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b-Generalized Derivations on Prime Rings

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Abstract In the present paper, we have discussed the commutativity of prime rings and extended some well known results concerning derivation and generalized derivations to b-generalized derivations.

Keywords prime rings; b-generalized derivations; commutativity

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1. Introduction

Throughout this paper, unless otherwise mentioned, R will represent associative ring center $Z(R)$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. If R is a prime ring, the symbol $Q = Q_{mr}$ means it is maximal right Martindale quotient ring. The center of Q, denoted by C, is called the extended centroid of R and RC is said to be the central closure of R. It is remarked that R is a prime ring if and only if C is a field (we refer the reader to [1] for these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. We denote the identity mapping of a ring R by I_{id}, that is, the mapping $I_{id} : R \longrightarrow R$ is defined as $I_{id}(x) = x$ for all $x \in R$. By a derivation on R we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is called an inner derivation induced by an element $a \in R$, if $d(x) = [a, x] = I_a(x)$ for all $x \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Note that if $F = d$, then F is just an ordinary derivation. In a recent paper of Kosan and Lee [2], the authors propose the definition of b-generalized derivation: An additive mapping $F : R \longrightarrow Q$ is called a b-generalized derivation of R with the associated map d if $F(xy) = F(x)y + bxd(y)$ holds for all $x, y \in R$ and $b \in Q$. In the same paper, it is proved that if R is a prime ring and $b \neq 0$, then d is a derivation of R. Clearly, a generalized derivation is a 1-generalized derivation. The following example demonstrates that b-generalized derivations in rings do exist. Let S be any ring and $R = \left\{\begin{pmatrix} x & y \\ m & n \end{pmatrix} | x, y, m, n \in S\right\}$. Define maps $F : R \to R$ by $F\begin{pmatrix} x & y \\ m & n \end{pmatrix} = \begin{pmatrix} x+n & y \\ 0 & 0 \end{pmatrix}$. Then it is easy to check that F is a b-generalized derivation of R,

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where $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. From the above, one may observe that the concept of b-generalized derivations includes the concept of derivations and generalized derivations. The b-generalized derivations appeared canonically in [3] and were studied by several authors [4, 5]. A classical problem in ring theory is to investigate and extend conditions under which a ring becomes commutative. So far the best tools found for this purpose are the derivations on rings and also on their modules. Many results in literature indicate that the global structure of a ring R is often lightly connected to the behavior of additive mappings defined on R. The first result in this topic is due to Divinsky [6] who proved that the simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner proved in [7] that if a prime ring R admits a nonzero derivation d satisfying $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. As is well known, this theorem was regarded as the starting point of many papers concerning the study of various kinds of additive mappings satisfying appropriate algebraic conditions on some subsets of prime and semiprime rings.

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations acting on appropriate subsets of the rings [8–12], where further references can be found). Moreover, many of the results obtained extended other ones proven previously just for the action of the considered mappings on the entire ring. In the year 2002, Ashraf and Rehman [13] proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) \pm x \circ y = 0$ for all $x, y \in I$, then R is commutative. Later, in [14] this result was also extended to the case of generalized derivations. More precisely, Rehman proved that if a prime ring R admits a generalized derivation F associated with derivation d such that $F(x \circ y) \pm x \circ y = 0$ for all $x, y \in I$, a nonzero ideal I of R, and if $F = 0$ or $d \neq 0$, then R is commutative. In the above results, the authors applied an elementary calculation to obtain commutativity theorems instead of differential identities. Therefore, it should be interesting to extend some results concerning derivations and generalized derivations to b-generalize derivations. Motivated by these observations, in the present paper we would like to extend some commutativity results for derivations and generalized derivations to b-generalized derivations.

2. Some preliminaries

In the present paper, we shall make some extensive use of the following basic identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z,$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

We begin with several known results which will be used in the sequel to prove our theorems.

Lemma 2.1 ([15, Lemma 2]) *Let $F : R \longrightarrow RC$ be an additive map such that $F(xy) = F(x)y$*

for all $x, y \in R$. Then there exists $q \in Q_{mr}(RC)$ such that $F(x) = qx$ for all $x \in R$.

Lemma 2.2 ([16, Lemma 3]) *If a prime R contains a nonzero commutative right ideal, then R is commutative.*

Lemma 2.3 ([17, Theorem 2]) *Let R be a prime ring with maximal right Martindale quotient ring Q, and I a two-sided ideal of R. Then I, R and Q satisfy the same differential identities.*

Lemma 2.4 ([1, Proposition 2.5.1]) *Any derivation d of a semiprime ring R can be extended uniquely to a derivation of Q_{mr}.*

Lemma 2.5 ([7, Lemma 1]) *Let d be a derivation of a prime ring and a be an element of R. If $ad(x) = 0$ for all $x \in R$, then either $a = 0$ or d is zero.*

3. Main results

We are now in a position to prove our main result.

Theorem 3.1 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that F is a nonzero b-generalized derivation of R such that $F(x \circ y) = 0$ for all $x, y \in I$, then R is commutative.*

Proof By the definition of F, we have $F(xy) = F(x)y + bxd(y)$ for all $x, y \in I$ and some $b \in Q$. If $b = 0$ or $d = 0$, then $F(xy) = F(x)y$ for all $x, y \in I$. Since $F \neq 0$, by Lemma 2.1, there exists $q \in Q_{mr}(RC) - \{0\}$ such that $F(x) = qx$ for all $x \in I$. In this case, $F(x \circ y) = 0$ implies that $q(x \circ y) = 0$ for all $x, y \in I$. Replacing y by yz in the last equation, we find that $0 = q(x \circ (yz)) = q((x \circ y)z - y[x, z]) = q(x \circ y)z - qy[x, z]$ and hence $-qy[x, z] = 0$ for all $x, y, z \in I$. This means that $qI[x, z] = 0$ for all $x, z \in I$. The primeness of I forces that either $q = 0$ or $[x, z] = 0$ for all $x, z \in I$. Since $q \neq 0$, then $[I, I] = 0$. Hence, by application of Lemma 2.2, R is commutative. Hence onward, we assume that both $b \neq 0$ and $d \neq 0$. We are given that

$$F(x \circ y) = 0 \text{ for all } x, y \in I. \tag{3.1}$$

Replacing y by yx in (3.1), we obtain that $0 = F((x \circ y)x) = F(x \circ y)x + b(x \circ y)d(x)$ for all $x, y \in I$. Using the fact that $F(x \circ y) = 0$, the above equation reduces to

$$b(x \circ y)d(x) = 0 \text{ for all } x, y \in I. \tag{3.2}$$

Replace y by yz in (3.2) to get

$$by(x \circ z)d(x) + b[x, y]zd(x) = 0 \text{ for all } x, y, z \in I. \tag{3.3}$$

By Lemma 2.3, I and Q satisfy the same differential identities, so (3.3) is also satisfied by Q. Replacing y by b in (3.3), we get

$$b^2(x \circ z)d(x) + b[x, b]zd(x) = 0 \text{ for all } x, z \in Q. \tag{3.4}$$

Combining (3.2) with (3.4), we find that $b[x, b]zd(x) = 0$ for all $x, z \in Q$.

$$b[x, b]zd(x) = 0 \text{ for all } x, z \in Q. \tag{3.5}$$

The above relation yields that $b[x,b]Qd(x)=0$ for all $x\in Q$. The primeness of Q forces that, for each $x\in Q$, either $b[x,b]=0$ or $d(x)=0$. The set of $x\in Q$ for which these two properties hold are additive subgroups of Q. It is well known that a group cannot be the union of two proper subgroups and d can also be regarded as a derivation of Q by Lemma 2.4. Therefore, $b[x,b]=0$ for all $x\in Q$ or $d(x)=0$ for all $x\in Q$. In the first case, we have $bI_b(x)=0$ for all $x\in Q$. By Lemma 2.5, we arrive at $b\in C-\{0\}$. Now recalling (3.2), we find that $(x\circ y)d(x)=0$ for all $x,y\in Q$ since b is invertible in C. Replace y by yz in the last relation to get $0=(x\circ(yz))d(x)=y(x\circ z)d(x)+[x,y]zd(x)=[x,y]zd(x)$ for all $x,y,z\in Q$. This implies that $[x,y]Qd(x)=0$ for all $x,y\in Q$. Hence, either $[x,y]=0$ or $d(x)=0$. Let $Q_1=\{x\in Q|[x,y]=0\}$ and $Q_2=\{x\in Q|d(x)=0\}$. Then, Q_1 and Q_2 are both additive subgroups of Q such that $Q=Q_1\cup Q_2$. By Brauer's trick, we have either $Q_1=Q$ or $Q_2=Q$. On the one hand, if $Q_1=Q$, then $[x,y]=0$ for all $x,y\in Q$, that is, $[Q,Q]=0$. By virtue of Lemma 2.2, Q is commutative and so R. On the other hand, if $Q_2=Q$, then $d(Q)=0$. Therefore, $d=0$, a contradiction. $\Box$

Corollary 3.2 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that F is a nonzero b-generalized derivation of R such that $F(x^2)=0$ for all $x\in I$, then R is commutative.*

Proof By the assumption, we get $F(x^2)=0$ for all $x\in I$. Linearization of the above equation gives that $F(x^2)+F(y^2)+F(x\circ y)=0$ for all $x,y\in I$. This equation can reduce to $F(x\circ y)=0$ for all $x\in I$. It follows from Theorem 3.1 that R is commutative. $\Box$

Theorem 3.3 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that F is a nonzero b-generalized derivation of R such that $[F(x),x]=0$ for all $x\in I$, then R is commutative.*

Proof We are given that

$$[F(x),x]=0 \text{ for all } x\in I. \tag{3.6}$$

Replacing x by $x+y$ in (3.6) and using (3.6), we obtain

$$[F(x),y]+[F(y),x]=0 \text{ for all } x,y\in I. \tag{3.7}$$

Substitute yx for y in (3.7) and use (3.7) to get

$$by[d(x),x]+[by,x]d(x)=0 \text{ for all } x,y\in I. \tag{3.8}$$

Once again replacing y by yz in (3.8) and using (3.8), we have $[b,x]bzd(x)=0$ for all $x,z\in I$. This is the same as Eq. (3.5). Hence, repeating the same process, we get the required result. $\Box$

Theorem 3.4 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F\neq I_{id}$ is a b-generalized derivation of R such that $F(x\circ y)=x\circ y$ for all $x,y\in I$, then R is commutative.*

Proof We divide the proof into two cases.

Case 1. If $F=0$, then $x\circ y=0$ for all $x,y\in I$. Taking y instead of yz, we get $0=x\circ(yz)=(x\circ y)z-y[x,z]$ for all $x,y,z\in I$. This implies that $y[x,z]=0$ for all $x,y,z\in I$, namely, $IR[I,I]=0$. The primeness of R forces that $[I,I]=0$ since $I\neq 0$. Accordingly, we are done by Lemma 2.2.

Case 2. Now, let $F \neq 0$. If $b = 0$ or $d = 0$, then $F(xy) = F(x)y$ for all $x, y \in I$. Since $F \neq 0$, by Lemma 2.1, there exists $q \in Q_{mr}(RC) - \{0\}$ such that $F(x) = qx$ for all $x \in I$. Therefore, $(q-1)(x \circ y) = 0$ for all $x, y \in I$. Note that the arguments given in the proof of Theorem 3.1 are still valid in the present situation and we can conclude that either $q - 1 = 0$ or $[x, z] = 0$ for all $x, z \in I$. If $q - 1 = 0$, then $F(x) = x$ for all $x \in I$. For all $x \in I$ and $r \in R$, we get $rx = F(rx) = F(r)x + brd(x) = F(r)x$ since $b = 0$ or $d = 0$. Thus, $(F(r) - r)x = 0$ for all $x \in I$ and $r \in R$. That is $(F(r) - r)RI = 0$ for all $r \in R$. This forces that $F(r) = r$ for all $r \in R$, a contradiction. If $b \neq 0$ and $d \neq 0$, we have

$$F(x \circ y) = x \circ y \text{ for all } x, y \in I. \tag{3.9}$$

Writing yx instead of y in (3.9), we find that

$$F(x \circ y)x + b(x \circ y)d(x) = (x \circ y)x \text{ for all } x, y \in I. \tag{3.10}$$

Application of (3.9) gives that $b(x \circ y)d(x) = 0$ for all $x, y \in I$, which is the same as Eq. (3.2), proving the theorem. □

Corollary 3.5 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F \neq I_{id}$ is a b-generalized derivation of R such that $F(x^2) = x^2$ for all $x \in I$, then R is commutative.*

Theorem 3.6 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F \neq I_{id}$ is a b-generalized derivation of R such that $F(x \circ y) + x \circ y = 0$ for all $x, y \in I$, then R is commutative.*

Proof Since F is a b-generalized derivation of R such that $F(x \circ y) + x \circ y = 0$, then $-F(x \circ y) - x \circ y = 0$ for all $x, y \in I$. This implies that $-F$ is also a b-generalized derivation of R satisfying $(-F)(x \circ y) = x \circ y$ for all $x, y \in I$. In view of Theorem 3.4, we get the required result. □

Corollary 3.7 *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F \neq I_{id}$ is a b-generalized derivation of R such that $F(x^2) + x^2 = 0$ for all $x \in I$, then R is commutative.*

The following example shows that the above results cannot be proved for semiprime rings.

Example 3.8 Let R_1 and R_2 be prime rings, where R_1 is noncommutative and put $R = R_1 \oplus R_2$. It is clear that R is a semiprime ring. Let $F_1 : R_1 \longrightarrow R_1$ be a nonzero b-generalized derivation satisfying the hypothesis of Theorem 3.1, that is, $F_1(x \circ y) = 0$ for all $x, y \in R_1$. Then the mapping $F : R \longrightarrow R$ defined by $F((r_1, r_2)) = (F_1(r_1), 0)$ is a nonzero $(b, 0)$-generalized derivation of R such that $F(X \circ Y) = 0$ for all $X, Y \in R$. However, R is not commutative.

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