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Harmonic Univalent Functions Related with Generalized (p, q)-Post Quantum Calculus Operators

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Abstract In this paper, we introduce certain subclasses of harmonic univalent functions associated with the Janowski functions, which are defined by using generalized (p, q)-post quantum calculus operators. Sufficient coefficient conditions, extreme points, distortion bounds and partial sums properties for the functions belonging to the subclasses are obtained.

Keywords Harmonic univalent function; Janowski function; subordination; partial sums; (p, q)-post quantum calculus operator

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1. Introduction and preliminaries

An analytic function $s : \mathbb{U} = \{z : |z| < 1\} \to \mathbb{C}$ is subordinate to an analytic function $t : \mathbb{U} \to \mathbb{C}$, if there is a function ν satisfying $\nu(0) = 0$ and $|\nu(z)| < 1$ $(z \in \mathbb{U})$, such that $s(z) = t(\nu(z))$ $(z \in \mathbb{U})$. Note that $s(z) \prec t(z)$. Especially, if t is univalent in $\mathbb{U}$, then the following conclusion is true [1]:

$$s(z) \prec t(z) \Longleftrightarrow s(0) = t(0) \text{ and } s(\mathbb{U}) \subset t(\mathbb{U}).$$

Let $\mathcal{A}$ define the class of functions h that are analytic in $\mathbb{U}$ of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The theory of (p, q)-calculus (or post quantum calculus [2]) operators is widely used in many fields of science. In recent years, there are also related researches in the theory of geometric functions. In 1991, Chakrabarti and Jagannathan [3] introduced the (p, q)-derivative operator $D_{p,q}$ by

$$D_{p,q}h(z) = \begin{cases} \frac{h(pz)-h(qz)}{(p-q)z}, & p \neq q, z \neq 0, \\ \lim\limits_{q\to p} \frac{h(pz)-h(qz)}{(p-q)z}, & p = q, z \neq 0, \end{cases} \tag{1.2}$$

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where $0 < q \leq p \leq 1$ and $h \in \mathcal{A}$.

Substituting (1.1) into (1.2), we obtain

$$D_{p,q}h(z) = 1 + \sum_{k=2}^{\infty}[k]_{p,q}a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_{p,q} = \begin{cases} \sum\limits_{\ell=1}^{k} p^{\ell-1}q^{k-\ell}, & p \neq q, \\ kp^{k-1}, & p = q. \end{cases}$$

From (1.3), we have

$$\lim_{z\to 0} D_{p,q}h(z) = 1, \quad \lim_{p\to 1^-} D_{p,p}h(z) = D_{1,1}h(z) = h'(z),$$

$$D_{1,t}h(z) = D_t h(z),\ t \in (0,1) \ \ \text{(see Jackson [4,5], Aral et al. [6]).}$$

Recently, many scholars have discussed the properties of some geometric analytic function classes with the help of the operators $D_t, D_{p,q}$ and achieved some important results (see [7–17]).

Let $\lambda \geq 0$, $0 < q \leq p \leq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $h \in \mathcal{A}$. We introduce the generalized normalization (p,q)-post quantum calculus operators $\mathcal{J}_{p,q}^{m,\lambda} : \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned} &\mathcal{J}_{p,q}^{0,0}h(z) = h(z), \\ &\mathcal{J}_{p,q}^{1,\lambda}h(z) = (1-\lambda)zD_{p,q}h(z) + \lambda z(zD_{p,q}h(z))' := \mathcal{J}_{p,q}^{\lambda}h(z), \\ &\mathcal{J}_{p,q}^{2,\lambda}h(z) = \mathcal{J}_{p,q}^{\lambda}(\mathcal{J}_{p,q}^{\lambda}h(z)) \end{aligned} \tag{1.4}$$

and in general,

$$\mathcal{J}_{p,q}^{m,\lambda}h(z) = \mathcal{J}_{p,q}^{\lambda}(\mathcal{J}_{p,q}^{m-1,\lambda}h(z)), \quad z \in \mathbb{U}. \tag{1.5}$$

After a simple calculation, we can obtain the following conclusion,

$$\mathcal{J}_{p,q}^{m,\lambda}h(z) = z + \sum_{k=1}^{\infty}\{[1+(k-1)\lambda][k]_{p,q}\}^m a_k z^k. \tag{1.6}$$

For ease of notations, we let

$$\omega_k(\lambda; p, q) = [1+(k-1)\lambda][k]_{p,q}. \tag{1.7}$$

Obviously, $\mathcal{J}_{p,p}^{m,0}h(z) = D^m h(z)$ (see Sălăgean operator [18]).

For analytic functions $h(z)$ and $g(z)$ $(z \in \mathbb{U})$, let S_H denote the class of harmonic univalent functions $f = h + \overline{g}$, which are sense preserving in $\mathbb{U}$, that is

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}. \tag{1.8}$$

It is well known that the necessary and sufficient condition for $f = h + \overline{g}$ to be locally univalent and sense preserving in $\mathbb{U}$ is $|h'(z)| > |g'(z)|$ $(z \in \mathbb{U})$ (see [19,20]).

A harmonic function F is given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k}, |B_1| < 1.$$

We define the convolution (or Hadamard product) of f and F by

$$(f*F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} \overline{b_k B_k z^k} = (F*f)(z).$$

Let $f \in S_H$. We now define the operator $\mathcal{J}_{p,q}^{m,\lambda} : S_H \to S_H$ as

$$\mathcal{J}_{p,q}^{m,\lambda} f(z) = \mathcal{J}_{p,q}^{m,\lambda} h(z) + (-1)^m \overline{\mathcal{J}_{p,q}^{m,\lambda} g(z)}, \tag{1.9}$$

where

$$\mathcal{J}_{p,q}^{m,\lambda} h(z) = z + \sum_{k=2}^{\infty} \omega_k^m(\lambda;p,q) a_k z^k, \quad \mathcal{J}_{p,q}^{m,\lambda} g(z) = \sum_{k=1}^{\infty} \omega_k^m(\lambda;p,q) b_k z^k, \tag{1.10}$$

with $\omega_k(\lambda;p,q)$ defined by (1.7).

Let

$$\phi(z) = z + \sum_{k=2}^{\infty} u_k z^k + \sum_{k=1}^{\infty} \overline{v_k z^k} \tag{1.11}$$

be harmonic in $\mathbb{U}$ with $u_k > 0$ and $v_k > 0$.

Take

$$\mathcal{L}_H f(z) = zh'(z) - \overline{zg'(z)}, \ \mathcal{L}_H^2 f(z) = \mathcal{L}_H(\mathcal{L}_H f(z)), \quad f \in S_H.$$

Throughout this paper, we assume $m \in \mathbb{N}_0$, $\lambda \geq 1$, $0 < q < p \leq 1$, $-1 \leq B < A < -B \leq 1$; $u_k > 0$ and $v_k > 0$.

Now, using the operator $\mathcal{J}_{p,q}^{m,\lambda} f(z)$ and Janowski functions [21], we define the following two classes.

Definition 1.1 *Let $f \in S_H$ be of the form (1.8). Then $f(z) \in S_\phi^{p,q}(\lambda, m, A, B)$ iff*

$$\frac{\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda} f * \phi)(z)}{\mathcal{J}_{p,q}^{m,\lambda} f * \phi(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.12}$$

and also $f(z) \in K_\phi^{p,q}(\lambda, m, A, B)$ iff

$$\frac{\mathcal{L}_H^2(\mathcal{J}_{p,q}^{m,\lambda} f * \phi)(z)}{\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda} f * \phi)(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.13}$$

where

$$\mathcal{J}_{p,q}^{m,\lambda}(f*\phi)(z) = z + \sum_{k=2}^{\infty} \omega_k^m(\lambda;p,q) u_k a_k z^k + (-1)^m \sum_{k=1}^{\infty} \omega_k^m(\lambda;p,q) v_k \bar{b}_k \bar{z}^k \tag{1.14}$$

and $\omega_k(\lambda;p,q)$ is defined by (1.7).

The classes $S_\phi^{p,q}(\lambda, m, A, B)$ and $K_\phi^{p,q}(\lambda, m, A, B)$ reduce to the well-known classes of S_H as well as many new ones. For example,

$$S^{1,1}_{\frac{z}{1-z}+\overline{\frac{z}{1-z}}}(0,0,1-2\beta,-1) = HS^*(\beta) = \left\{f \in S_H : \mathrm{Re}\left[\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right] > \beta\right\},$$

$$K^{1,1}_{\frac{z}{1-z}+\overline{\frac{z}{1-z}}}(0,1,1-2\beta,-1) = CH(\beta) = \left\{f \in S_H : \mathrm{Re}\left[\frac{zh''(z) + h'(z) + \overline{zg''(z) + g'(z)}}{h'(z) - \overline{g'(z)}}\right] > \beta\right\},$$

where $\beta \in [0,1)$.

In particular, the classes $HS^* = HS^*(0)$ (harmonic starlike functions) and $CH = CH(0)$ (harmonic convex functions) were studied by Jahangiri [22–27].

Also, we denote by T^m the subclass of S_H, the function f of which is expressed as

$$f(z) = h(z) + \overline{g(z)} = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^m \sum_{k=1}^{\infty} |b_k| \overline{z}^k, \quad |b_1| < 1,\ z \in \mathbb{U}. \tag{1.15}$$

We let

$$\overline{S}_{\phi}^{p,q}(\lambda, m, A, B) = T^m \cap S_{\phi}^{p,q}(\lambda, m, A, B),$$

$$\overline{K}_{\phi}^{p,q}(\lambda, m, A, B) = T^{m+1} \cap K_{\phi}^{p,q}(\lambda, m, A, B).$$

In this paper, the necessary and sufficient conditions of coefficient are obtained. As what we have hoped, distortion estimates, extreme points and properties of partial sums for the above-defined classes are also obtained.

2. Basic properties

First of all, we provide the sufficient conditions of coefficients for the classes defined in Definition 1.1.

Theorem 2.1 *Let $f = h + \overline{g}$ be given by (1.8) and $\omega_k(\lambda; p, q)$ given by (1.7).*

(i) The sufficient condition for f to be sense-preserving and harmonic univalent in $\mathbb{U}$ and $f \in S_{\phi}^{p,q}(\lambda, m, A, B)$ is

$$\sum_{k=1}^{\infty} (\xi_k^m |a_k| + \mu_k^m |b_k|) \le 2, \tag{2.1}$$

where $a_1 = u_1 = 1$ and

$$\begin{cases} k \le \xi_k^m := \frac{u_k \omega_k^m(\lambda; p, q)[k(1-B) - (1-A)]}{A-B}, \\ k \le \mu_k^m := \frac{v_k \omega_k^m(\lambda; p, q)[k(1-B) + (1-A)]}{A-B}. \end{cases} \tag{2.2}$$

(ii) The sufficient condition for f to be sense-preserving and harmonic univalent in $\mathbb{U}$ and $f \in K_{\phi}^{p,q}(\lambda, m, A, B)$ is

$$\sum_{k=1}^{\infty} k(\xi_k^m |a_k| + \mu_k^m |b_k|) \le 2, \tag{2.3}$$

where $a_1 = u_1 = 1, \xi_k^m$ and μ_k^m are given by (2.2).

Proof (i) Let $f = h + \overline{g} \in S_H$ be of the form (1.8). In 1999, Jahangiri [23, Theorem 1, $\alpha = 0$] obtained that f is univalent and sense-preserving in $\mathbb{U}$ if $\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \le 2$.

Using (2.1) and (2.2), we have

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \le \sum_{k=1}^{\infty} (\xi_k^m |a_k| + \mu_k^m |b_k|) \le 2.$$

Therefore, it can be deduced that $f(z)$ is univalent and sense-preserving in $\mathbb{U}$. By means of Definition 1.1 and the relationship of subordination, the function $f \in S_{\phi}^{p,q}(\lambda, m, A, B)$ iff there exists an analytic function $\varpi(z)$ satisfying $\varpi(0)=0,\ |\varpi(z)|<1\ (z\in\mathbb{U})$ such that

$$\frac{\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)}{\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)}=\frac{1+A\varpi(z)}{1+B\varpi(z)},$$

or equivalently

$$\left|\frac{\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)-\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)}{A\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)-B\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)}\right|<1. \tag{2.4}$$

We only need to show that

$$|A\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)-B\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)|-|\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)-\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)|>0,\quad z\in\mathbb{U}. \tag{2.5}$$

Let

$$\begin{cases}\vartheta_{k,j}=Bk\omega_k^m(\lambda;p,q)+(-1)^jA\omega_k^m(\lambda;p,q), & j=1,2,\\ \theta_{k,j}=k\omega_k^m(\lambda;p,q)+(-1)^j\omega_k^m(\lambda;p,q), & j=1,2.\end{cases} \tag{2.6}$$

Therefore, from (2.1), we get

$$\begin{aligned}&|A\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)-B\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)|-|\mathcal{L}_H(\mathcal{J}_{p,q}^{m,\lambda}f*\phi)(z)-\mathcal{J}_{p,q}^{m,\lambda}f*\phi(z)|\\ &=\Big|z-\sum_{k=2}^{\infty}\vartheta_{k,1}u_ka_kz^k+(-1)^m\sum_{k=1}^{\infty}\vartheta_{k,2}v_k\overline{b_kz^k}\Big|-\Big|\sum_{k=2}^{\infty}\theta_{k,1}u_ka_kz^k+(-1)^{m+1}\sum_{k=1}^{\infty}\theta_{k,2}v_k\overline{b_kz^k}\Big|\\ &\geq(A-B)|z|+\sum_{k=2}^{\infty}\vartheta_{k,1}u_k|a_k||z|^k+\sum_{k=1}^{\infty}\vartheta_{k,2}v_k|b_k||z|^k-\sum_{k=2}^{\infty}\theta_{k,1}u_k|a_k||z|^k-\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k||z|^k\\ &=(A-B)|z|\Big[1-\sum_{k=2}^{\infty}\xi_k^m|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\mu_k^m|b_k||z|^{k-1}\Big]\\ &>(A-B)|z|\Big[1-\sum_{k=2}^{\infty}\xi_k^m|a_k|-\sum_{k=1}^{\infty}\mu_k^m|b_k|\Big]\geq 0.\end{aligned}$$

Hence, we complete the proof of (i). Also, applying the same method as (i), we can obtain (ii). □

Theorem 2.2 *Let $a_1=u_1=1$ and $f=h+\overline{g}$ be given by (1.15). Then*

(i) $f\in\overline{S}_{\phi}^{p,q}(\lambda,m,A,B)$ iff (2.1) holds true.

(ii) $f\in\overline{K}_{\phi}^{p,q}(\lambda,m,A,B)$ iff (2.3) holds true.

Proof (i) It appears from (1.15) that $\overline{S}_{\phi}^{p,q}(\lambda,m,A,B)\subset S_{\phi}^{p,q}(\lambda,m,A,B)$. In view of Theorem 2.1, it is straightforward to show that if $f\in\overline{S}_{\phi}^{p,q}(\lambda,m,A,B)$, then (2.1) holds true. Next, we use the methods in [28] to prove.

Let $f\in\overline{S}_{\phi}^{p,q}(\lambda,m,A,B)$. Then it satisfies (1.12) or equivalently

$$\left|\frac{\sum_{k=2}^{\infty}\theta_{k,1}u_k|a_k|z^{k-1}+\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k|\bar{z}^{k-1}}{(A-B)+\sum_{k=2}^{\infty}\vartheta_{k,1}u_k|a_k|z^{k-1}+\sum_{k=1}^{\infty}\vartheta_{k,2}v_k|b_k|\bar{z}^{k-1}}\right|<1,\quad z\in\mathbb{U}, \tag{2.7}$$

where $\vartheta_{k,j},\theta_{k,j}$ are given by (2.6).

From (2.7), we get

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty}\theta_{k,1}u_k|a_k|z^{k-1}+\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k|\bar{z}^{k-1}}{(A-B)+\sum_{k=2}^{\infty}\vartheta_{k,1}u_k|a_k|z^{k-1}+\sum_{k=1}^{\infty}\vartheta_{k,2}v_k|b_k|\bar{z}^{k-1}}\right\}<1, \tag{2.8}$$

which holds for all $z \in \mathbb{U}$. Taking $z = r$ $(0 < r < 1)$ in (2.8), we get

$$\frac{\sum_{k=2}^{\infty} \theta_{k,1} u_k |a_k| r^{k-1} + \sum_{k=1}^{\infty} \theta_{k,2} v_k |b_k| r^{k-1}}{(A-B) + \sum_{k=2}^{\infty} \vartheta_{k,1} u_k |a_k| r^{k-1} + \sum_{k=1}^{\infty} \vartheta_{k,2} v_k |b_k| r^{k-1}} < 1. \tag{2.9}$$

Thus, from (2.9) we have

$$\sum_{k=1}^{\infty} (\xi_k^m |a_k| + \mu_k^m |b_k|) r^{k-1} < 2, \quad 0 < r < 1, \tag{2.10}$$

where ξ_k^m and μ_k^m are given by (2.2).

Let $S_n = \sum_{k=1}^{n} (\xi_k^m |a_k| + \mu_k^m |b_k|)$.

For the series $\sum_{k=1}^{\infty} \xi_k^m |a_k| + \sum_{k=1}^{\infty} \mu_k^m |b_k|$, $\{S_n\}$ is the nondecreasing sequence of partial sums of it. Moreover, by (2.10) it is bounded by 2. Therefore, it is convergent and

$$\sum_{k=1}^{\infty} (\xi_k^m |a_k| + \mu_k^m |b_k|) = \lim_{n\to\infty} S_n \le 2.$$

Thus, we get the inequality (2.1). Similarly, it is easy to prove (ii) of Theorem 2.2. □

Clearly, from Theorem 2.2, we have

$$\bar{K}_{\phi}^{p,q}(\lambda, m, A, B) \subset \bar{S}_{\phi}^{p,q}(\lambda, m, A, B). \tag{2.11}$$

Next, we give the extreme points of these classes.

Theorem 2.3 *Let $X_k \ge 0, Y_k \ge 0, \sum_{k=1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1, \xi_k^m$ and μ_k^m be given by (2.2).*

(i) If $f \in \bar{S}_{\phi}^{p,q}(\lambda, m, A, B)$, then $f \in \mathrm{clco}\bar{S}_{\phi}^{p,q}(\lambda, m, A, B)$ iff

$$f(z) = \sum_{k=1}^{\infty} [X_k h_k + Y_k g_k], \quad z \in \mathbb{U}, \tag{2.12}$$

where

$$\begin{cases} h_1 = z \ h_k = z - \frac{1}{\xi_k^m} z^k, & k \ge 2, \\ g_k = z + \frac{(-1)^m}{\mu_k^m} \bar{z}^k, & k \ge 1. \end{cases} \tag{2.13}$$

(ii) If $f \in \bar{K}_{\phi}^{p,q}(\lambda, m, A, B)$, then $f \in \mathrm{clco}\bar{K}_{\phi}^{p,q}(\lambda, m, A, B)$ iff the condition (2.12) holds and

$$\begin{cases} h_1 = z \ h_k = z - \frac{1}{k\xi_k^m} z^k, & k \ge 2, \\ g_k = z + \frac{(-1)^{m+1}}{k\mu_k^m} \bar{z}^k, & k \ge 1. \end{cases} \tag{2.14}$$

Proof (i) From (2.12), we get

$$f(z) = \Big(\sum_{k=1}^{\infty} [X_k + Y_k]\Big) z - \sum_{k=2}^{\infty} \frac{1}{\xi_k^m} X_k z^k + (-1)^m \sum_{k=1}^{\infty} \frac{1}{\mu_k^m} Y_k \overline{z^k}.$$

Since, $0 \le X_k \le 1$ $(k = 1, 2, \ldots)$, we obtain

$$\sum_{k=2}^{\infty} \xi_k^m \frac{1}{\xi_k^m} X_k + \sum_{k=1}^{\infty} \mu_k^m \frac{1}{\mu_k^m} Y_k = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \le 1.$$

It follows from (i) of Theorem 2.2 that $f \in \bar{S}_{\phi}^{p,q}(\lambda, m, A, B)$.

Conversely, if $f\in \bar{S}^{p,q}_{\phi}(\lambda,m,A,B)$, then

$$|a_k|\le \frac{1}{\xi_k^m},\quad |b_k|\le \frac{1}{\mu_k^m}.$$

Putting $X_k=\xi_k^m|a_k|, Y_k=\mu_k^m|b_k|$ and $\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1$ $(0\le X_1\le 1)$, we obtain

$$\begin{aligned}f(z)&=z-\sum_{k=2}^{\infty}|a_k|z^k+(-1)^m\sum_{k=1}^{\infty}|b_k|\overline{z}^k\\&=\sum_{k=1}^{\infty}(X_k+Y_k)z-\sum_{k=2}^{\infty}\frac{1}{\xi_k^m}X_kz^k+(-1)^m\sum_{k=1}^{\infty}\frac{1}{\mu_k^m}Y_k\overline{z}^k\\&=\sum_{k=1}^{\infty}[h_k(z)X_k+g_k(z)Y_k].\end{aligned}$$

Thus f can be expressed in the form of (2.12). The remainder of the proof is analogous to (i) in Theorem 2.3 and so we omit. □

Next, using Theorems 2.2, we proceed to give the distortion theorems for functions of these classes.

Theorem 2.4 *Let $f=h+\overline{g}$ be of the form (1.15), $|z|=r$, ξ_k^m and μ_k^m are defined by (2.2), $\{\xi_k^m\}$ and $\{\mu_k^m\}$ are non-decreasing sequences.*

(i) If $f\in \bar{S}^{p,q}_{\phi}(\lambda,m,A,B)$, then

$$(1-|b_1|)r-\frac{r^2}{\min\{\xi_2^m,\mu_2^m\}}\le |f(z)|\le (1+|b_1|)r+\frac{r^2}{\min\{\xi_2^m,\mu_2^m\}}.$$

(ii) If $f\in \bar{K}^{p,q}_{\phi}(\lambda,m,A,B)$, then

$$(1-|b_1|)r-\frac{r^2}{2\min\{\xi_2^m,\mu_2^m\}}\le |f(z)|\le (1+|b_1|)r+\frac{r^2}{2\min\{\xi_2^m,\mu_2^m\}}.$$

Proof (i) For $f\in \bar{S}^{p,q}_{\phi}(\lambda,m,A,B)$, using Theorem 2.2 and (2.1), we have

$$\begin{aligned}|f(z)|&=\Big|z-\sum_{k=2}^{\infty}|a_k|z^k+(-1)^m\sum_{k=1}^{\infty}|b_k|\overline{z^k}\Big|\\&\le (1+|b_1|)r+\frac{1}{\min\{\xi_2^m,\mu_2^m\}}\sum_{k=2}^{\infty}(\xi_k^m|a_k|+\mu_k^m|b_k|)r^2\\&\le (1+|b_1|)r+\frac{1}{\min\{\xi_2^m,\mu_2^m\}}r^2\end{aligned}$$

and

$$|f(z)|\ge (1-|b_1|)r-\Big(\sum_{k=2}^{\infty}|a_k|+\sum_{k=2}^{\infty}|b_k|\Big)r^2\ge (1-|b_1|)r-\frac{1}{\min\{\xi_2^m,\mu_2^m\}}r^2.$$

The result is sharp and the extremal function is

$$f(z)=z-\frac{1}{\min\{\xi_2^m,\mu_2^m\}}z^2+|b_1|\overline{z}.$$

Similarly, it is easy to prove the remainder of Theorem 2.4. □

Using Theorem 2.4, it is trivial to show the covering results.

Theorem 2.5 *Let ξ_k^m and μ_k^m be given by (2.2).*

(i) If $f \in \bar{S}_\phi^{p,q}(\lambda, m, A, B)$, then

$$\{w : |w| < 1 - |b_1| - \frac{1}{\min\{\xi_2^m, \mu_2^m\}}\} \subset f(\mathbb{U}).$$

(ii) If $f \in \bar{K}_\phi^{p,q}(\lambda, m, A, B)$, then

$$\{w : |w| < 1 - |b_1| - \frac{1}{2\min\{\xi_2^m, \mu_2^m\}}\} \subset f(\mathbb{U}).$$

Theorem 2.6 *The classes $\bar{S}_\phi^{p,q}(\lambda, m, A, B)$ and $\bar{K}_\phi^{p,q}(\lambda, m, A, B)$ are closed under convex combinations.*

Remark 2.7 *By taking the special value of the parameters λ, p, q, m, A, B and ϕ in Theorems 2.1–2.6, it is easy to show the corresponding results for the classes $HS^*(\beta)$ and $CH(\beta)$.*

3. Partial sums properties

Next, we will consider the properties of partial sums of the classes studied in this article. The partial sums of the harmonic function classes of the form (1.8) are defined as follows.

$$f_\rho(z) = h_\rho(z) + \overline{g(z)} = z + \sum_{k=2}^{\rho} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k},$$

$$f_\sigma(z) = h(z) + \overline{g_\sigma(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\sigma} \overline{b_k z^k}$$

and

$$f_{\rho,\sigma}(z) = h_\rho(z) + \overline{g_\sigma(z)} = h_\rho(z) + \overline{g_\sigma(z)} = z + \sum_{k=2}^{\rho} a_k z^k + \sum_{k=1}^{\sigma} \overline{b_k z^k},$$

where $|b_1| < 1,\ \rho, \sigma \in \mathbb{N}$ and $\rho \geq 2$ (see [29–33]).

In order to obtain the properties of partial sums for functions belonging to the classes $\bar{S}_\phi^{p,q}(\lambda, m, A, B)$ and $\bar{K}_\phi^{p,q}(\lambda, m, A, B)$, we introduce a new class of harmonic functions as follows.

Definition 3.1 *Let $\delta \in \mathbb{N}_0$ and $f = h + \overline{g}$ be given by (1.15). Then $f \in \overline{L}_\phi^{\delta,p,q}(\lambda, m, A, B)$ if and only if*

$$\sum_{k=2}^{\infty} k^\delta \xi_k^m |a_k| + \sum_{k=1}^{\infty} k^\delta \mu_k^m |b_k| \leq 1, \tag{3.1}$$

where ξ_k^m and μ_k^m are defined by (2.2).

Obviously, for any positive integer δ, we have the following inclusion relation:

$$\begin{aligned}\overline{L}_\phi^{\delta,p,q}(\lambda, m, A, B) &\subset \overline{L}_\phi^{1,p,q}(\lambda, m, A, B) = \overline{K}_\phi^{p,q}(\lambda, m, A, B)\\ &\subset \overline{L}_\phi^{0,p,q}(\lambda, m, A, B) = \overline{S}_\phi^{p,q}(\lambda, m, A, B).\end{aligned}$$

By Definition 3.1, we obtain the following conclusion.

Theorem 3.2 *Let $f(z) \in \bar{L}_{\phi}^{\delta,p,q}(\lambda, m, A, B), \xi_k^m$ and μ_k^m be given by (2.2) with $a_1 = u_1 = v_1 = 1$.*

(i) If $\{k^\delta \xi_k^m\}$ is a non-decreasing sequence, then

$$\mathrm{Re}\{\frac{f(z)}{f_\rho(z)}\} > 1 - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m}, \tag{3.2}$$

$$\mathrm{Re}\{\frac{f_\rho(z)}{f(z)}\} > \frac{(\rho+1)^\delta \xi_{\rho+1}^m}{1+(\rho+1)^\delta \xi_{\rho+1}^m}. \tag{3.3}$$

(ii) If $\{k^\delta \mu_k^m\}$ is a non-decreasing sequence, then

$$\mathrm{Re}\{\frac{f(z)}{f_\sigma(z)}\} > 1 - \frac{1}{(\sigma+1)^\delta \mu_{\sigma+1}^m}, \tag{3.4}$$

$$\mathrm{Re}\{\frac{f_\sigma(z)}{f(z)}\} > \frac{(\sigma+1)^\delta \mu_{\sigma+1}^m}{1+(\sigma+1)^\delta \mu_{\sigma+1}^m}. \tag{3.5}$$

The estimates of (3.2) and (3.3) are sharp for the function given by

$$f(z) = z - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m} z^{\rho+1}. \tag{3.6}$$

Also, the estimates of (3.4) and (3.5) are sharp for the function given by

$$f(z) = z + \frac{(-1)^m}{(\sigma+1)^\delta \mu_{\sigma+1}^m} \bar{z}^{\sigma+1}. \tag{3.7}$$

Proof (i) Let

$$\begin{aligned} F_1(z) &= (\rho+1)^\delta \xi_{\rho+1}^m [\frac{f(z)}{f_\rho(z)} - (1 - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m})] \\ &= 1 - \frac{(\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k| z^k}{z - \sum_{k=2}^\rho |a_k| z^k + (-1)^m \sum_{k=1}^\infty |b_k| \bar{z}^k}. \end{aligned}$$

To prove that the inequality (3.2) is true, we only need to show that $F_1(z)$ satisfies the following condition $|\frac{F_1(z)-1}{F_1(z)+1}| \le 1$ $(z \in \mathbb{U})$. Since

$$|\frac{F_1(z)-1}{F_1(z)+1}| \le \frac{(\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k|}{2 - 2(\sum_{k=2}^\rho |a_k| + \sum_{k=1}^\infty |b_k|) - (\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k|}, \tag{3.8}$$

the inequality (3.8) is bounded above by 1, if and only if

$$\sum_{k=2}^{\rho} |a_k| + \sum_{k=1}^\infty |b_k| + (\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k| \le 1. \tag{3.9}$$

According to Definition 3.1 and the increasing sequence $\{k^\delta \xi_k^m\}$ with $\xi_k^m \ge k$ and $\mu_k^m \ge k$ for $k \ge 1$, we have

$$\sum_{k=2}^{\rho} |a_k| + \sum_{k=1}^\infty |b_k| + (\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k| \le \sum_{k=2}^\infty k^\delta \xi_k^m |a_k| + \sum_{k=1}^\infty k^\delta \mu_k^m |b_k| \le 1. \tag{3.10}$$

For the function $f(z) = z - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m} z^{\rho+1}$ given by (3.6), let $z = re^{\frac{2\pi i}{\rho}}$ and $r \to 1^-$. We have

$$\frac{f(z)}{f_\rho(z)} = 1 - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m} r^\rho e^{2\pi i} \to 1 - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m},$$

which shows that the bound in (3.2) is sharp.

Similarly, let

$$F_2(z) = (1+(\rho+1)^\delta \xi_{\rho+1}^m)[\frac{f_\rho(z)}{f(z)} - \frac{(\rho+1)^\delta \xi_{\rho+1}^m}{1+(\rho+1)^\delta \xi_{\rho+1}^m}]$$
$$= 1 + \frac{(1+(\rho+1)^\delta \xi_{\rho+1}^m)\sum_{k=\rho+1}^\infty |a_k| z^k}{z - \sum_{k=2}^\infty |a_k| z^k + (-1)^m \sum_{k=1}^\infty |b_k| \bar{z}^k}.$$

We have

$$|\frac{F_2(z)-1}{F_2(z)+1}| \le \frac{(1+(\rho+1)^\delta \xi_{\rho+1}^m)\sum_{k=\rho+1}^\infty |a_k|}{2-2(\sum_{k=2}^\rho |a_k| + \sum_{k=1}^\infty |b_k|) - ((\rho+1)^\delta \xi_{\rho+1}^m - 1)\sum_{k=\rho+1}^\infty |a_k|}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^\rho |a_k| + \sum_{k=1}^\infty |b_k| + (\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k| \le 1.$$

According to Definition 3.1 and the increasing sequence $\{k^\delta \xi_k^m\}$ with $\xi_k^m \ge k$ and $\mu_k^m \ge k$ for $k \ge 1$, we have

$$\sum_{k=2}^\rho |a_k| + \sum_{k=1}^\infty |b_k| + (\rho+1)^\delta \xi_{\rho+1}^m \sum_{k=\rho+1}^\infty |a_k| \le \sum_{k=2}^\infty k^\delta \xi_k^m |a_k| + \sum_{k=1}^\infty k^\delta \mu_k^m |b_k| \le 1.$$

For the function $f(z) = z - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m} z^{\rho+1}$ given by (3.6), let $z = re^{\frac{\pi i}{\rho}}$ and $r \to 1^-$. We have

$$\frac{f_\rho(z)}{f(z)} = \frac{1}{1 - \frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m} r^\rho e^{\pi i}} \to \frac{(\rho+1)^\delta \xi_{\rho+1}^m}{1+(\rho+1)^\delta \xi_{\rho+1}^m},$$

which shows that the bound in (3.3) is sharp result.

Proof of (ii) is similar to that of (i), it is not difficult but too lengthy to give here. This completes the proof. □

Using the analogous methods to the proof in Theorem 3.2, we obtain the following Theorem.

Theorem 3.3 *Let ξ_k^m and μ_k^m be given by (2.2) and $f(z) \in \bar{L}_\phi^{\delta,p,q}(\lambda, m, A, B)$. If $\{k^\delta \xi_k^m\}$ and $\{k^\delta \mu_k^m\}$ are non-decreasing sequence, then*

$$\mathrm{Re}\{\frac{f(z)}{f_{\rho,\sigma}(z)}\} > 1 - \frac{1}{\min\{(\rho+1)^\delta \xi_{\rho+1}^m, (\sigma+1)^\delta \mu_{\sigma+1}^m\}}, \tag{3.11}$$

$$\mathrm{Re}\{\frac{f_{\rho,\sigma}(z)}{f(z)}\} > \frac{\min\{(\rho+1)^\delta \xi_{\rho+1}^m, (\sigma+1)^\delta \mu_{\sigma+1}^m\}}{1+\min\{(\rho+1)^\delta \xi_{\rho+1}^m, (\sigma+1)^\delta \mu_{\sigma+1}^m\}}. \tag{3.12}$$

Proof Note that $M = \min\{(\rho+1)^\delta \xi_{\rho+1}^m, (\sigma+1)^\delta \mu_{\sigma+1}^m\}$.

(i) Let

$$F_3(z) = M[\frac{f(z)}{f_{\rho,\sigma}(z)} - (1 - \frac{1}{M})] = 1 + \frac{M(-\sum_{k=\rho+1}^\infty |a_k| z^k + (-1)^m \sum_{k=\sigma+1}^\infty |b_k| z^k)}{z - \sum_{k=2}^\rho |a_k| z^k + (-1)^m \sum_{k=1}^\sigma |b_k| \bar{z}^k}.$$

To prove inequality (3.11), we only need to show that

$$|\frac{F_3(z)-1}{F_3(z)+1}| \le 1, \quad z \in \mathbb{U}.$$

Since

$$|\frac{F_3(z)-1}{F_3(z)+1}| \leq \frac{M(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|)}{2-2(\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|)-M(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|)}, \tag{3.13}$$

the inequality (3.13) is bounded above by 1 if and only if

$$\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|+M\Big(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|\Big) \leq 1. \tag{3.14}$$

According to Definition 3.1 and the increasing sequence $\{k^\delta \xi_k^m\}$ and $\{k^\delta \mu_k^m\}$ with $\xi_k^m \geq k$ and $\mu_k^m \geq k$ for $k \geq 1$, we have

$$\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|+M\Big(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|\Big) \leq \sum_{k=2}^{\infty}k^\delta \xi_k^m|a_k|+\sum_{k=1}^{\infty}k^\delta \mu_k^m|b_k| \leq 1.$$

If $\min\{(\rho+1)^\delta \xi_{\rho+1}^m,(\sigma+1)^\delta \mu_{\sigma+1}^m\}=(\rho+1)^\delta \xi_{\rho+1}^m$, we take the function $f(z)=z-\frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m}z^{\rho+1}$ and let $z=re^{\frac{2\pi i}{\rho}}$ with $r\to 1^-$. Then

$$\frac{f(z)}{f_{\rho,\sigma}(z)}=1-\frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m}z^\rho \to 1-\frac{1}{(\rho+1)^\delta \xi_{\rho+1}^m}.$$

If $\min\{(\rho+1)^\delta \xi_{\rho+1}^m,(\sigma+1)^\delta \mu_{\sigma+1}^m\}=(\sigma+1)^\delta \mu_{\sigma+1}^m$, we take the function

$$f(z)=z+(-1)^m\frac{1}{(\sigma+1)^\delta \mu_{\sigma+1}^m}\overline{z}^{\sigma+1}$$

and let $z=re^{\frac{(m-1)\pi i}{\sigma+2}}$ with $r\to 1^-$. Then

$$\frac{f(z)}{f_{\rho,\sigma}(z)}=\frac{z+(-1)^m\frac{1}{(\sigma+1)^\delta \mu_{\sigma+1}^m}\overline{z}^{\sigma+1}}{z}\to 1-\frac{1}{(\sigma+1)^\delta \mu_{\sigma+1}^m}.$$

It shows that the bound in (3.11) is sharp result.

(ii) Let

$$F_4(z)=(1+M)[\frac{f(z)}{f_{\rho,\sigma}(z)}-(\frac{M}{1+M})]=1+\frac{M(-\sum_{k=\rho+1}^{\infty}|a_k|z^k+(-1)^m\sum_{k=\sigma+1}^{\infty}|b_k|z^k)}{z-\sum_{k=2}^{\rho}|a_k|z^k+(-1)^m\sum_{k=1}^{\sigma}|b_k|\bar{z}^k}.$$

To prove inequality (3.12), we only need to show that

$$|\frac{F_4(z)-1}{F_4(z)+1}| \leq 1, \quad z\in\mathbb{U}.$$

Since

$$|\frac{F_4(z)-1}{F_4(z)+1}| \leq \frac{(1+M)(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|)}{2-2(\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|)-(M-1)(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|)}, \tag{3.15}$$

the inequality (3.15) is bounded above by 1 if and only if

$$\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|+M\Big(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|\Big) \leq 1. \tag{3.16}$$

According to Definition 3.1 and the increasing sequence $\{k^\delta \xi_k^m\}$ and $\{k^\delta \mu_k^m\}$ with $\xi_k^m \geq k$

and $\mu_k^m \geq k$ for $k \geq 1$, we have

$$\sum_{k=2}^{\rho}|a_k|+\sum_{k=1}^{\sigma}|b_k|+M\Big(\sum_{k=\rho+1}^{\infty}|a_k|+\sum_{k=\sigma+1}^{\infty}|b_k|\Big)\leq\sum_{k=2}^{\infty}k^{\delta}\xi_k^m|a_k|+\sum_{k=1}^{\infty}k^{\delta}\mu_k^m|b_k|\leq 1.$$

If $\min\{(\rho+1)^{\delta}\xi_{\rho+1}^m,(\sigma+1)^{\delta}\mu_{\sigma+1}^m\}=(\rho+1)^{\delta}\xi_{\rho+1}^m$, we take the function

$$f(z)=z-\frac{1}{(\rho+1)^{\delta}\xi_{\rho+1}^m}z^{\rho+1}$$

and let $z=re^{\frac{\pi i}{\rho}}$ with $r\to 1^-$. Then

$$\frac{f_{\rho,\sigma}(z)}{f(z)}=\frac{1}{1-\frac{1}{(\rho+1)^{\delta}\xi_{\rho+1}^m}z^{\rho}}\to\frac{(\rho+1)^{\delta}\xi_{\rho+1}^m}{1+(\rho+1)^{\delta}\xi_{\rho+1}^m}.$$

If $\min\{(\rho+1)^{\delta}\xi_{\rho+1}^m,(\sigma+1)^{\delta}\mu_{\sigma+1}^m\}=(\sigma+1)^{\delta}\mu_{\sigma+1}^m$, we take the function

$$f(z)=z+(-1)^m\frac{1}{(\sigma+1)^{\delta}\mu_{\sigma+1}^m}\overline{z}^{\sigma+1}$$

and let $z=re^{\frac{(m-2)\pi i}{\sigma+2}}$ with $r\to 1^-$. Then

$$\frac{f_{\rho,\sigma}(z)}{f(z)}=\frac{z}{z+(-1)^m\frac{1}{(\sigma+1)^{\delta}\mu_{\sigma+1}^m}\overline{z}^{\sigma+1}}\to\frac{(\sigma+1)^{\delta}\mu_{\sigma+1}^m}{1+(\sigma+1)^{\delta}\mu_{\sigma+1}^m}.$$

It shows that the bound in (3.12) is sharp result. This completes the proof. □

Remark 3.4 (i) Taking $\delta=0$ and $\delta=1$ in Theorems 3.2 and 3.3, we get partial sums properties of the classes $\bar{K}_{\phi}^{p,q}(\lambda,m,A,B)$ and $\bar{M}_{\phi}^{p,q}(\lambda,m,A,B)$, respectively.

(ii) Selecting different parameters λ,p,q,m,A,B and ϕ in Theorems 3.2 and 3.3, we can deduce new results for univalent harmonic function classes $HS^*(\beta)$ and $CH(\beta)$ mentioned in Section 1.

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