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Minimum k-Path Vertex Cover in Cartesian Product Graphs

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Abstract For the subset $S \subseteq V(G)$, if every path with k vertices in a graph G contains at least one vertex from S, we call that S is a k-path vertex cover set of the graph G. Obviously, the subset is not unique. The cardinality of the minimum k-path vertex cover set of a graph G is called the k-path vertex cover number, we denote it by $\psi_k(G)$. In this paper, a lower or upper bound of ψ_k for some Cartesian product graphs is presented.

Keywords k-path vertex cover; Cartesian product graphs; bound

MR(2020) Subject Classification 05C69; 05C70; 05C76

1. Introduction

All graphs we consider in this paper are finite, undirected, and simple. For a graph $G = (V, E)$, the vertex set and edge set of G are denoted by $V = V(G)$ and $E = E(G)$, respectively. For a subset $S \subset V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A path with k vertices is called a k-path. For a graph G, the k-path vertex cover problem in G is to find a subset $S \subset V$ (which is called a k-path vertex cover set) such that the removal of S from G results in a graph that does not contain any k-path, i.e., any path in $G[V\backslash S]$ only contains less than k vertices. The k-path vertex cover number of a graph G, denoted by $\psi_k(G)$, is the cardinality of a minimum k-path vertex cover set of G.

The motivation for this research is from the following two aspects. One is the communication of wireless sensor network, by K-generalized Canvas Scheme, and we use it to integrate the data in [1]; the other is the intelligent traffic control, such as installing multiple cameras problem in the crossing roads [2].

For the k-path vertex cover problem, we ignore the problem for $k = 1$ because $\psi_1(G) = |V(G)|$. For $k = 2$, it is easy to see that $\psi_2(G)$ is related to maximum independent set problem, i.e.,

$$\psi_2(G) = |V(G)| - \alpha(G),$$

where $\alpha(G)$ denotes the maximum independent number of G. There are many related studies about independent number in [3]. In particular, we calculate $\psi_2(G)$ by finding its maximum

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matching when a graph is a bipartite graph. For $k = 3$, there is also a known concept of dissociation number related to $\psi_3(G)$ in [4–6]. The so-called dissociation number is the maximum cardinality of an induced subgraph of G whose maximum degree is less than or equal to 1, and we denote it by $\mathrm{diss}(G)$. Hence,

$$\psi_3(G) = |V(G)| - \mathrm{diss}(G).$$

At present, many researchers studied the value $\psi_3(G)$ in various ways [2, 7–9]. Tsur [10] gave a parameterized algorithm with time complexity $O^*(1.713^k)$. In [11], the authors presented an efficient polynomial time approximation scheme for the 3-path vertex cover problem (VCP_3) on planar graphs. In 2011, Tu et al. gave a 2-approximation algorithm, and proposed a primal-dual approximation algorithm [2]; In 2013, they mainly studied cubic graphs and gave a 1.57-approximation algorithm for the VCP_3 problem. Later, Zhang et al. [7] improved Tu's upper bound and obtained a 1.5-approximation algorithm.

For $k \geq 3$, Brešar et al. [12] gave some efficient algorithms for trees, outerplanar graphs, and obtained a general result of connected graphs. Brešar et al. [13] also presented $\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V(G)|$ for $d \geq k-1$, where G is a d-regular graph. Later, Liu et al. [14] added connectivity to the k-path vertex cover problem and presented a polynomial time approximation scheme of unit disk graphs. In 2014, the authors added a new factor, weight, and studied weighted versions of minimum k-path vertex cover problem for trees, cycles and complete graphs in [15]. Then, Li et al. [16] gave an approximation algorithm for minimum (weighted) connected k-path vertex cover problem.

Recently, the k-path vertex cover problem was studied in many classes of graph products, such as [13], and [17–21], respectively. For the lexicographic product of arbitrary two graphs, Tu and Zhou [2] gave an approximation result of $\psi_k(G)$. For the Cartesian product of two paths, Brešar et al. [13] gave an exact value of $\psi_3(G)$. Li et al. [18] studied Cartesian product of path and cycle, wheel, complete bipartite graph, respectively. Then, Jakovac [19] obtained a lower and an upper bound of the k-path vertex cover number of rooted product graphs.

In this paper, we mainly make a further research based on [18]. In the next section, we show some known related results. In Section 3, we obtain an exact value of $\psi_3(P_m \square C_n)$ for positive m and n, provide a better way for calculating $\psi_k(P_m \square C_n)$, correct $\psi_k(P_m \square W_{n+1})$'s upper bound, and generalize the result in [18, Theorem 2.16] to m dimension.

2. Preliminary knowledge

A cycle of length n is denoted by C_n. A wheel W_{n+1} is obtained from C_n by adding a new vertex, say v_0, such that v_0 is connected to every vertex of C_n. A bipartite graph is a graph whose vertex set can be divided into two disjoint subsets called bipartition, denoted by (X, Y), so that the two ends of every edge are contained in different subsets. A complete bipartite graph with the bipartition (X, Y), denoted by $K_{m,n}$, is a bipartite graph such that $|X| = m$, $|Y| = n$, and every vertex of X is connected to every vertex of Y.

For two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the Cartesian product $G \square H$ has vertex set $V(G)\times V(H)$, and vertices (u,v),(x,y) are adjacent whenever $u=x$ and $vy\in E(H)$ or $ux\in E(G)$ and $v=y$ (see [17]).

Some known results are shown as follows.

Lemma 2.1 ([12]) *If a graph G does not contain isolated vertices, then*

$$\psi_k(G)\le |V(G)|-\frac{k-1}{k}\sum_{u\in V(G)}\frac{2}{1+d(u)}.$$

Lemma 2.2 ([13]) *For positive integers $k\ge 2$ and $n\ge k$, the following results hold:*

$$\psi_k(C_n)=\lceil\frac{n}{k}\rceil,\quad \psi_k(P_n)=\lfloor\frac{n}{k}\rfloor.$$

Lemma 2.3 ([13]) *For positive integers $n,k\ge 1$, we have*

$$\psi_3(P_{2n+1}\Box P_{2k})=2nk+\lfloor\frac{2k}{3}\rfloor,$$
$$\psi_3(P_{2n}\Box P_{2k})=2nk,$$
$$\psi_3(P_{2n+1}\Box P_{2k+1})=n(2k+1)+\lfloor\frac{2k+1}{3}\rfloor\ (1\le n\le k).$$

Lemma 2.4 ([18]) *If $m\ge 4$, $m+2\le k<2m$, then $\psi_k(K_{m,m})=m+1-\lfloor\frac{k}{2}\rfloor$.*

Lemma 2.5 ([18]) *If $k\ge 2$ and G' is a subgraph of G, then $\psi_k(G)\ge\psi_k(G')$.*

Lemma 2.6 ([18]) *If $2\le k<n$, then $\psi_k(W_{n+1})=\lceil\frac{n}{k}\rceil+1$.*

3. Main results

Let us first introduce the following notions. For any vertex $u\in V(G)$, the set $u\times V(H)$ is called H-layer of $G\Box H$, denoted by ${}^uH$. Analogously, for any vertex $v\in V(H)$, the set $v\times V(G)$ is called G-layer of $G\Box H$, denoted by G^v. Obviously, in Cartesian product graphs, every H-layer(G-layer) is isomorphic to $H(G)$.

Theorem 3.1 *For positive integer $1\le n\le k$, we have*

(i) $\psi_3(P_{2n+1}\Box C_{3k})=3nk+k$, $\psi_3(P_{2n}\Box C_{3k})=3nk$.

(ii) $\psi_3(P_{2n+1}\Box C_{3k+1})=n(3k+1)+k+1$, $\psi_3(P_{2n}\Box C_{3k+1})=n(3k+1)$.

(iii) $\psi_3(P_{2n+1}\Box C_{3k+2})=n(3k+2)+k+1$, $\psi_3(P_{2n}\Box C_{3k+2})=n(3k+2)$.

Proof Use $u_1,u_2,\ldots,u_m$ to mark the vertices of P_m, and $v_1,v_2,\ldots,v_n$ to mark the vertices of C_n, respectively. With the mark, the vertex $(u_i,v_j)\in V(P_m\Box C_n)$ is both on the i-th ${}^{u_i}C_n$-layer and the j-th ${}^{v_j}P_m$-layer $(1\le i\le m, 1\le j\le n)$.

(i) Let

$$S=\{(u_{2i-1},v_{3j}),(u_{2i},v_{3j-2}),(u_{2i},v_{3j-1})|\text{ for applicable indices } i \text{ and } j\}.$$

The set S is a 3-path vertex cover of $P_{2n+1}\Box C_{3k}$ since every 3-path contains at least one vertex from S (see Figure 1). Then $\psi_3(P_{2n+1}\Box C_{3k})\le 3nk+k$.

Due to $P_{2n+1}\Box P_{3k} \subset P_{2n+1}\Box C_{3k}$, we can obtain that $\psi_3(P_{2n+1}\Box C_{3k}) \geq \psi_3(P_{2n+1}\Box P_{3k})$ by Lemma 2.5. And $\psi_3(P_{2n+1}\Box P_{3k}) = 3nk + k$ in Lemma 2.3, then $\psi_3(P_{2n+1}\Box C_{3k}) \geq 3nk + k$. Hence, $\psi_3(P_{2n+1}\Box C_{3k}) = 3nk + k$.

Similarly, we follow the same way as shown above, then $\psi_3(P_{2n}\Box C_{3k}) = 3nk$.

(ii) According to (i), we can get the 3-path vertex cover set S of $P_{2n+1}\Box C_{3k+1}$ by adding vertices (u_{2i-1}, v_1) and deleting vertices (u_{2i}, v_1) (for suitable i) as follows

$$S = \{(u_{2i-1}, v_1), (u_{2i-1}, v_{3j}), (u_{2i}, v_2), (u_{2i}, v_{3j+1}), (u_{2i}, v_{3j+2})| \text{ for applicable indices } i \text{ and } j\}.$$

$$|S| = n(3k+1) + \lfloor\frac{3k+1}{3}\rfloor + 1 = n(3k+1) + k + 1.$$

Obviously, the set S is a 3-path vertex cover of $P_{2n+1}\Box C_{3k+1}$ since every 3-path contains at least one vertex from S (see Figure 2). Then

$$\psi_3(P_{2n+1}\Box C_{3k+1}) \leq n(3k+1) + k + 1.$$

Due to $P_{2n+1}\Box P_{3k+1} \subset P_{2n+1}\Box C_{3k+1}$, then

$$\psi_3(P_{2n+1}\Box C_{3k+1}) \geq \psi_3(P_{2n+1}\Box P_{3k+1}) = n(3k+1) + k$$

by Lemma 2.3. Assume that it has a 3-path vertex cover T', $|T'| \leq n(3k+1) + k$, we can always find a path with order greater than or equal to 3, a contradiction, then

$$\psi_3(P_{2n+1}\Box C_{3k+1}) \geq n(3k+1) + k + 1.$$

Therefore,

$$\psi_3(P_{2n+1}\Box C_{3k+1}) = n(3k+1) + k + 1.$$

Similarly, we follow the same way as shown above, then $\psi_3(P_{2n}\Box C_{3k+1}) \leq n(3k+1)$. By Lemmas 2.3 and 2.5, we have

$$\psi_3(P_{2n}\Box C_{3k+1}) \geq \psi_3(P_{2n}\Box P_{3k+1}) = n(3k+1).$$

Therefore,

$$\psi_3(P_{2n}\Box C_{3k+1}) = n(3k+1).$$

(iii) The proof process is as (ii). □

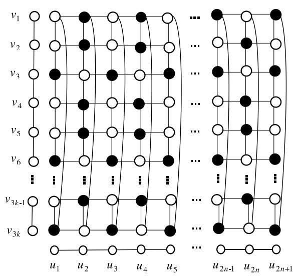

Figure 1 3-path vertex cover of $P_{2n+1}\Box C_{3k}$

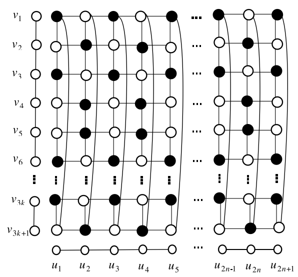

Figure 2 3-path vertex cover of $P_{2n+1}\Box C_{3k+1}$

Theorem 3.2 *For integer $k \geq 3$, let (a, b) be the middle D_{k-1} pair. We have*

$$\psi_k(P_m\Box C_n) \leq \min\{(\lfloor\frac{n-1}{a+1}\rfloor + 1)m + \lfloor\frac{m}{b+1}\rfloor(n-1) - 2\lfloor\frac{n-1}{a+1}\rfloor\lfloor\frac{m}{b+1}\rfloor,$$

$$(\lfloor\frac{n-1}{b+1}\rfloor+1)m+\lfloor\frac{m}{a+1}\rfloor(n-1)-2\lfloor\frac{n-1}{b+1}\rfloor\lfloor\frac{m}{a+1}\rfloor\}.$$

For the notion of D_i pair, the introduction is as follows. The D_i denotes the set of all divisors of i. Let $a \le b$, where a is the largest factor in D_i less than or equal to $\sqrt{i}$, and b is the smallest factor in D_i greater than or equal to $\sqrt{i}$. The pair (a, b) is called the middle D_i pair [17]. Generally, we can see that $a \cdot b = i$.

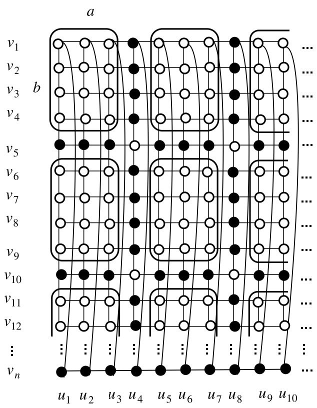

Figure 3 k-path vertex cover of $P_m \Box C_n$

Proof Use $u_1, u_2, \dots, u_m$ to mark the vertices of P_m, and $v_1, v_2, \dots, v_n$ to mark the vertices of C_n, respectively.

Let

$$S_1 = \{(u_i, v_j) \in P_m \Box C_n | j \in [1, n-1] \text{ and } i \equiv 0 \pmod{a+1}\},$$

$$S_2 = \{(u_i, v_j) \in P_m \Box C_n | i \in [1, m] \text{ and } j \equiv 0 \pmod{b+1}\}.$$

The set $S = (S_1 \bigcup S_2 \bigcup (u_i, v_n)) \backslash (S_1 \bigcap S_2)$ $(i = 1, 2, \dots, m)$ is a k-path vertex cover of $P_m \Box C_n$, because the remaining largest connected subgraph of $P_m \Box C_n$ is isomorphic to $P_a \Box P_b$. What's more, $P_a \Box P_b$ contains the longest path with $a \cdot b \le k-1$ vertices (see Figure 3).

In every P_m-layer, we choose to cover every $(a+1)$-th vertex. There are $n-1$ layers, then $|S_1| \le \lfloor\frac{m}{a+1}\rfloor(n-1)$. Similarly, we cover every $(b+1)$-th vertex in every C_n-layer, then $|S_2| \le \lfloor\frac{n-1}{b+1}\rfloor m$. The vertices (u_i, v_j) in $S_1 \cap S_2$ cannot be covered, because its neighbor vertices are all in S. $|S_1 \cap S_2| \le \lfloor\frac{m}{a+1}\rfloor\lfloor\frac{n-1}{b+1}\rfloor$ and we count such (u_i, v_j) in the intersection twice. Thus,

$$|S| \le \lfloor\frac{m}{a+1}\rfloor(n-1)+(\lfloor\frac{n-1}{b+1}\rfloor+1)m-2\lfloor\frac{n-1}{b+1}\rfloor\lfloor\frac{m}{a+1}\rfloor.$$

Similarly, we can also construct a k-path vertex cover with

$$\lfloor\frac{m}{b+1}\rfloor(n-1)+(\lfloor\frac{n-1}{a+1}\rfloor+1)m-2\lfloor\frac{n-1}{a+1}\rfloor\lfloor\frac{m}{b+1}\rfloor$$

vertices.

When $a+2b+2 \le m-1$, we have the following formula

$$\lfloor\frac{m}{b+1}\rfloor(n-1)+(\lfloor\frac{n-1}{a+1}\rfloor+1)m-2\lfloor\frac{n-1}{a+1}\rfloor\lfloor\frac{m}{b+1}\rfloor$$

$$\leq \frac{mn}{b+1} + \frac{n+a}{a+1} \cdot m - 2\lfloor \frac{n-1}{a+1} \rfloor \lfloor \frac{m}{b+1} \rfloor$$
$$\leq \frac{n(a+1)+(n+a)(b+1)}{(a+1)(b+1)} \cdot m - 2\lfloor \frac{n-1}{a+1} \rfloor \lfloor \frac{m}{b+1} \rfloor$$
$$< \frac{mn}{k} \cdot (a+2b+2) < \lceil \frac{m^2 n}{k} \rceil.$$

Hence, the upper bound is always better than the result of [18]. □

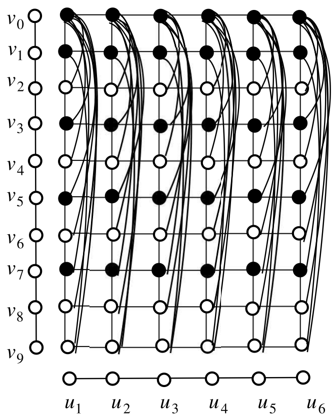

Figure 4 4-path vertex cover of $P_6 \square W_{10}$

In [18], if $2 \leq k \leq \lceil \frac{n+1}{2} \rceil$ and integer $m \geq 2$, then the set $S = \sum_{i=1}^{m} S_i$ is a k-path vertex cover of $P_m \square W_{n+1}$. Moreover

$$S_i = \{(u_i, v_0), (u_i, v_j), (u_i, v'_j) \in V(P_m \square W_{n+1}) | j = 1 + lk, j' = \lceil \frac{k+2}{2} \rceil + lk, 0 \leq l \leq \lceil \frac{n}{k} \rceil - 1\}$$

for odd i, and

$$S_i = \{(u_i, v_0), (u_i, v_j), (u_i, v'_j) \in V(P_m \square W_{n+1}) | j = 1 + lk, j' = \lfloor \frac{k+2}{2} \rfloor + lk, 0 \leq l \leq \lceil \frac{n}{k} \rceil - 1\}$$

for even i, where $i = 1, 2, \ldots, m$, $\psi_k(P_m \square W_{n+1}) \leq m(2\lceil \frac{n}{k} \rceil + 1)$. According to the above method, when $m = 6$, $n = 9$, $\psi_4(P_6 \square W_{10}) \leq 30$, there exists the path with more than four vertices (see Figure 4), a contradiction.

Next, we will research it again.

Theorem 3.3 *For $2 < m < k$, we have $m(1 + \lceil \frac{n}{k} \rceil) \leq \psi_k(P_m \square W_{n+1}) \leq \lceil \frac{m^2 n}{k} \rceil + m$.*

Proof Use $u_1, u_2, \ldots, u_m$ to mark the vertices of P_m, and $v_0, v_1, \ldots, v_n$ to mark the vertices of W_{n+1}, respectively. Take any vertex (u_1, v_j) in ${}^{u_1}W_{n+1}$ and choose another vertex (u_1, v_{j+t+1}) in ${}^{u_1}W_{n+1}$ with $t = \lfloor \frac{k}{m} \rfloor$. Construct $S' = S_1 \bigcup S_2 \bigcup (u_i, v_0)$ $(i = 1, 2, \ldots, m)$ with

$$S_1 = \{(u_i, v_j) \in V(P_m \square W_{n+1}) | \ i \text{ is odd}\}$$
$$S_2 = \{(u_i, v_{j+t+1}) \in V(P_m \square W_{n+1}) | \ i \text{ is even}\}.$$

It is a k-path vertex cover set of $P_m \square W_{n+1}$, and $|S_1 \bigcup S_2| = m$. By this same step, there are

$\frac{mn}{k}$ sets that are copies of every k-path in the same layer. $|S'| = |S_1| + |S_2| + m \leq \lceil \frac{m^2 n}{k} \rceil + m$, then the upper $\psi_k(P_m \square W_{n+1}) \leq \lceil \frac{m^2 n}{k} \rceil + m$.

We get the lower bound of $\psi_k(P_m \square W_{n+1})$ by partitioning $P_m \square W_{n+1}$ into m disjoint subgraphs that are isomorphic to W_{n+1}. It is easy to get

$$\psi_k(P_m \square W_{n+1}) \geq m(1 + \lceil \frac{n}{k} \rceil)$$

by Lemma 2.5. The result above corrects the one in [18, Theorem 2.8]. □

Theorem 3.4 *If integer $m, n \geq 4$, and m is an even, we have $\psi_2(P_m \square K_{n,n}) = mn$.*

Proof Mark the vertices of P_m with u_i $(i = 1, 2, \ldots, m)$ and $K_{n,n}$ with x_j $(j = 1, 2, \ldots, n)$ and y_j $(j = 1, 2, \ldots, n)$, respectively.

By the definition of $K_{n,n}$, we can easily get that C_{2mn} is a subgraph of $P_m \square K_{n,n}$. Then

$$\psi_2(P_m \square K_{n,n}) \geq \psi_2(C_{2mn}) = \lceil \frac{2mn}{2} \rceil = mn$$

by Lemma 2.5. Next, construct a 2-path vertex cover to prove that $\psi_2(P_m \square K_{n,n}) \leq mn$ in the following way. Let

$$S_i = \{(u_i, y_j) \in P_m \square K_{n,n} \ (j = 1, 2, \ldots, n) | \text{ for even } i\},$$

$$S_i = \{(u_i, x_j) \in P_m \square K_{n,n} \ (j = 1, 2, ...n) | \text{ for odd } i\}.$$

Let $S = \bigcup_{i=m}^{m} S_i$. Clearly, the set S is a 2-path vertex cover of $P_m \square K_{n,n}$ since the degrees of the remaining uncovered vertices are all 0. Therefore, $\psi_2(P_m \square K_{n,n}) = mn$. □

Theorem 3.5 *If integer $m, n \geq 4$, and m is an even, we have*

(i) For integer $3 \leq k \leq n + 1$, then

$$\psi_k(P_m \square K_{n,n}) \leq mn - \frac{m}{2}(\lfloor \frac{k}{2} \rfloor - 1).$$

(ii) For integer $n + 2 \leq k \leq 2n + 1$, then

$$m(n + 1 - \lfloor \frac{k}{2} \rfloor) \leq \psi_k(P_m \square K_{n,n}) \leq \frac{3}{4} mn.$$

(iii) For integer $k \geq 2n + 1$, then

$$\psi_k(P_m \square K_{n,n}) \geq \lceil \frac{2mn}{k} \rceil.$$

Proof (i) Let

$$S_{2i} = \{(u_{2i}, y_j) \in P_m \square K_{n,n} | j = 1, 2, \ldots, n\},$$
$$S_{2i-1} = \{(u_{2i-1}, x_j) \in P_m \square K_{n,n} | j = 1, 2, \ldots, n - \lfloor \frac{k}{2} \rfloor + 1\},$$
$$S_{2i+1} = \{(u_{2i+1}, x_j) \in P_m \square K_{n,n} | j = \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \ldots, n\},$$

where

$$|S_{2i}| = n, \ |S_{2i-1}| = |S_{2i+1}| = n - \lfloor \frac{k}{2} \rfloor + 1, \quad i \in [1, \frac{m}{2}].$$

It is easy to see that $S=\bigcup_{i=m}^{m} S_i$ is a k-path vertex cover set since the largest uncovered path has at most $2(\lfloor\frac{k}{2}\rfloor-1)+1\le k-1$ vertices. Therefore,

$$\psi_k(P_m\Box K_{n,n})\le mn-\frac{m}{2}(\lfloor\frac{k}{2}\rfloor-1).$$

If m is odd, we follow the same way as shown above, then

$$\psi_k(P_m\Box K_{n,n})\le mn-\frac{m+1}{2}(\lfloor\frac{k}{2}\rfloor-1).$$

(ii) Let

$$S_{2i}=\{(u_{2i},y_j)\in P_m\Box K_{n,n}|j=1,2,\ldots,n\},$$
$$S_{2i-1}=\{(u_{2i-1},x_j)\in P_m\Box K_{n,n}|j=1,2,\ldots,\lfloor\frac{n}{2}\rfloor\},$$
$$S_{2i+1}=\{(u_{2i+1},x_j)\in P_m\Box K_{n,n}|j=\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+1,\ldots,n\}.$$

The set $S=\bigcup_{i=1}^{m} S_i$ is the k-path vertex cover of $P_m\Box K_{n,n}$, because the largest uncovered path has $2(n-\lfloor\frac{n}{2}\rfloor)+1\le n+1\le k-1$ vertices. $|S_{2i}|=n$, $|S_{2i-1}|+|S_{2i+1}|=n$, $i\in[1,\frac{m}{2}]$, $|S_{2i}|$ is an arithmetic sequence, so the upper $\psi_k(P_m\Box K_{n,n})\le\frac{m}{2}(n+\lfloor\frac{n}{2}\rfloor)\le\frac{3}{4}mn$. If m is odd, we follow the same way as shown above similarly, then

$$\psi_k(P_m\Box K_{n,n})\le\frac{m+1}{2}(n+\lfloor\frac{n}{2}\rfloor)-n.$$

For the lower, we can partition $P_m\Box K_{n,n}$ into m subgraphs $K_{n,n}$, thus we will get the lower bound $\psi_k(P_m\Box K_{n,n})\ge m(n+1-\lfloor\frac{k}{2}\rfloor)$ by using Lemma 2.4.

(iii) For $k\ge 2n+1$, it is clear that C_{2mn} is a subgraph of $P_m\Box K_{n,n}$, then

$$\psi_k(P_m\Box K_{n,n})\ge\psi_k(C_{2mn})=\lceil\frac{2mn}{k}\rceil.$$

All of proofs are completed. □

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