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Three-Way Granular Approximations Based on Bisimulations

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Abstract In this paper, we propose three-way granular approximations (3WGAs) based on bisimulations. We discover the relationships between 3WGAs based on underlying relations and 3WGAs based on bisimilarity (the largest bisimulation induced by an underlying relation).

Keywords three-way decision; bisimulation; three-way approximation

MR(2020) Subject Classification 03E20; 03E47

1. Introduction

Yao [1, 2] initiated three-way decision theory as a thinking method and a guiding theory. Thinking in three is a key idea of the theory. Three-way decision (3WD) originates from rough set [3], but it has gone beyond rough set now.

3WD has rapidly developed in both theory and application though it is a new theory. For example, Hu [4] gave the axiomatic definition of three-way decision spaces by analyzing the commonness of existing three-way decision. Li et al. [5] proposed 3WD based on subset-evaluation and 3WD matroids. Liu and Liang [6] proposed a new 3WD model in order decision system. Yao [7] proposed a trisecting-acting-outcome (TAO) model of 3WD and discussed the application of 3WD in granular computing. Liang et al. [8] proposed a 3WD model with decision-theoretic rough set under Pythagorean fuzzy information and gave its application. Chen et al. [9] applied three-way decision to diagnosis of focal liver lesions. Zhang and Miao [10] studied three-way attribute reducts.

In [11], Zhu et al. pointed out that rough approximations based on an arbitrary binary relation R are the utility of "one step" (the ordered pair (m, n) are called one step if $(m, n) \in R$) information which may not be adequate to characterize indiscerniblity. Zhu et al. noticed that

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bisimulation [12–14] can reflect "multi-step" information, then they [11] proposed a method of approximating unknown concepts by bisimulations which can make up the defect caused by "one step" information. As continuations of Zhu et al.'s work in [11], Du and Zhu [15, 16] discussed fuzzy lower and upper approximations of fuzzy relational structures and labeled fuzzy lower and upper approximations of fuzzy sets by using bisimulations.

It is an interesting topic that one constructs rough set model from the view point of granule. In general, rough set model based on granule is not equivalent to corresponding rough set model based on element. Many researchers have done many efforts for the study of rough set model based on granule [17–19]. Existing granular (variable precision) rough set models based on arbitrary binary relation only depend on "one step" information of relation. Since bisimulation reflects "multi-step" information, in this paper, by the idea of 3WD, we attempt to study 3WRSAs based on bisimulations from the angle of granule.

This paper is structured as follows. In the next section, some preliminaries are reviewed. In Section 3, we introduce 3WGAs based on bisimulations and explore the relationships between the 3WGAs based on underlying relation and 3WGAs based on bisimilarity. In the end, we summarize the paper and give a few prospects for future research.

2. Preliminaries

We review some notions and results.

2.1. Pawlak rough set and three-way decision

(OB, R) denotes an approximation space (AS), where OB is a finite nonempty set of objects and R is an equivalence relation on OB. $\forall m \in OB$, $[m]_R = \{n \in OB | (m, n) \in R\}$ represents the equivalence class of m. We review two kinds of definitions of Pawlak rough set [3]: element based definition and granule based definition.

- Element based definition: $\forall D \subseteq OB$, the lower approximation $\underline{R}^E(D)$ and upper approximation $\overline{R}^E(D)$ of D are defined as

$$\underline{R}^E(D) = \{m \in OB | [m]_R \subseteq D\};\quad \overline{R}^E(D) = \{m \in OB | [m]_R \cap D \neq \emptyset\}.$$

- Granule based definition: $\forall D \subseteq OB$, the lower approximation $\underline{R}^G(D)$ and upper approximation $\overline{R}^G(D)$ are defined as

$$\underline{R}^G(D) = \cup\{[m]_R | m \in OB, [m]_R \subseteq D\};\quad \overline{R}^G(D) = \cup\{[m]_R | m \in OB, [m]_R \cap D \neq \emptyset\}.$$

As we know, the two kinds of definitions above are equivalent. But, the result is not right when R is not an equivalence relation.

3WD was initiated by Yao [1, 2]. The idea of 3WD is to divide one universe into three pairwise disjoint regions according to certain criterion, then take corresponding action strategy w.r.t. different region.

By lower and upper approximations of Pawlak rough set, we can obtain three regions:

$$\mathrm{Pos}(D) = \underline{R}^E(D) = \underline{R}^G(D),\quad \mathrm{Neg}(D) = (\overline{R}^E(D))^c = (\overline{R}^G(D))^c,$$

$$\mathrm{Bou}(D) = (\mathrm{Pos}(D) \cup \mathrm{Neg}(D))^c,$$

$\mathrm{Pos}(D)$, $\mathrm{Neg}(D)$, $\mathrm{Bou}(D)$ are called positive region, negative region, and boudary region of D, respectively.

Conversely, by giving three pair-wise disjoint regions: positive region $\mathrm{Pos}(D)$, negative region $\mathrm{Neg}(D)$, and boudary region $\mathrm{Bou}(D)$ of D, we can also get lower and upper approximations of D:

$$\underline{R}(D) = \mathrm{Pos}(D), \quad \overline{R}(D) = (\mathrm{Neg}(D))^c = \mathrm{Pos}(D) \cup \mathrm{Bou}(D).$$

Many theoretic and application-oriented literatures on 3WD published in influential international journals have explored the strong strength of 3WD (see [20, 21]).

2.2. Bisimulations

Bisimulations take on different forms over different structures [22–25]. This subsection reviews the concept of bisimulations.

Let (OB, R) be a generalized approximation space (GAS), where R is an arbitrary binary relation on OB. $\forall m \in OB$, $R(m) = \{n \in OB | (m, n) \in R\}$ denotes the successor neighborhood of m. A relation R is called serial, if $\forall m \in OB$, $\exists n \in OB$, s.t. $(m, n) \in R$. A GAS (OB, R) can be regarded as a discrete event system, where the objects in OB are viewed as states, the steps (m, n) (belonging to R) as moves, and R as an underlying relation. We refer to [11–14, 26] for details.

Based on a GAS, Zhu et al. [11] presented bisimulations as follows.

Definition 2.1 ([11]) *Let (OB, R) be a GAS. A binary relation $B \subseteq OB \times OB$ is called a bisimulation if $\forall (m, n) \in B$,*

(1) $(m, m') \in R \Longrightarrow (n, n') \in R$ for some $n' \in OB$ satisfying $(m', n') \in B$;

(2) $(n, n') \in R \Longrightarrow (m, m') \in R$ for some $m' \in OB$ satisfying $(m', n') \in B$.

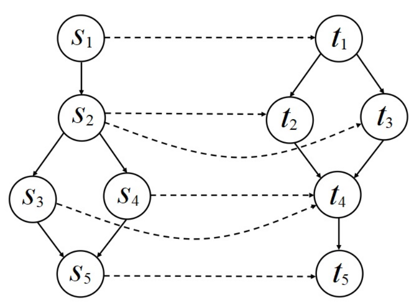


Figure 1 The GAS (OB_1, R_1)

The identity relation $Id_{OB} = \{(o, o) \mid o \in OB\}$ is a bisimulation [11]. For a GAS (OB, R), its all bisimulations' union makes up a largest bisimulation, called bisimilarity induced by R, denoted as $\sim_R$, which is an equivalence relation [11, 12]. $\forall o_1, o_2 \in OB$, if $o_1 \sim_R o_2$, then they are said bisimilar.

Next, we provide an example which comes from [11, 27].

Example 2.2 ([11,27]) Suppose $OB_1 = \{s_1, s_2, s_3, s_4, s_5, t_1, t_2, t_3, t_4, t_5\}$ and R_1 is displayed by solid arrows (see Figure 1). Then (OB_1, R_1) is a GAS. We can verify $B_1 = \{(s_1, t_1), (s_2, t_2), (s_4, t_4), (s_5, t_5), (s_2, t_3), (s_3, t_4)\}$ (described by dotted arrows in Figure 1) and $B_2 = \{(s_4, t_4), (s_5, t_5)\}$ are two bisimulations.

Furthermore, $OB/\sim_{R_1} = \{\{s_1, t_1\}, \{s_5, t_5\}, \{s_2, t_2, t_3\}, \{s_3, s_4, t_4\}\}$.

3. 3WGAs based on bisimulations

We will raise 3WGAs based on bisimulations, and expatiate the relationships between the 3WGAs based on underlying relation and 3WGAs based on bisimilarity.

3.1. The notion of 3WGAs based on bisimulations

We fist give the notion of 3WGAs based on bisimulations as follows.

Definition 3.1 *Let (OB, R) be a GAS and $B \subseteq OB \times OB$ be a bisimulation. $\forall D \subseteq OB$, the positive region $\mathrm{Pos}_B^G(D)$, negative region $\mathrm{Neg}_B^G(D)$ and boundary region $\mathrm{Bou}_B^G(D)$ of D w.r.t. B are defined as:*

$$\mathrm{Pos}_B^G(D) = \cup\{B(m) | m \in OB, B(m) \subseteq D\};$$

$$\mathrm{Neg}_B^G(D) = OB - \cup\{B(m) | m \in OB, B(m) \cap D \neq \emptyset\};$$

$$\mathrm{Bou}_B^G(D) = \cup\{B(m) | m \in OB, B(m) \cap D \neq \emptyset\} - \cup\{B(m) | m \in OB, B(m) \subseteq D\};$$

$\mathrm{Pos}_B^G(D)$, $\mathrm{Neg}_B^G(D)$, and $\mathrm{Bou}_B^G(D)$ are called 3WGAs based on bisimulations of D.

Apparently, $\mathrm{Pos}_B^G(D) \subseteq (\mathrm{Neg}_B^G(D))^c$ and $\mathrm{Bou}_B^G(D) = (\mathrm{Pos}_B^G(D) \cup \mathrm{Neg}_B^G(D))^c$. For the three regions, any two are disjoint and the three regions' union is OB.

By using above three regions, we can get the granular lower approximation $\underline{\mathrm{App}}_B^G(D)$ and upper approximation $\overline{\mathrm{App}}_B^G(D)$ based on bisimulations of D:

$$\underline{\mathrm{App}}_B^G(D) = \mathrm{Pos}_B^G(D); \quad \overline{\mathrm{App}}_B^G(D) = (\mathrm{Neg}_B^G(D))^c = \mathrm{Pos}_B^G(D) \cup \mathrm{Bou}_B^G(D).$$

Remark 3.2 In general, a bisimulation may not be an equivalence relation, so the granular rough approximations above are not equivalent to rough approximations in [11]. Of course, they are equivalent if $B = \sim_R$.

We give an example on 3WGAs.

Example 3.3 For the GAS (OB_1, R_1) in Example 2.2, $D = \{s_3, s_5, t_5\}$.

(1) Take $B = \{(s_1, t_1), (s_2, t_2), (s_4, t_4), (s_5, t_5), (s_2, t_3), (s_3, t_4)\}$. We have $\mathrm{Pos}_B^G(D) = \{t_5\}$, $\mathrm{Neg}_B^G(D) = \{s_1, s_2, s_3, s_4, s_5, t_1, t_2, t_3, t_4\}$, $\mathrm{Bou}_B^G(D) = \emptyset$.

(2) Take $B = \sim_{R_1}$. We have $\mathrm{Pos}_{\sim_{R_1}}^G(D) = \{s_5, t_5\}$, $\mathrm{Neg}_{\sim_{R_1}}^G(D) = \{s_1, s_2, t_1, t_2, t_3\}$, $\mathrm{Bou}_{\sim_{R_1}}^G(D) = \{s_3, s_4, t_4\}$.

Proposition 3.4 *Let (OB, R) be a GAS. Then $\forall D \subseteq OB$, $\mathrm{Pos}_{Id_{OB}}^G(D) \cup \mathrm{Neg}_{Id_{OB}}^G(D) = OB$.*

Proof $\forall m \in OB$, $Id_{OB}(m) = \{m\}$. Therefore,

$$\mathrm{Pos}_{Id_{OB}}^G(D) = \cup\{Id_{OB}(m) | m \in OB, Id_{OB}(m) \subseteq D\}$$

$$= \cup\{\{m\} \mid m \in OB, \{m\} \subseteq D\} = D,$$

and

$$\begin{aligned}\mathrm{Neg}^G_{Id_{OB}}(D) =& OB - \cup\{Id_{OB}(m) | m \in OB, Id_{OB}(m) \cap D \neq \emptyset\}\\ =& OB - \cup\{\{m\} | m \in OB, \{m\} \cap D \neq \emptyset\} = OB - D.\end{aligned}$$

So $\mathrm{Pos}^G_{Id_{OB}}(D) \cup \mathrm{Neg}^G_{Id_{OB}}(D) = OB$. □

3.2. Relationships between two kinds of 3WGAs

In [11], Zhu et al. discussed the relationships between rough approximations based on underlying relation and rough approximations based on bisimilarity. In Definition 3.1, by replacing B with R, we can obtain 3WGAs based on underlying relation R. We will reveal relationships between two kinds of 3WGAs–based on R and based on $\sim_R$.

Example 3.5 Let (OB_1, R_1) be the GAS in Example 2.2. (OB_2, R_2) is the GAS of Example 4.1 in [11], where $OB_2 = \{M_1, M_2, B_1, B_2, F_1, F_2, F_3, F_4, C_1, \ldots, C_k\}$ ("M_i" stands for "MSA_i" ($i = 1, 2$), "B_i" stands for "BSA_i" ($i = 1, 2$), "F_i" stands for "FA_i" ($i = 1, 2, 3, 4$)), R_2 is described by solid arrows (see Figure 2).

$$OB_2/ \sim_{R_2} = \{\{M_1, M_2\}, \{B_1, B_2\}, \quad \{F_1, F_2, C_1, C_2\}, \{F_3, F_4\}, \{C_3, \ldots, C_k\}\}.$$

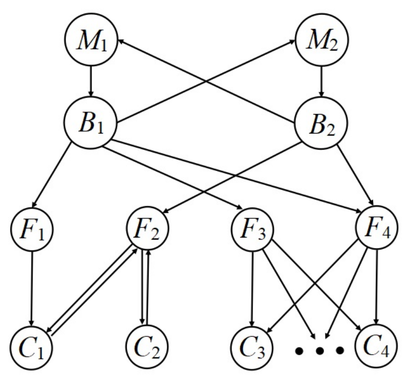


Figure 2 The GAS (OB_2, R_2)

(1) Suppose $D_1 = \{s_1, t_1, s_5\}$, then we have

$$\mathrm{Pos}^G_{R_1}(D_1) = \{s_5\}, \quad \mathrm{Pos}^G_{\sim_{R_1}}(D_1) = \{s_1, t_1\},$$

which means that $\mathrm{Pos}^G_{R_1}(D_1)$ and $\mathrm{Pos}^G_{\sim_{R_1}}(D_1)$ have no inclusion relation.

(2) Suppose $D_2 = \{s_1, s_2, t_1, t_2, t_3\}$, then we have

$$\mathrm{Pos}^G_{R_1}(D_2) = \{s_2, t_2, t_3\}, \quad \mathrm{Pos}^G_{\sim_{R_1}}(D_2) = \{s_1, s_2, t_1, t_2, t_3\},$$

which means that $\mathrm{Pos}^G_{R_1}(D_2) \subset \mathrm{Pos}^G_{\sim_{R_1}}(D_2)$, where "$a \subset b$" means $a \subseteq b$ but $a \neq b$.

(3) Suppose $D_3 = \{s_2, t_2, t_3, t_4\}$, then we have

$$\mathrm{Pos}^G_{R_1}(D_3) = \{s_2, t_2, t_3, t_4\}, \quad \mathrm{Pos}^G_{\sim_{R_1}}(D_3) = \{s_2, t_2, t_3\},$$

which means that $\mathrm{Pos}^G_{\sim_{R_1}}(D_3) \subset \mathrm{Pos}^G_{R_1}(D_3)$.

(4) Suppose $D_4 = \{s_3, s_5, t_5\}$, then we have

$$\mathrm{Neg}^G_{R_1}(D_4) = \{s_1, s_2, t_1, t_2, t_3, t_4\}, \quad \mathrm{Neg}^G_{\sim_{R_1}}(D_4) = \{s_1, s_2, t_1, t_2, t_3\},$$

which implies that $\mathrm{Neg}^G_{\sim_{R_1}}(D_4) \subset \mathrm{Neg}^G_{R_1}(D_4)$.

$$\mathrm{Bou}^G_{R_1}(D_4) = \{s_3, s_4\}, \quad \mathrm{Bou}^G_{\sim_{R_1}}(D_4) = \{s_3, s_4, t_4\},$$

which implies that $\mathrm{Bou}^G_{R_1}(D_4) \subset \mathrm{Bou}^G_{\sim_{R_1}}(D_4)$.

(5) Suppose $D_5 = \{F_1, F_2, C_1, C_2\}$, then we have

$$\mathrm{Neg}^G_{R_2}(D_5) = \{B_1, B_2, C_3, C_4, \dots, C_k\},$$

$$\mathrm{Neg}^G_{\sim_{R_2}}(D_5) = \{F_3, F_4, B_1, B_2, M_2, M_1, C_3, C_4, \dots, C_k\}.$$

Then $\mathrm{Neg}^G_{R_2}(D_5) \subset \mathrm{Neg}^G_{\sim_{R_2}}(D_5)$.

$$\mathrm{Bou}^G_{R_2}(D_5) = \{M_1, M_2, F_1, F_3, F_4\}, \quad \mathrm{Bou}^G_{\sim_{R_2}}(D_5) = \emptyset.$$

Then $\mathrm{Bou}^G_{\sim_{R_2}}(D_5) \subset \mathrm{Bou}^G_{R_2}(D_5)$.

(6) Suppose $D_6 = \{M_2, B_1, F_3\}$, then we have

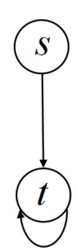
$$\mathrm{Neg}^G_{R_2}(D_6) = \{M_1, B_2, F_2, C_1, \dots, C_k\}, \quad \mathrm{Neg}^G_{\sim_{R_2}}(D_6) = \{F_1, F_2, C_1, \dots, C_k\}.$$

$$\mathrm{Bou}^G_{R_2}(D_6) = \{M_1, F_1, F_3, F_4\}, \quad \mathrm{Bou}^G_{\sim_{R_2}}(D_6) = \{M_1, M_2, B_1, B_2, F_3, F_4\}.$$

Therefore, $\mathrm{Neg}^G_{R_2}(D_6)$ and $\mathrm{Neg}^G_{\sim_{R_2}}(D_6)$ have no inclusion relation as well as $\mathrm{Bou}^G_{R_2}(D_6)$ and $\mathrm{Bou}^G_{\sim_{R_2}}(D_6)$.

If does there exist a set D such that $\mathrm{Pos}^G_R(D) = \mathrm{Pos}^G_{\sim_R}(D)$, $\mathrm{Neg}^G_R(D) = \mathrm{Neg}^G_{\sim_R}(D)$ or $\mathrm{Bou}^G_R(D) = \mathrm{Bou}^G_{\sim_R}(D)$? The answer is positive.

Example 3.6 For the GAS (OB_3, R_3), where $OB_3 = \{s, t\}$ and $R_3 = \{(s,t), (t,t)\}$ (see Figure 3). Obviously, $\sim_{R_3} = Id_{OB_3}$. Take $D = \{t\}$, then $\mathrm{Pos}^G_{R_3}(D) = \mathrm{Pos}^G_{\sim_{R_3}}(D) = D$.



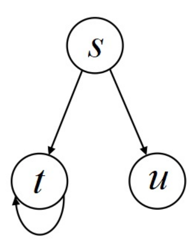


Figure 3 The GAS $(OB_3,\ R_3)$ Figure 4 The GAS $(OB_4,\ R_4)$

Example 3.7 Consider the GAS (OB_4, R_4) in [11, Example 5.1 (9)], where $OB_4 = \{t, s, u\}$, $R_4 = \{(t,t), (s,u), (s,t)\}$ (see Figure 4) and $\sim_{R_4} = Id_{OB_4}$. For $D = \{t, u\}$, $\mathrm{Neg}^G_{R_4}(D) = \mathrm{Neg}^G_{\sim_{R_4}}(D) = \{s\}$.

Example 3.8 (Continuation of Example 3.7) For $D = \{t, u\}$, $\mathrm{Bou}^G_{R_4}(D) = \mathrm{Bou}^G_{\sim_{R_4}}(D) = \emptyset$.

From Examples 3.5–3.8, we could see that there is no desired relationships between the two kinds of 3WGAs. To excavate relationships between the two kinds of 3WGAs for special underlying relations, we recall the following concepts.

Definition 3.9 ([11, 12]) *Let (OB, R) be a GAS. $\forall o \in OB$, o is steady if there does not exist o' such that $(o, o') \in R$. o is unsteady if there does not exist a path $o_1 o_2 \cdots o_i$ ending up with a steady state o_i, where $o_1 = o$ and $(o_j, o_{j+1}) \in R$ $(j = 1, \ldots, i-1)$.*

Based on the concepts of bisimulations and unsteady states, Zhu et al. [11] obtained the following result.

Lemma 3.10 ([11]) *Let (OB, R) be a GAS and $T = \{(o_1, o_2) | o_1, o_2$ are unsteady$\}$. Then T is a bisimulation.*

By Lemma 3.10, all unsteady states are bisimilar [11] and we have the following proposition.

Proposition 3.11 *Let (OB, R) be a GAS. If R is reflexive, then $\forall D \subseteq OB$, $\mathrm{Pos}^G_{\sim_R}(D) \subseteq \mathrm{Pos}^G_R(D)$.*

Proof If R is reflexive, then all states in OB are unsteady. Therefore, they are bisimilar, which means that $\sim_R = OB \times OB$. Then $[m]_{\sim_R} = OB$ $(\forall m \in OB)$. According to Definition 3.1, $\forall D \subseteq OB$,

$$\mathrm{Pos}^G_{\sim_R}(D) = \cup\{[m]_{\sim_R} | m \in OB, [m]_{\sim_R} \subseteq D\} = \begin{cases} OB, & \text{if } D = OB \\ \emptyset, & \text{otherwise} \end{cases}$$

As R is reflexive, $\forall m \in OB$, $m \in R(m)$. Then

$$\mathrm{Pos}^G_R(OB) = \cup\{R(m) | m \in OB, R(m) \subseteq OB\} = OB.$$

Hence, $\mathrm{Pos}^G_{\sim_R}(D) \subseteq \mathrm{Pos}^G_R(D)$. □

Proposition 3.12 *Let (OB, R) be a GAS. If R is symmetric and transitive, then $\forall D \subseteq OB$, $\mathrm{Neg}^G_{\sim_R}(D) \subseteq \mathrm{Neg}^G_R(D)$.*

Proof $\forall m \notin \mathrm{Neg}^G_R(D)$, then $\exists n \in OB$, s.t. $m \in R(n)$ and $R(n) \cap D \neq \emptyset$. As R is symmetric, $n \in R(m)$. By the transitivity of R, $R(n) \subseteq R(m)$. Then $R(m) \cap D \neq \emptyset$. Suppose $z \in R(m) \cap D$. As R is symmetric, $(m, z) \in R$ and $(z, m) \in R$. By Lemma 3.10, $m \sim_R z$. Then $z \in [m]_{\sim_R} \cap D \neq \emptyset$. Therefore, $m \in [m]_{\sim_R} \subseteq OB - \mathrm{Neg}^G_{\sim_R}(D)$. Then $m \notin \mathrm{Neg}^G_{\sim_R}(D)$. So $\mathrm{Neg}^G_{\sim_R}(D) \subseteq \mathrm{Neg}^G_R(D)$. □

Corollary 3.13 *Let (OB, R) be an AS. Then $\forall D \subseteq OB$, $\mathrm{Pos}^G_{\sim_R}(D) \subseteq \mathrm{Pos}^G_R(D)$, $\mathrm{Neg}^G_{\sim_R}(D) \subseteq \mathrm{Neg}^G_R(D)$, and $\mathrm{Bou}^G_R(D) \subseteq \mathrm{Bou}^G_{\sim_R}(D)$.*

Proof It follows from Propositions 3.11 and 3.12. □

Proposition 3.14 *Let (OB, R) be a GAS. If R is serial, then $\forall D \subseteq OB$, $\mathrm{Pos}^G_R(D) = \mathrm{Pos}^G_{\sim_R}(D)$, $\mathrm{Neg}^G_R(D) = \mathrm{Neg}^G_{\sim_R}(D)$ and $\mathrm{Bou}^G_R(D) = \mathrm{Bou}^G_{\sim_R}(D)$ hold iff $R = OB \times OB$.*

Proof " $\Longleftarrow$ ". If $R = OB \times OB$, then all the states are bisimilar, which implies $\sim_R = OB \times OB$.

Thus $\mathrm{Pos}_R^G(D)=\mathrm{Pos}_{\sim_R}^G(D)$, $\mathrm{Neg}_R^G(D)=\mathrm{Neg}_{\sim_R}^G(D)$ and $\mathrm{Bou}_R^G(D)=\mathrm{Bou}_{\sim_R}^G(D)$ $(\forall D\subseteq OB)$.

" $\Longrightarrow$ ". Suppose $\forall D\subseteq OB$, $\mathrm{Pos}_R^G(D)=\mathrm{Pos}_{\sim_R}^G(D)$, $\mathrm{Neg}_R^G(D)=\mathrm{Neg}_{\sim_R}^G(D)$ and $\mathrm{Bou}_R^G(D)=\mathrm{Bou}_{\sim_R}^G(D)$. Since R is serial, then all states are unsteady. By Lemma 3.10, $\sim_R= OB\times OB$.

(1) If OB is a single point set $\{o\}$, then $R=\{(o,o)\}=OB\times OB$.

(2) If OB is not a single point set. Assume that $R\neq OB\times OB$. Then there exists $(p,q)\notin R$. Take $Y=OB-\{q\}$, then $\emptyset\neq Y\subset OB$. Note that $\sim_R= OB\times OB$, then

$$\mathrm{Pos}_R^G(Y)=\mathrm{Pos}_{\sim_R}^G(Y)=\emptyset.$$

Since R is serial, $R(m)\neq\emptyset$ $(\forall m\in OB)$. Then $\forall m\in OB, R(m)\nsubseteq Y$ by $\mathrm{Pos}_R^G(Y)=\emptyset$. Especially, $R(p)\nsubseteq Y=OB-\{q\}$. It means that $q\in R(p)$, i.e., $(p,q)\in R$, which is a contradiction. So $R=OB\times OB$. $\square$

Corollary 3.15 *Let (OB,R) be an AS. Then $\forall D\subseteq OB$, $\mathrm{Pos}_R^G(D)=\mathrm{Pos}_{\sim_R}^G(D)$, $\mathrm{Neg}_R^G(D)=\mathrm{Neg}_{\sim_R}^G(D)$ and $\mathrm{Bou}_R^G(D)=\mathrm{Bou}_{\sim_R}^G(D)$ hold iff $R=OB\times OB$.*

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