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# The Existence and Non-Existence of Positive Steady State Solutions for a Cross-Diffusion Predator-Prey Model with Holling Type II Functional Response

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**Abstract** In this paper, we consider the positive steady state solutions of a predator-prey model with Holling type II functional response and cross-diffusion, where two cross-diffusion rates represent the tendency of prey to keep away from its predator and the tendency of the predator to chase its prey, respectively. Applying the fixed point index theory, some sufficient conditions for the existence of positive steady state solutions are established. Furthermore, the non-existence of positive steady state solutions is studied.

**Keywords** predator-prey model; Holling type II functional response; cross-diffusion; coexistence states

**MR(2020) Subject Classification** 35A01; 35B45; 35K57

## 1. Introduction

In ecosystems, whether different species can coexist or not is determined by the combination of various factors, such as natural environments, interactions between the species, and behavioral patterns. Therefore, it is important to investigate what effect the above factors will have on coexistence problems. In this work, we study the effect of cross-diffusion on the existence and non-existence of positive steady state solutions to the following cross-diffusion predator-prey system with Holling type II functional response

$$\begin{cases} u_t - \Delta u - \gamma\Delta v = u(a-u) - \dfrac{buv}{e+u}, & x \in \Omega, t > 0, \\ v_t + \beta\Delta u - \Delta v = -cv + \dfrac{duv}{e+u}, & x \in \Omega, t > 0, \\ u = v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $u$ and $v$ stand for the densities of prey and predator, $\Omega$ is a bounded domain of $\mathrm{R}^N$ with smooth boundary $\partial\Omega$, and $a$, $b$, $c$, $d$, $e$ are all positive constants, $a$ is the growth rate of the prey, $b$ and $d$ represent the strength of the relative effect of the interaction on the two species; the function $u/(e+u)$ denotes the functional response of the predator to the prey, $c$ is the death

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rate of the predator. The positive constants $\gamma$ and $\beta$ are cross-diffusion rates. Biologically, the introduced cross-diffusion rate $\gamma$ in (1.1) represents the tendency of the prey to keep away from its predator, and $\beta$ represents the tendency of the predator to chase its prey, see for example [1–3]. The initial values $u_0$ and $v_0$ are nonnegative smooth functions which are not identically zero.

The system (1.1) is proposed on the basis of a model with Holling type II functional response

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a-u) - \dfrac{buv}{e+u}, & t>0, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -cv + \dfrac{duv}{e+u}, & t>0, \end{cases} \tag{1.2}$$

which has been extensively studied by many authors in either qualitative or numerical analysis, see for example [4–7]. When the densities of the prey and predator are spatially inhomogeneous in a bounded domain with smooth boundary, one introduces diffusion into system (1.2) to get the following reaction-diffusion system

$$\begin{cases} u_t - d_1\Delta u = u(a-u) - \dfrac{buv}{e+u}, & x\in\Omega,\ \ t>0, \\ v_t - d_2\Delta v = -cv + \dfrac{duv}{e+u}, & x\in\Omega,\ \ t>0. \end{cases} \tag{1.3}$$

For the system (1.3) with homogeneous Neumann boundary condition, the properties of solutions such as stability, bifurcation and spatiotemporal patterns have been well researched [8–11]. For the system (1.3) with homogeneous Dirichlet boundary condition, there are only a few results.

In [12], Zhou and Mu considered the steady state problem

$$\begin{cases} -\Delta u = u(a-u) - \dfrac{buv}{e+u}, & x\in\Omega, \\ -\Delta v = -cv + \dfrac{duv}{e+u}, & x\in\Omega, \\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{1.4}$$

By fixed point index theory and bifurcation theory, they proved that system (1.4) admits a coexistence state if and only if $a>\lambda_1$ and $-\lambda_1<c<-\lambda_1(-\frac{d\Theta}{\Theta+e})$, where $\lambda_1$ denotes the principle eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary condition and $\Theta$ is the unique positive solution of

$$\begin{cases} -\Delta\phi = \phi(a-\phi), & x\in\Omega, \\ \phi=0, & x\in\partial\Omega. \end{cases} \tag{1.5}$$

In the present paper, we introduce the cross-diffusion to system (1.4) and study the strongly coupled reaction-diffusion system (1.1). We point out that the cross-diffusion terms introduced into system (1.1) are different from the cross-diffusion rates of the forms $-\Delta[(1+\gamma v)u]$ and $-\Delta[(1+\beta u)v]$, which are introduced in the previous work [13-15], our main interest focuses on the effects of cross-diffusion on the existence and non-existence of positive steady state solutions of system (1.1). The steady state system of (1.1) is

$$\begin{cases} -\Delta u - \gamma\Delta v = u(a-u) - \dfrac{buv}{e+u}, & x \in \Omega, \\ \beta\Delta u - \Delta v = -cv + \dfrac{duv}{e+u}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

Our results show that system (1.6) has at least one positive solution provided that the cross diffusion rate $\beta$ is sufficiently small, $a > \lambda_1$ and $c$ is small appropriately (see Corollary 3.5).

This paper is organized as follows. In Section 2, some preliminaries are prepared. In Section 3, the sufficient conditions for the existence of positive solutions of system (1.6) are found. In Section 4, the non-existence theorem for positive solutions to system (1.6) is obtained.

## 2. Preliminaries and a priori estimates

In this section, some fundamental results are obtained. For simplicity, let

$$f_1(u,v) = u(a-u) - \frac{buv}{e+u}, \quad f_2(u,v) = -cv + \frac{duv}{e+u}.$$

Then system (1.6) is equivalent to the following system

$$\begin{cases} -\Delta u = \dfrac{1}{1+\gamma\beta}[f_1(u,v) - \gamma f_2(u,v)], & x \in \Omega, \\ -\Delta v = \dfrac{1}{1+\gamma\beta}[\beta f_1(u,v) + f_2(u,v)], & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We multiply the first equation by $\delta$ and subtract it from the second equation in system (2.1), to get

$$-\Delta(v - \delta u) = \frac{1}{1+\gamma\beta}[(\beta-\delta)f_1(u,v) + (1+\gamma\delta)f_2(u,v)],$$

where $\delta$ is a positive constant which will be determined later. Letting $\omega := v - \delta u$, system (2.1) is equivalent to

$$\begin{cases} -\Delta u = \dfrac{1}{1+\gamma\beta}\big[-(1+\dfrac{\delta(b+\gamma d)}{e+u})u^2 - \dfrac{(b+\gamma d)\omega u}{e+u} + au + c\gamma\delta u + c\gamma\omega\big], \ x \in \Omega, \\ -\Delta\omega = \dfrac{1}{1+\gamma\beta}\big[-c\omega(1+\gamma\delta) + \dfrac{d(1+\gamma\delta) - (\beta-\delta)b}{e+u}u\omega + u(M_1 + M_2 u)\big], \ x \in \Omega, \\ u = \omega = 0, x \in \partial\Omega, \end{cases} \tag{2.2}$$

where

$$M_1 := -c(1+\gamma\delta)\delta + a(\beta-\delta), \quad M_2 := \frac{d(1+\gamma\delta)\delta}{e+u} - \frac{b\delta(\beta-\delta)}{e+u} - (\beta-\delta). \tag{2.3}$$

For convenience, we denote the first and second equations in (2.2) by

$$-\Delta u = \tilde{f}_1(u,\omega), \quad -\Delta\omega = \tilde{f}_2(u,\omega).$$

Obviously, if system (2.2) has a positive solution $(u,\omega)$, then system (1.6) has a positive solution $(u,v)$. Thus we can prove the existence of positive solution of system (2.2) to show that system (2.1) has one positive solution at least. For $\theta \in (0,1]$, assume that $(u,\omega)$ is a positive solution of the following system

$$\begin{cases} -\Delta u = \theta \tilde{f}_1(u,\omega), & x \in \Omega, \\ -\Delta \omega = \theta \tilde{f}_2(u,\omega), & x \in \Omega, \\ u = \omega = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

we can obtain a priori estimates as follows.

**Lemma 2.1** *Any positive solution $(u,\omega)$ of system (2.4) satisfies*

$$u \le \frac{ae}{b\delta + e - a}(1 + \frac{\gamma(b\delta + e)}{b(1+\gamma\delta)}) := Q_1, \quad \omega \le \frac{ae}{b\delta + e - a}(\frac{1}{\gamma} + \frac{e}{b} + 2\delta) := Q_2.$$

**Proof** For any $\theta \in (0,1]$, we multiply the first equation in system (2.4) by $1+\gamma\delta$, the second by $\gamma$, and add them to get

$$-\Delta[(1+\gamma\delta)u + \gamma\omega] = u\big[a - (1 + \frac{b\delta}{e+u})u - \frac{b\omega}{e+u}\big], \quad x \in \Omega.$$

If $(1+\gamma\delta)u + \gamma\omega$ achieves its positive maximum at $x_0 \in \Omega$, then

$$-\Delta[(1+\gamma\delta)u(x_0) + \gamma\omega(x_0)] = u(x_0)\big[a - (1 + \frac{b\delta}{e+u(x_0)})u(x_0) - \frac{b\omega(x_0)}{e+u(x_0)}\big] \ge 0, \quad x \in \Omega.$$

Therefore,

$$a - (1 + \frac{b\delta}{e+u(x_0)})u(x_0) - \frac{b\omega(x_0)}{e+u(x_0)} \ge 0,$$

so that

$$(1 + \frac{b\delta}{e+u(x_0)})u(x_0) \le a, \quad \frac{b\omega(x_0)}{e+u(x_0)} \le a,$$

and thus

$$u(x_0) \le \frac{ae}{b\delta + e - a}, \quad \omega(x_0) \le \frac{a(e+u(x_0))}{b} \le \frac{ae}{b}(\frac{b\delta + e}{b\delta + e - a}).$$

These facts imply

$$\begin{aligned} \max_{x\in\bar{\Omega}}\{(1+\gamma\delta)u(x) + \gamma\omega(x)\} &= (1+\gamma\delta)u(x_0) + \gamma\omega(x_0) \\ &\le (1+\gamma\delta)\frac{ae}{b\delta+e-a} + \frac{\gamma ae}{b}(\frac{b\delta+e}{b\delta+e-a}), \end{aligned}$$

so we get the desired inequalities

$$\max_{x\in\bar{\Omega}} u \le \frac{ae}{b\delta + e - a}(1 + \frac{\gamma(b\delta + e)}{b(1+\gamma\delta)}), \quad \max_{x\in\bar{\Omega}} \omega \le \frac{ae}{b\delta + e - a}(\frac{1}{\gamma} + \frac{e}{b} + 2\delta).$$

It is easy to see that $Q_1 > 0$ and $Q_2 > 0$ if the parameter $e$ is larger than $a$. □

Now, we state the fixed point index theory, which is a fundamental tool in our proofs.

Let $E$ be a real Banach space and $W$ is the natural positive cone of $E$. For $y \in W$, define

$$\begin{aligned} W_y &= \{x \in E : y + rx \in W \text{ for some } r > 0\}, \\ S_y &= \{x \in \overline{W}_y : -x \in \overline{W}_y\}. \end{aligned}$$

Let $y_*$ be a fixed point of a compact operator $\mathcal{A}: W \to W$ and $\mathcal{L} = \mathcal{A}'(y_*)$ be the Fréchet derivative of $\mathcal{A}$ at $y_*$. We say that $\mathcal{L}$ has property $\alpha$ on $\overline{W}_{y_*}$, if there exists a $t \in (0,1)$ and a $y \in \overline{W}_{y_*} \setminus S_{y_*}$ such that $y - t\mathcal{L}y \in S_{y_*}$. For an open subset $U \subset W$, let $\mathrm{ind}_W(\mathcal{A}, U)$ be the Leray-Schauder degree $\deg_W(I - \mathcal{A}, U, 0)$, where $I$ is the identity map, the fixed of $\mathcal{A}$ at $y_*$ in $W$ is defined by

$$\mathrm{ind}_W(\mathcal{A}, y_*) := \mathrm{ind}(\mathcal{A}, U(y_*), W),$$

where $U(y_*)$ is a small open neighborhood of $y_*$ in $W$.

The following theorem follows from Lemma 4.1 of [16] (see also [17]).

**Theorem 2.2** *Assume that $I - \mathcal{L}$ is invertible on $\overline{W}_{y_*}$. If $\mathcal{L}$ has property $\alpha$ on $\overline{W}_{y_*}$, then*

$$\mathrm{ind}_W(\mathcal{A}, y_*) = 0.$$

We also introduce the following notations.

**Notation 2.3** *(i) $\lambda_1$ denotes the principal eigenvalue of $-\Delta$ on $\Omega$ corresponding to homogeneous Dirichlet boundary condition, $\varphi_1 > 0$ is the principal eigenfunction corresponding to $\lambda_1$.*

*(ii) $E := C_D(\bar{\Omega}) \oplus C_D(\bar{\Omega})$, where $C_D(\bar{\Omega}) := \{\phi \in C(\bar{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}$.*

*(iii) $N := N_Q \oplus N_Q$, where $N_Q := \{\phi \in C_D(\bar{\Omega}) : \phi < \max\{Q_1, Q_2\} + 1 \text{ in } \bar{\Omega}\}$.*

*(iv) $W := K \oplus K$, where $K := \{\phi \in C_D(\bar{\Omega}) : 0 \le \phi(x), x \in \bar{\Omega}\}$.*

*(v) $N' := N \cap W$.*

## 3. The existence of positive steady-state solution

In this section, we get some sufficient conditions for system (1.6) has a coexistence state by applying fixed point index theory.

Choose

$$\delta := \frac{-(a+c) + \sqrt{(a+c)^2 + 4ac\gamma\beta}}{2\gamma c} \tag{3.1}$$

in (2.3) such that $M_1 = 0$. Therefore,

$$\beta - \delta = \frac{c(1+\gamma\delta)\delta}{a},$$
$$M_2 = \frac{d(1+\gamma\delta)\delta}{e+u} - \frac{b\delta(\beta-\delta)}{e+u} - (\beta-\delta) = \frac{(1+\gamma\delta)\delta}{a(e+u)}[ad - (b\delta + e + u)c].$$

Take $P$ to be sufficiently large positive constant with

$$P \ge \frac{1}{1+\gamma\beta} \max\big\{[1 + \frac{\delta(b+\gamma d)}{e}]Q_1 + \frac{(b+d\gamma)Q_2}{e}, c(1+\gamma\delta) + \frac{(\beta-\delta)bQ_1}{e}\big\},$$

such that $\tilde{f}_1(u,\omega) + Pu$ and $\tilde{f}_2(u,\omega) + P\omega$ are respectively monotone increasing with respect to $u$ and $\omega$ for all $(u,\omega) \in [0, Q_1] \times [0, Q_2]$. Define a compact operator $\mathcal{A}: W \to W$ by

$$\mathcal{A}(u,\omega) := (-\Delta + PI)^{-1} \begin{pmatrix} \tilde{f}_1(u,\omega) + Pu \\ \\ \tilde{f}_2(u,\omega) + P\omega \end{pmatrix}.$$

Here, $W := K \oplus K$ and $K := \{\phi \in C_D(\bar{\Omega}) : 0 \le \phi(x), x \in \bar{\Omega}\}$.

If the condition

$$\beta \le \frac{ad}{bc} - \frac{e + Q_1}{b} \tag{H}$$

holds, then $\beta - \delta = \frac{c(1+\gamma\delta)\delta}{a} > 0$, and thus $M_2 \ge 0$. This implies that the operator $\mathcal{A}$ is positive in $N'$.

**Remark 3.1** Note that system (2.2) is equivalent to $(u, \omega) = \mathcal{A}(u, \omega)$. Therefore, it suffices to prove that $\mathcal{A}$ has a positive fixed point in $N'$ to show that system (2.2) has a positive solution.

Since system (2.2) has no semi-trivial solutions, we only need to calculate $\text{ind}_W(\mathcal{A}, N')$ and $\text{ind}_W(\mathcal{A}, (0,0))$ to discuss the existence of positive steady-state solutions.

**Lemma 3.2** *Assume that (H) holds, then* $\deg(I - \mathcal{A}, N', 0) = 1$.

**Proof** Define a homotopy $\mathcal{A}_\theta$: $E \to E$ by

$$\mathcal{A}_\theta(u, \omega) = (-\Delta + PI)^{-1} \begin{pmatrix} \theta[\tilde{f}_1(u, \omega) + Pu] \\ \theta[\tilde{f}_2(u, \omega) + P\omega] \end{pmatrix}$$

for $\theta \in [0, 1]$. From Lemma 2.1, it is easy to see that $\mathcal{A}_\theta$ has no fixed point on $\partial N'$, so $\deg(I - \mathcal{A}_\theta, N', 0)$ is well defined and $\deg(I - \mathcal{A}_\theta, N', 0)$ is independent of $\theta$. Therefore,

$$\deg(I - \mathcal{A}_0, N', 0) = \deg(I - \mathcal{A}, N', 0).$$

From the normalization properties of the degree, $\deg(I - \mathcal{A}_0, N', 0) = 1$, which yields the desired result. □

**Lemma 3.3** *Assume that (H) holds. If* $\lambda_1 < \frac{a+c\gamma\delta}{1+\gamma\beta}$, *then* $\text{ind}_W(\mathcal{A}, (0,0)) = 0$.

**Proof** A straightforward calculation shows that $\overline{W}_{(0,0)} = W$, $S_{(0,0)} = \{(0,0)\}$. Define

$$\mathcal{L} := \mathcal{A}'(0,0) = (-\Delta + PI)^{-1} \begin{pmatrix} \dfrac{a + c\gamma\delta}{1 + \gamma\beta} + P & \dfrac{\gamma c}{1 + \gamma\beta} \\ 0 & -\dfrac{c(1 + \gamma\delta)}{1 + \gamma\beta} + P \end{pmatrix}.$$

Firstly, we show that $I - \mathcal{L}$ is invertible on $W$. Assume $\mathcal{L}(\phi, \psi)^\top = (\phi, \psi)^\top$ for some $(\phi, \psi)^\top \in W$, then

$$\begin{cases} -\Delta\phi = \dfrac{a + \gamma c\delta}{1 + \gamma\beta}\phi + \dfrac{\gamma c}{1 + \gamma\beta}\psi, & x \in \Omega, \\ -\Delta\psi = -\dfrac{c(1 + \gamma\delta)}{1 + \gamma\beta}\psi, & x \in \Omega, \\ (\phi, \psi) = (0, 0), & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Since all eigenvalues of $-\Delta$ under the homogeneous Dirichlet boundary condition are positive, we conclude that $\psi \equiv 0$ in $\Omega$ from the second equation of system (3.2). Substituting $\psi \equiv 0$ in the first equation of system (3.2) and multiplying $\varphi_1$, and then integrating it on $\Omega$, we have

$$0 = \int_\Omega \varphi_1 \big(\Delta\phi + \frac{a + \gamma c\delta}{1 + \gamma\beta}\phi\big) dx = \int_\Omega \phi \big(\Delta\varphi_1 + \frac{a + \gamma c\delta}{1 + \gamma\beta}\varphi_1\big) dx$$

$$= \int_\Omega \varphi_1 \phi\big( -\lambda_1 + \frac{a+\gamma c\delta}{1+\gamma\beta}\big)\mathrm{d}x. \tag{3.3}$$

From the facts $\lambda_1 < (a+\gamma c\delta)/(1+\gamma\beta)$ and $\varphi_1 > 0$ in $\Omega$, we can see that $\phi \equiv 0$ in $\Omega$. This implies that $I - \mathcal{L}$ is invertible on $W$.

Furthermore, we prove that $\mathcal{L}$ has property $\alpha$. In fact, choosing $y = (\varphi_1, 0)$ and

$$t_1 := \frac{\lambda_1 + P}{\frac{a+\gamma c\delta}{1+\gamma\beta} + P},$$

it is easy to check that $t_1 \in (0,1)$, $(\varphi_1, 0) \in \overline{W}_{(0,0)} \setminus S_{(0,0)}$ and $(\varphi_1, 0)^\top - t_1 \mathcal{L}(\varphi_1, 0)^\top \in S_{(0,0)}$. Thus, $\mathrm{ind}_W(\mathcal{A}, (0,0)) = 0$ by Theorem 2.2. □

Now, we can prove that (H) and $\lambda_1 < \frac{a+\gamma c\delta}{1+\gamma\beta}$ are the sufficient conditions for system (1.6) to be of a coexistence state.

**Theorem 3.4** *If (H) and $\lambda_1 < \frac{a+\gamma c\delta}{1+\gamma\beta}$ hold, then system (1.6) has at least one positive solution, where $\delta$ is give in (3.1).*

**Proof** If (H) and $\lambda_1 < \dfrac{a+\gamma c\delta}{1+\gamma\beta}$ hold. By Lemmas 3.2 and 3.3,

$$\deg(\mathcal{I} - \mathcal{A}, N', 0) \neq \mathrm{ind}_W(\mathcal{A}, (0,0)) = 0,$$

which implies that system (1.6) has at least one positive solution. □

**Corollary 3.5** *(i) If $\beta \le \min\{\frac{ad}{bc} - \frac{e+Q_1}{b}, \frac{a-\lambda_1}{\gamma\lambda_1}\}$, then system (1.6) has at least one positive solution.*

*(ii) If $\frac{a-\lambda_1}{\gamma\lambda_1} < \beta < \min\{\frac{ad}{bc} - \frac{e+Q_1}{b}, \frac{(\lambda_1+c)(a-\lambda_1)}{\gamma\lambda_1^2}\}$, then system (1.6) has at least one positive solution.*

**Proof** (i) If $\beta \le \dfrac{a-\lambda_1}{\gamma\lambda_1}$, then $(1+\gamma\beta)\lambda_1 \le a$, which implies $\lambda_1 < \frac{a+\gamma c\delta}{1+\gamma\beta}$. From Theorem 3.4, we can get the desired result.

(ii) If $\frac{a-\lambda_1}{\gamma\lambda_1} < \beta$, then $2\lambda_1(1+\gamma\beta) > 2a > a - c$ and $2\lambda_1(1+\gamma\beta) - a + c > 0$. By a direct calculation, we find that

$$\lambda_1(1+\gamma\beta) - a - \gamma c\delta = \frac{1}{2}(2\lambda_1(1+\gamma\beta) - a + c - \sqrt{(a+c)^2 + 4ac\gamma\beta})$$

and

$$[2\lambda_1(1+\gamma\beta) - a + c]^2 - (a+c)^2 - 4ac\gamma\beta = 4(1+\gamma\beta)[\lambda_1^2\gamma\beta - (a-\lambda_1)(\lambda_1+c)].$$

Note that the assumption

$$\beta < \frac{(\lambda_1+c)(a-\lambda_1)}{\gamma\lambda_1^2} \Leftrightarrow \lambda_1^2\gamma\beta - (a-\lambda_1)(\lambda_1+c) < 0,$$

so we get

$$\lambda_1(1+\gamma\beta) - a - \gamma c\delta < 0.$$

From Theorem 3.4, we see that system (1.6) has at least one positive solution. The proof of Corollary 3.5 is completed. □

**Remark 3.6** Corollary 3.5 (i) implies that there exists a positive constant $\hat{\beta} := \hat{\beta}(a, c, \gamma, b, d, \lambda_1)$ such that system (1.6) has a positive solution provided that $\beta < \hat{\beta}$. Biologically, this means that the prey and predator species may coexist when the intrinsic growth rate of prey is greater than some level (i.e., $a > \max\{\lambda_1, (e + Q_1)c/d\}$), provided that the cross-diffusion $\beta$ is sufficiently small.

## 4. The no-existence of positive steady-state solution

In this section, we derive some sufficient conditions which make system (1.1) have no positive steady state solution.

**Theorem 4.1** *(i) If $\lambda_1 \geq \max\{a, \frac{ad}{e-a}(1 + \frac{\gamma e}{b})\}$, then system (1.1) has no positive steady state solution.*

*(ii) There exists a positive constant $\bar{\beta} := \bar{\beta}(a, c, \gamma, \lambda_1)$ such that system (1.1) has no positive steady state solution provided that $\beta \geq \bar{\beta}$.*

*(iii) There exists a positive constant $\bar{\gamma} := \bar{\gamma}(a, c, \beta, \lambda_1)$ such that system (1.1) has no positive steady state solution provided that $\gamma \geq \bar{\gamma}$.*

**Proof** (i) Assume $(u, v)$ is a coexistence state of system (1.6). Multiplying both sides of the first equation in system (1.6) by $u$, and integrating by parts on $\Omega$, we have

$$\gamma \int_\Omega \nabla u \nabla v \mathrm{d}x = \int_\Omega u^2 (a - u - \frac{bv}{e + u}) \mathrm{d}x - \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{4.1}$$

Similarly, we can get

$$-\beta \int_\Omega \nabla u \nabla v \mathrm{d}x = \int_\Omega v^2 (-c + \frac{du}{e + u}) \mathrm{d}x - \int_\Omega |\nabla v|^2 \mathrm{d}x. \tag{4.2}$$

Note that system (1.6) is exactly equivalent to system (2.2) for $\delta = 0$, so Lemma 2.1 implies that $(u, v)$ satisfies

$$u \leq \frac{ae}{e - a}(1 + \frac{\gamma e}{b}) := \tilde{Q}_1, \quad v \leq \frac{ae}{e - a}(\frac{1}{\gamma} + \frac{e}{b}) := \tilde{Q}_2.$$

Applying the Poincaré inequality to (4.1) and using the given assumption $a \leq \lambda_1$, we get

$$\begin{aligned} \gamma \int_\Omega \nabla u \nabla v \mathrm{d}x &\leq \int_\Omega u^2 (a - u) \mathrm{d}x - \int_\Omega \frac{bu^2 v}{e + u} \mathrm{d}x - \lambda_1 \int_\Omega u^2 \mathrm{d}x \\ &= \int_\Omega (a - \lambda_1) u^2 \mathrm{d}x - \int_\Omega (u + \frac{bv}{e + u}) u^2 \mathrm{d}x < 0. \end{aligned} \tag{4.3}$$

Then again using the Poincaré inequality for (4.2), we obtain

$$-\beta \int_\Omega \nabla u \nabla v \mathrm{d}x \leq \int_\Omega (\frac{du}{e + u} - \lambda_1 - c) v^2 \mathrm{d}x. \tag{4.4}$$

If $\frac{du}{e+u} - \lambda_1 \leq 0$ holds, then the contradiction can be derived from (4.3) and (4.4). In fact, the given assumption

$$\lambda_1 \geq \frac{ad}{e - a}(1 + \frac{\gamma e}{b})$$

implies $\lambda_1 \geq \frac{d\tilde{Q}_1}{e}$, which is a sufficient condition for

$$\frac{du}{e+u} - \lambda_1 \leq 0.$$

(ii) To be contrary, assume that system (1.6) has a positive solution $(u, v)$. Multiplying the first and second equations in system (1.6) by $v$ and $u$, respectively, and then integrating these equations over $\Omega$, we get

$$\begin{cases} \int_\Omega \nabla u \nabla v \mathrm{d}x = \int_\Omega uv(a - u - \frac{bv}{e+u})\mathrm{d}x - \gamma \int_\Omega |\nabla v|^2 \mathrm{d}x, \\ \int_\Omega \nabla u \nabla v \mathrm{d}x = \int_\Omega uv(-c + \frac{du}{e+u})\mathrm{d}x + \beta \int_\Omega |\nabla u|^2 \mathrm{d}x. \end{cases} \tag{4.5}$$

Therefore,

$$\gamma \int_\Omega |\nabla v|^2 \mathrm{d}x + \beta \int_\Omega |\nabla u|^2 \mathrm{d}x - (a+c) \int_\Omega uv \mathrm{d}x = -\int_\Omega uv(u + \frac{bv + du}{e+u})\mathrm{d}x. \tag{4.6}$$

By using the Poincaré inequality and Young inequality to (4.6), we obtain

$$\begin{aligned} &\gamma \int_\Omega |\nabla v|^2 \mathrm{d}x + \beta \int_\Omega |\nabla u|^2 \mathrm{d}x - (a+c) \int_\Omega uv \mathrm{d}x \\ &\geq (\gamma\lambda_1 - \frac{(a+c)\epsilon}{2}) \int_\Omega v^2 \mathrm{d}x + (\beta\lambda_1 - \frac{a+c}{2\epsilon}) \int_\Omega u^2 \mathrm{d}x, \end{aligned} \tag{4.7}$$

where $\epsilon$ is a positive constant. Choosing $\epsilon_0$ small enough such that

$$\gamma\lambda_1 - \frac{(a+c)\epsilon_0}{2} \geq 0,$$

then the left-hand side of (4.7) is nonnegative for

$$\beta \geq \hat{\beta} := \frac{a+c}{2\epsilon_0\lambda_1}.$$

This contradicts the fact that the right-hand side of (4.6) is negative.

(iii) Through some similar arguments as the proof of (ii), we can get the result, and so we omit here. □

**Remark 4.2** In view of Theorem 4.1, we may conclude that if the cross-diffusion rate of the prey or its predator is large enough, then the prey and predator species cannot coexist. In other words, the large cross-diffusion coefficients $\gamma$ and $\beta$ tend to mean no positive coexistence.

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