) \\ 0 & e^{-t(s)} & y(s) \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that the elements in the Lie algebra of Sol are

$$X = \begin{pmatrix} t & 0 & a \\ 0 & -t & b \\ 0 & 0 & 0 \end{pmatrix}.$$

We take an ordered (linear) basis for the Lie algebra as follows:

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any $\alpha = (x_0, y_0, t_0) \in \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$, we have

$$\tau_\alpha((x, y, t)) = (x_0 + e^{t_0}x - e^t x_0, y_0 + e^{-t_0}y - e^{-t}y_0, t).$$

So

$$Ad(\alpha) = (\tau_\alpha)_* = \begin{pmatrix} e^{t_0} & 0 & -x_0 \\ 0 & e^{-t_0} & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\sigma(Ad(\alpha)) = \{e^{t_0}, e^{-t_0}, 1\}$ and the Lie group Sol is solvable Lie group of type (R) which is not isomorphic to nilpotent Lie group by [13, Proposition 2.1]. Let Γ be the subgroup of Sol which is generated by

$$(\frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0), (\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}, 0), (0, 0, \ln\frac{3+\sqrt{5}}{2}).$$

Then Γ is isomorphic to the group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ where

$$\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is an element of $SL(2, \mathbb{Z})$ with eigenvalues $\frac{3\pm\sqrt{5}}{2}$ and in fact Γ is a lattice of Sol.

Let $a = (0, 0, \frac{1}{2}\ln\frac{3+\sqrt{5}}{2}) \in \mathrm{Sol}$ and A: Sol$\to$ Sol be the automorphism of Sol given by

$$A((x, y, t)) := (-x, -y, t).$$

Then A has period 2, and $(a, A)^2 = ((0, 0, \ln\frac{3+\sqrt{5}}{2}), I) \in \mathrm{Sol} \rtimes \mathrm{Aut(Sol)}$, where I is the identity automorphism of Sol. The subgroup

$$\pi := \langle \Gamma, (a, A) \rangle \subset \mathrm{Sol} \rtimes \mathrm{Aut}\,(\mathrm{Sol})$$

generated by the lattice Γ and the element (a, A) is discrete and torsion free, and Γ is a normal subgroup of π of index 2. Thus π is a torsion-free finite extension of the lattice Γ, and $\pi\backslash\mathrm{Sol}$ is an infra-solvmanifold, which has a double covering $\Gamma\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ by its holonomy group $H = \pi/\Gamma = \{I, A\} \cong \mathbb{Z}_2$.

Let $B : \mathrm{Sol} \to \mathrm{Sol}$ be the automorphism of Sol given by

$$B(x, y, t) := (mx, my, -t),$$

where m is any nonzero integer. Then $BA = AB$ and the conjugation by $((0, 0, 0), B) \in \mathrm{Sol} \rtimes \mathrm{Aut}\,(\mathrm{Sol})$ maps π into π (and Γ into Γ). Thus, the affine endomorphism

$$\Phi = ((0, 0, 0), B) : \mathrm{Sol} \to \mathrm{Sol}$$

induces $\phi_B : \mathrm{Sol}/\Gamma \to \mathrm{Sol}/\Gamma$ and $\Phi_B : \pi\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ so that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Sol} & \xrightarrow{((0,0,0),B)} & \mathrm{Sol} \\
\downarrow & & \downarrow \\
\mathrm{Sol}/\Gamma & \xrightarrow{\phi_B} & \mathrm{Sol}/\Gamma \\
\downarrow & & \downarrow \\
\pi\backslash\mathrm{Sol} & \xrightarrow{\Phi_B} & \pi\backslash\mathrm{Sol}
\end{array}$$

Diagram 3 Commutative diagram of the Lie group Sol and its orbit spaces

On other hand, with respect to the basis $\{e_1, e_2, e_3\}$, the differentials of A and B are

$$A_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_* = \begin{pmatrix} 0 & m & 0 \\ m & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since $\sigma(B_*) = \{-1, \pm m\}$, for any $f \simeq \Phi_B$ we have

$$\inf\{h(g) | g \simeq f : M = \pi\backslash Sol \to M\} = \log N^\infty(f),$$

by Theorem 3.5 (i) and (ii).

Acknowledgements I would like to express my gratitude to Professor Xuezhi ZHAO at Capital Normal University for proposing this work to me and for providing me with valuable suggestions and references. I also thank the referee for carefully reading the manuscript and contributing many useful comments.

References

[1] N. V. IVANOV. *Entropy and the Nielsen numbers.* Sov. Nath. Dokl., 1982, **26**: 63–66.

[2] Boju JIANG. *Estimation of the number of periodic orbits*. Pacific J. Math., 1996, **172**(1): 151–185.

[3] Baojun HUANG. *On the topological entropy of homotopy class for maps*. Adv. in Math., 1997, **26**: 241–244. (in Chinese)

[4] Baojun HUANG. *Infimum of topological entropies of homotopy classes of maps on infra-nilmanifolds*. http://engine.scichina.com/doi/10.1360/SSM-2019-0297.

[5] J. B. LEE, K. B. LEE. *Averaging formula for Nielsen numbers of maps on infra-solvmanifolds of type (R)*. Nagoya Math. J., 2009, **196**: 117–134.

[6] K. DEKIMPE, B. DEROCK, W. MALFAIT. *The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds*. Monatsh. Math., 2007, **150**(1): 1–10.

[7] K. Y. HA, J. B. LEE. *Corrigendum to 'Averaging formula for Nielsen numbers of maps on infra-solvmanifolds of type (R)'*. Nagoya Math. J., 2016, **221**: 207–212.

[8] J. B. LEE, K. B. LEE. *Lefschetz numbers for continuous maps, and periods for expanding maps on infra-nilmanifolds*. J. Geom. Phys., 2006, **56**(10): 2011–2023.

[9] W. MARZANTOWICZ, F. PRZYTYCKI. *Estimates of the topological entropy from below for continuous self-maps on some compact manifolds*. Discrete Contin. Dyn. Syst., 2008, **21**(2): 501–512.

[10] A. FATHI, M. SHUB. *Some dynamics of pseudo-Anosov diffeomorphisms*. Exposé 10, Travaux de Thurston sur les Surfaces, Asterique, No. 66-67, Soc. Math. France, Paris, 1979, 181–207.

[11] R. BOWEN. *Entropy for group endomorphisms and homogeneous spaces*. Trans. Amer. Math. Soc., 1971, **153**: 401–414.

[12] P. SCOTT. *The geometries of 3-manifolds*. Bull. London Math. Soc., 1983, **15**(5): 401–487.

[13] J. H. JO, J. B. LEE. *Nielsen fixed point theory on infra-solvmanifolds of Sol*. Topol. Methods Nonlinear Anal., 2017, **49**(1): 325–350.