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The Global Attraction of Logistic Equation with Lévy Noise

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Abstract The logistic equation with Lévy noise is considered. Under suitable conditions, the global existence and uniqueness is obtained; it is shown that the unique positive equilibrium is globally attractive if the initial value is less than the carrying capacity.

Keywords Logistic equation; Lévy noise; global attraction; equilibrium

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1. Introduction

The one-dimensional logistic population model is described by the ordinary differential equation

$$\dot{X}(t) = rX(t)[1 - \frac{X(t)}{K}]. \tag{1.1}$$

Here $X(t)$ is the population size of a certain species at time t whose members usually live in proximity, share the same basic requirements, and compete for resources, food, habitat, or territory; $r > 0$ is a constant which represents the rate of growth and $K > 0$ is carrying capacity of the environment. It is well-known that, for any positive initial value, the population will survive and there is a stable and globally attractive equilibrium point, see May [1] for the details of this model.

Population systems in the real world are inevitably affected by environmental noise, so it is significative and interesting to reveal how the noise affects the population systems. One of the most usual environmental noise is Gaussian noise. Many authors have discussed population systems under perturbation of Gaussian noise. Mao et al. [2] found the phenomenon that Gaussian noise can suppress a potential population explosion. Jiang and Shi [3] studied the periodic solution of nonautonomous logistic model perturbed by Gaussian noise; Jiang et al. [4] studied the existence and uniqueness of Gaussian noise perturbed logistic model and global attraction of the unique equilibrium. Li and Yin [5] and Wang et al. [6] considered Logistic models and Lotka-Volterra models respectively with switching Gaussian noise.

To arrive at a more feasible and realistic model, it would be helpful for us to take into account of some sudden or discontinuous change of environmental phenomena, which could be described by jump processes, or more general Lévy noise. Bao et al. [7, 8] investigated the

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Lotka-Volterra system perturbed by Lévy noise; under suitable conditions, they obtained the existence of unique global positive solution and the estimate of sample Lyapunov exponent and the boundedness of the moment of solution. Huang and Cao [9] studied ergodicity and bifurcations for stochastic logistic equation driven by Lévy noise. Besides Lévy noise, Xu [10] recently studied the phenomenological bifurcation for a stochastic logistic model with correlated colored noise.

For a population system, the existence of stable positive equilibrium means longtime survival of species, so it is of great biological significance to investigate this interesting problem when the system is perturbed by Lévy noise. In this paper, we study the logistic equation with Lévy noise

$$\begin{aligned}\mathrm{d}X(t) =& X(t-)[1-\frac{X(t-)}{K}]\times\\ &\Big[r\mathrm{d}t+\alpha\mathrm{d}B(t)+\beta\int_{|x|<1}H(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)+\gamma\int_{|x|\ge 1}D(t,x)N(\mathrm{d}t,\mathrm{d}x)\Big].\end{aligned} \tag{1.2}$$

By different method from [7,8], we obtain that there exists a unique positive solution to (1.2); furthermore, we show that (1.2) admits a unique positive equilibrium which is globally attractive when the initial population is less than the carrying capacity.

2. Preliminaries

In this section, we recall some preliminaries for later use. Firstly, let us review the definition of Lévy processes, see [11,12] for details.

Definition 2.1 *An $\mathbb{R}^n$-valued stochastic process $L=(L(t),t\ge 0)$ is called Lévy process if:*

(1) $L(0)=0$ almost surely;

(2) L has independent and stationary increments;

(3) L is stochastically continuous, i.e., for all $\epsilon>0$ and for all $s>0$

$$\lim_{t\to s}P(|L(t)-L(s)|>\epsilon)=0.$$

For a given Lévy process L, the associated jump process $\Delta L=(\Delta L(t),t\ge 0)$ is given by $\Delta L(t)=L(t)-L(t-)$ for each $t\ge 0$. For any Borel set D in $\mathbb{R}^n-\{0\}$, define the random counting measure

$$N(t,D)(\omega):=\sharp\{0\le s\le t:\Delta L(s)(\omega)\in D\}=\sum_{0\le s\le t}\chi_D(\Delta L(s)(\omega)),$$

where χ_D is the indicator function of D. We write $\nu(\cdot)=\mathbf{E}(N(1,\cdot))$ and call it the intensity measure associated with L. A Borel set D in $\mathbb{R}^n-\{0\}$ is bounded below if $0\notin\overline{D}$, the closure of D. If D is bounded below, then $N(t,D)<\infty$ almost surely for all $t\ge 0$ and $(N(t,D),t\ge 0)$ is a Poisson process with intensity $\nu(D)$. So N is called Poisson random measure. For each $t\ge 0$ and D bounded below, the corresponding compensated Poisson random measure is given by

$$\widetilde{N}(t,D):=N(t,D)-t\nu(D).$$

Proposition 2.2 [Lévy-Itô decomposition] *If L is an $\mathbb{R}^n$-valued Lévy process, then there exist*

$a \in \mathbb{R}^n$, an $\mathbb{R}^n$-valued Wiener process B, and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^n - \{0\})$ such that, for each $t \geq 0$

$$L(t) = at + B(t) + \int_{|x|<1} x\widetilde{N}(t, \mathrm{d}x) + \int_{|x|\geq 1} xN(t, \mathrm{d}x). \tag{2.1}$$

Here the Poisson random measure N has the intensity measure ν which satisfies

$$\int_{\mathbb{R}^n} (|y|^2 \wedge 1)\nu(\mathrm{d}y) < \infty \tag{2.2}$$

and $\widetilde{N}$ is the compensated Poisson random measure of N.

Consider the following differential equation in $\mathbb{R}^n$ perturbed by Lévy noise

$$\mathrm{d}X(t) = f(t, X(t-))\mathrm{d}t + g(t, X(t-))\mathrm{d}L(t),$$

where f is $\mathbb{R}^n$-valued and g is an $n \times n$ matrix-valued function. By the Lévy-Itô decomposition (2.1), the equation can be written as

$$\begin{aligned}\mathrm{d}X(t) =&(f(t, X(t-)) + g(t, X(t-))a)\mathrm{d}t + g(t, X(t-))\mathrm{d}B(t)+ \\ &\int_{|x|<1} g(t, X(t-))x\ \widetilde{N}(\mathrm{d}t, \mathrm{d}x) + \int_{|x|\geq 1} g(t, X(t-))xN(\mathrm{d}t, \mathrm{d}x).\end{aligned}$$

Hence we may consider stochastic differential equation with Lévy noise of more general form

$$\begin{aligned}\mathrm{d}X(t) =&f(t, X(t-))\mathrm{d}t + g(t, X(t-))\mathrm{d}B(t)+ \\ &\int_{|x|<1} F(t, X(t-), x)\ \widetilde{N}(\mathrm{d}t, \mathrm{d}x) + \int_{|x|\geq 1} G(t, X(t-), x)N(\mathrm{d}t, \mathrm{d}x),\end{aligned}$$

where F and G are $\mathbb{R}^n$-valued. Note that if the mathematical expectation $\mathbf{E}|L(t)|^p < \infty$ for some $p \geq 1$ and all $t \geq 0$, then $\int_{|x|\geq 1} |x|^p \nu(\mathrm{d}x) < \infty$ and hence $L(t)$ admits the Lévy-Itô decomposition of the form

$$L(t) = at + B(t) + \int_{\mathbb{R}^n} x\widetilde{N}(t, \mathrm{d}x).$$

In this case, we only need to consider the stochastic differential equation of the form

$$\mathrm{d}X(t) = f(t, X(t-))\mathrm{d}t + g(t, X(t-))\mathrm{d}B(t) + \int_{\mathbb{R}^n} F(t, X(t-), x)\ \widetilde{N}(\mathrm{d}t, \mathrm{d}x).$$

In this paper, we do not assume this condition, i.e., we consider the full form (1.2) and so very large jump is allowed with considerable probability.

The following well-known result will be used below.

Proposition 2.3 [Law of iterated logarithm] *Assume that $B(t)$ is an n-dimensional Wiener process, then*

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t \log\log t}} = 1, \quad a.s. \tag{2.3}$$

3. Existence, uniqueness and global attraction of the equilibrium

Consider the one dimensional logistic Eq. (1.1). If r is perturbed by a Lévy type noise, by Lévy-Itô decomposition, we consider the following more general form of perturbation

$$r \to r + \alpha \dot{B}(t) + \beta \int_{|x|<1} H(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)/\mathrm{d}t + \gamma \int_{|x|\geq 1} D(t,x)N(\mathrm{d}t,\mathrm{d}x)/\mathrm{d}t,$$

where B is standard one-dimensional Brownian motion, N is a Poisson random measure independent of B and $\widetilde{N}$ is the associated compensated Poisson random measure of N, H and D describe small and large jumps, respectively (a simple case is $H(t,x) = x = D(t,x)$, i.e., r is perturbed simply by standard Lévy noise $L(\cdot)$), and the constants α, β, γ denote the intensity of perturbations. Hence the associated stochastic differential equation reads as follows

$$\begin{aligned}\mathrm{d}X(t) =& X(t-)[1 - \frac{X(t-)}{K}]\times \\ & \Big[r\mathrm{d}t + \alpha\mathrm{d}B(t) + \beta \int_{|x|<1} H(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \gamma \int_{|x|\geq 1} D(t,x)N(\mathrm{d}t,\mathrm{d}x)\Big]. \end{aligned} \tag{3.1}$$

Note that $X(t) \equiv 0$ and $X(t) \equiv K$ are two solutions of Eq. (3.1).

Firstly we give a priori estimate for the solution of (3.1):

Theorem 3.1 *Consider the Eq. (3.1) with initial value $X(0) = X_0$. Assume that $0 < X_0 < K$ almost surely. Then the solution $X(\cdot)$ of (3.1) with initial value X_0 satisfies the property that $0 < X(t) < K$ almost surely for all $t \geq 0$.*

Proof When $X(t) \neq 0$ and $X(t) \neq K$, from Itô's theorem we have

$$\begin{aligned}
&\mathrm{d}\log|\frac{X(t)}{K - X(t)}| = \mathrm{d}\log|X(t)| - \mathrm{d}\log|K - X(t)| \\
&= \frac{1}{X(t-)}[X(t-)(1 - \frac{X(t-)}{K})r\mathrm{d}t + X(t-)(1 - \frac{X(t-)}{K})\alpha\mathrm{d}B(t)] + \\
&\frac{1}{2}(-\frac{1}{X(t-)^2})[X(t-)^2(1 - \frac{X(t-)}{K})^2\alpha^2]\mathrm{d}t + \\
&\int_{|x|\geq 1}[\log|X(t-) + X(t-)(1 - \frac{X(t-)}{K})\gamma D(t,x)| - \log|X(t-)|]N(\mathrm{d}t,\mathrm{d}x) + \\
&\int_{|x|<1}[\log|X(t-) + X(t-)(1 - \frac{X(t-)}{K})\beta H(t,x)| - \log|X(t-)|]\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \\
&\int_{|x|<1}\big[\log|X(t-) + X(t-)(1 - \frac{X(t-)}{K})\beta H(t,x)| - \log|X(t-)| - \\
&\frac{1}{X(t-)}X(t-)(1 - \frac{X(t-)}{K})\beta H(t,x)\big]\nu(\mathrm{d}x)\mathrm{d}t - \\
&\frac{1}{X(t-) - K}[X(t-)(1 - \frac{X(t-)}{K})r\mathrm{d}t + X(t-)(1 - \frac{X(t-)}{K})\alpha\mathrm{d}B(t)] - \\
&\frac{1}{2}[-\frac{1}{(X(t-) - K)^2}][X(t-)^2(1 - \frac{X(t-)}{K})^2\alpha^2]\mathrm{d}t - \\
&\int_{|x|\geq 1} E\ N(\mathrm{d}t,\mathrm{d}x) - \int_{|x|<1} A\ \widetilde{N}(\mathrm{d}t,\mathrm{d}x) -
\end{aligned}$$

$$\int_{|x|<1}[A-\frac{1}{X(t-)-K}X(t-)(1-\frac{X(t-)}{K})\beta H(t,x)]\nu(\mathrm{d}x)\mathrm{d}t$$
$$=(r-\frac{1}{2}\alpha^2)\mathrm{d}t+\frac{\alpha^2}{K}X(t-)\mathrm{d}t+\alpha\mathrm{d}B(t)+\int_{|x|\geq 1}\log|1+\frac{K\gamma D(t,x)}{K-X(t-)\gamma D(t,x)}|N(\mathrm{d}t,\mathrm{d}x)+$$
$$\int_{|x|<1}\log|1+\frac{K\beta H(t,x)}{K-X(t-)\beta H(t,x)}|\widetilde{N}(\mathrm{d}t,\mathrm{d}x)+$$
$$\int_{|x|<1}[\log|1+\frac{K\beta H(t,x)}{K-X(t-)\beta H(t,x)}|-\beta H(t,x)]\nu(\mathrm{d}x)\mathrm{d}t, \tag{3.2}$$

where

$$A=\log|K-(X(t)+X(t)(1-\frac{X(t)}{K})\beta H(t,x))|-\log|K-X(t)|,$$
$$E=\log|K-(X(t)+X(t)(1-\frac{X(t)}{K})\gamma D(t,x))|-\log|K-X(t)|.$$

So there exists a random variable $C=C(\omega)$ such that

$$\frac{X(t)}{K-X(t)}=C\exp\Big\{rt-\frac{1}{2}\alpha^2t+\alpha B(t)+\frac{\alpha^2}{K}\int_0^t X(s-)\mathrm{d}s+\int_0^t\int_{|x|\geq 1}FN(\mathrm{d}s,\mathrm{d}x)+$$
$$\int_0^t\int_{|x|<1}M\widetilde{N}(\mathrm{d}s,\mathrm{d}x)+\int_0^t\int_{|x|<1}[M-\beta H(s,x)]\nu(\mathrm{d}x)\mathrm{d}s\Big\}=:C\mathrm{e}^I, \tag{3.3}$$

where

$$M=\log|1+\frac{K\beta H(s,x)}{K-X(s)\beta H(s,x)}|,\quad F=\log|1+\frac{K\gamma D(s,x)}{K-X(s)\gamma D(s,x)}|.$$

It follows that

$$X(t)=\frac{K}{1+\frac{1}{C}\mathrm{e}^{-I}}.$$

Since $0<X_0<K$, $C=\frac{X_0}{K-X_0}>0$, we have

$$X(t)=\frac{K}{1+(\frac{K}{X_0}-1)\mathrm{e}^{-I}}, \tag{3.4}$$

which yields that $0<X(t)<K$ almost surely for all $t\geq 0$. □

Now we establish the existence and uniqueness result for Eq. (3.1).

Theorem 3.2 *Assume that* $|\beta H(t,x)|\leq\eta<1$ *for* $(t,x)\in[0,\infty)\times B_1$ *and* $\nu(B_1)<\infty$, *where* η *is a constant and* B_1 *represents the interval* $(-1,1)$. *Then for arbitrary initial value* $X(0)=X_0$ *with* $0<X_0<K$ *almost surely, (3.1) admits a unique solution* $X(\cdot)$.

Proof Let

$$X(t)=K\frac{e^{Q(t)}}{1+e^{Q(t)}},\quad \text{for all}\ \ t\geq 0. \tag{3.5}$$

Then

$$Q(t)=\log\frac{X(t)}{K-X(t)}. \tag{3.6}$$

From (3.2) and (3.6), the Eq. (3.1) can be transformed into

$$\mathrm{d}Q(t)=(r-\frac{1}{2}\alpha^2+\frac{e^{Q(t-)}}{1+e^{Q(t-)}}\alpha^2)\mathrm{d}t+\alpha\mathrm{d}B(t)+$$

$$\int_{|x|<1}[\log(1+\frac{\beta H(t,x)(1+e^{Q(t-)})}{1+e^{Q(t-)}-e^{Q(t-)}\beta H(t,x)})-\beta H(t,x)]\nu(\mathrm{d}x)\mathrm{d}t+$$
$$\int_{|x|\geq 1}\log[1+\frac{\gamma D(t,x)(1+e^{Q(t-)})}{1+e^{Q(t-)}-e^{Q(t-)}\gamma D(t,x)}]N(\mathrm{d}s,\mathrm{d}x)+$$
$$\int_{|x|<1}\log[1+\frac{\beta H(t,x)(1+e^{Q(t-)})}{1+e^{Q(t-)}-e^{Q(t-)}\beta H(t,x)}]\widetilde{N}(\mathrm{d}s,\mathrm{d}x) \quad (3.7)$$

with initial value $Q(0)=\log\frac{X_0}{K-X_0}$. Let

$$f(Q):=r-\frac{1}{2}\alpha^2+\frac{e^Q}{1+e^Q}\alpha^2,$$
$$g(t,Q):=\int_{|x|<1}[\log(1+\frac{\beta H(t,x)(1+e^Q)}{1+e^Q-e^Q\beta H(t,x)})-\beta H(t,x)]\nu(\mathrm{d}x),$$
$$h(t,x,Q):=\log[1+\frac{\gamma D(t,x)(1+e^Q)}{1+e^Q-e^Q\gamma D(t,x)}],$$
$$p(t,x,Q):=\log[1+\frac{\beta H(t,x)(1+e^Q)}{1+e^Q-e^Q\beta H(t,x)}].$$

It is easy to see that the function $f(\cdot)$ meets Lipschitz and linear growth conditions. For the function g, since $|\beta H(t,x)|\leq\eta<1$ and $\nu(B_1)<\infty$, we have

$$\begin{aligned}\frac{\partial g(t,Q)}{\partial Q}=&\Big(\int_{|x|<1}\big[\log(1+\frac{\beta H(t,x)(1+e^Q)}{1+e^Q-e^Q\beta H(t,x)})-\beta H(t,x)\big]\nu(\mathrm{d}x)\Big)'_Q\\ =&\int_{|x|<1}\big[\log(1+\frac{\beta H(t,x)(1+e^Q)}{1+e^Q-e^Q\beta H(t,x)})-\beta H(t,x)\big]'_Q\nu(\mathrm{d}x)\\ =&\int_{|x|<1}\frac{e^Q\beta^2H^2(t,x)}{[1+e^Q-e^Q\beta H(t,x)][1+e^Q+\beta H(t,x)]}\nu(\mathrm{d}x)\\ \leq&\int_{|x|<1}\frac{\eta^2}{1-\eta^2}\nu(\mathrm{d}x)\\ =&\frac{\eta^2}{1-\eta^2}\nu(B_1).\end{aligned} \quad (3.8)$$

That is, $\frac{\partial g(t,Q)}{\partial Q}$ is bounded, so g meets global Lipschitz condition in Q and hence linear growth condition. Note that the function h is continuous in Q. For the function p, by the assumption $|\beta H(t,x)|\leq\eta<1$ and $\nu(B_1)<\infty$ again, we have

$$\begin{aligned}\frac{\partial p(t,x,Q)}{\partial Q}=&\big[\log(1+\frac{\beta H(t,x)(1+e^Q)}{1+e^Q-e^Q\beta H(t,x)})\big]'_Q\\ =&\frac{e^Q\beta^2H^2(t,x)}{[1+e^Q-e^Q\beta H(t,x)][1+e^Q+\beta H(t,x)]}\\ \leq&\frac{\eta^2}{1-\eta^2},\end{aligned} \quad (3.9)$$

so the coefficient of $\widetilde{N}$ satisfies Lipschitz and linear growth conditions in Q. By standard existence and uniqueness theorem [11, Chapter 6], Eq. (3.7) has a unique solution; so it follows from (3.5) that the Eq. (3.1) has a unique solution. The proof is completed. □

Remark 3.3 Note that the assumption $|\beta H(t,x)| \leq \eta < 1$ naturally holds when the constant β satisfies $|\beta| < 1$ and the noise is simply the Lévy noise itself, i.e., $H(t,x) = x$. Finally we consider the global attraction property for the Eq. (3.1).

Theorem 3.4 *Assume that* $0 < \beta H(t,x) < 1$, $0 < \gamma D(t,x) < 1$ *and* $\nu(B_1) < \infty$. *If* $[r - \frac{\alpha^2}{2} - \nu(B_1)] > 0$, *then the unique solution of (3.1), with initial value* $X(0) = X_0$ *satisfying* $0 < X_0 < K$ *almost surely, is globally attractive to* K.

Proof Note that when $0 < \beta H(t,x) < 1$, $0 < \gamma D(t,x) < 1$ and $\nu(B_1) < \infty$, we have $M > 0, F > 0$ and by (3.4)

$$\begin{aligned} X(t) =& \frac{K}{1 + (\frac{K}{X_0} - 1)\mathrm{e}^{-I}} \\ =& \frac{K}{1 + (\frac{K}{X_0} - 1)\mathrm{e}^{-rt + \frac{1}{2}\alpha^2 t - \alpha B(t) - \frac{\alpha^2}{K}\int_0^t X(s-)\mathrm{d}s + J}} \\ >& \frac{K}{1 + (\frac{K}{X_0} - 1)\mathrm{e}^{-(r - \frac{\alpha^2}{2})t + \int_0^t \nu(B_1)\mathrm{d}s}} \\ =& \frac{K}{1 + (\frac{K}{X_0} - 1)\mathrm{e}^{-[r - \frac{\alpha^2}{2} - \nu(B_1)]t}}, \end{aligned}$$

where

$$J := -\int_0^t \int_{|x|\geq 1} FN(\mathrm{d}s, \mathrm{d}x) - \int_0^t \int_{|x|<1} MN(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x|<1} \beta H(s,x)\nu(\mathrm{d}x)\mathrm{d}s$$

and the inequality holds when $t \gg 1$ by the law of iterated logarithm in Proposition 2.3. Therefore, $X(t) \to K$ almost surely as $t \to \infty$. □

Remark 3.5 Note that the assumptions $0 < \beta H(t,x) < 1$ and $0 < \gamma D(t,x) < 1$ mean that the jump perturbation is appropriately small. This is intuitively natural since very large jumps may destroy the stability of dynamics associated to Eq. (3.1).

Finally, let us make some comments on the initial value assumption: $X_0 < K$ almost surely. The meaning is obvious: we only consider the case when the initial population size is smaller than the carrying capacity of environment. It is certainly interesting to consider the opposite situation: $X_0 > K$ almost surely, or even the mixed situation. But unfortunately our method does not apply for this case, because our arguments depend heavily on the transformation (3.5). We will investigate this problem further in our future work.

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