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Essential Norm of Toeplitz Operators on Dirichlet-Type Spaces

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Abstract In this paper, we show that, on the Dirichlet-type space of unit disk, the essential norm of a noncompact Toeplitz operator equals its distance to the set of compact Toeplitz operators, and moreover, this distance is realized by infinitely many compact Toeplitz operators, which is analogous to the case of the weighted Bergman space.

Keywords Essential norm; Toeplitz operator; Dirichlet-type space

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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and dA be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, we denote the weight by

$$\mathrm{d}A_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \mathrm{d}A(z).$$

The weighted Sobolev space $L^{2,\alpha}(\mathbb{D})$ is defined to be the collection of functions on $\mathbb{D}$ which satisfy

$$\|f\|_\alpha = \Big[\Big|\int_{\mathbb{D}} f \mathrm{d}A_\alpha(z)\Big|^2 + \int_{\mathbb{D}} (|\frac{\partial f}{\partial z}|^2 + |\frac{\partial f}{\partial \bar{z}}|^2) \mathrm{d}A_\alpha(z)\Big]^{\frac{1}{2}} < +\infty.$$

The space $L^{2,\alpha}(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f \mathrm{d}A_\alpha(z) \int_{\mathbb{D}} \bar{g} \mathrm{d}A_\alpha(z) + \int_{\mathbb{D}} (\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}}\overline{\frac{\partial g}{\partial \bar{z}}}) \mathrm{d}A_\alpha(z).$$

The Dirichlet-type space $\mathcal{D}_\alpha$ is the subspace of all analytic functions f in $L^{2,\alpha}(\mathbb{D})$ with $f(0) = 0$. It is easy to know that $\mathcal{D}_\alpha$ is a closed subspace of $L^{2,\alpha}(\mathbb{D})$. Let P_α be the orthogonal projection from $L^{2,\alpha}(\mathbb{D})$ onto $\mathcal{D}_\alpha$. In fact, P_α is an integral operator represented by

$$P_\alpha(f)(z) = \langle f, K_z^\alpha\rangle_\alpha = \int_{\mathbb{D}} \frac{\partial f}{\partial w}\overline{\frac{\partial K_z^\alpha}{\partial w}} \mathrm{d}A_\alpha(w),$$

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where $K_z^\alpha(w)$ is the reproducing kernel of Dirichlet-type space $\mathcal{D}_\alpha$. A calculation shows that

$$\|z^k\|_\alpha^2=\int_{\mathbb{D}}|kz^{k-1}|^2\mathrm{d}A_\alpha(z)=(1+\alpha)k^2B(k,1+\alpha),$$

where $k\in\mathbb{Z}$ and $B(x,y)$ is the Beta function. Therefore, the reproducing kernel has a form as follows

$$K_z^\alpha(w)=\sum_{k=1}^{\infty}\frac{w^k\bar{z}^k}{(1+\alpha)k^2B(k,1+\alpha)}.$$

Let $\widetilde{\mathcal{D}}_\alpha$ $(\alpha>-1)$ denote the subspace of all analytic function f in $L^{2,\alpha}(\mathbb{D})$. The weighted Bergman space A_α^2 is the subspace of all analytic function f in Lebesgue space $L^2(\mathrm{d}A_\alpha)$ with respect to measure $\mathrm{d}A_\alpha$. That is,

$$L^2(\mathrm{d}A_\alpha)=\Big\{f:\|f\|_2=\Big(\int_{\mathbb{D}}|f(z)|^2\mathrm{d}A_\alpha(z)\Big)^{\frac{1}{2}}<+\infty\Big\}.$$

It is well-known that if $\alpha>1$, then $\widetilde{\mathcal{D}}_\alpha=A_{\alpha-2}^2$. Hence $\widetilde{\mathcal{D}}_\alpha$ concludes A_α^2 as a subspace.

Let $L^\infty(\mathbb{D})$ denote the algebra of all essentially bounded measurable functions on $\mathbb{D}$, and $C(\mathbb{D})$ denote the algebra of all continuous functions on $\mathbb{D}$, and also let $H^\infty(\mathbb{D})$ denote the space of bounded analytic function on $\mathbb{D}$. Next we will introduce some familiar spaces as follows

$$L^{\infty,1}(\mathbb{D})=\{\varphi:\varphi,\frac{\partial\varphi}{\partial z},\frac{\partial\varphi}{\partial\bar{z}}\in L^\infty(\mathbb{D})\}.$$

Correspondingly, its norm is defined by

$$\|\varphi\|_{\infty,1}=\max\{\|\varphi\|_\infty,\|\frac{\partial\varphi}{\partial z}\|_\infty,\|\frac{\partial\varphi}{\partial\bar{z}}\|_\infty\}.$$

In the following, we introduce some notations for the subsequent use.

$$C^1(\bar{\mathbb{D}})=\{\varphi\in C(\bar{\mathbb{D}}):\varphi,\frac{\partial\varphi}{\partial z},\frac{\partial\varphi}{\partial\bar{z}}\in C(\bar{\mathbb{D}})\};$$
$$C_0^1(\bar{\mathbb{D}})=\{\varphi\in C^1(\bar{\mathbb{D}}):\varphi|_{\partial\mathbb{D}}=0\};$$
$$H^{\infty,1}(\mathbb{D})=\{\varphi\in H^\infty(\mathbb{D}):\varphi,\frac{\partial\varphi}{\partial z},\frac{\partial\varphi}{\partial\bar{z}}\in H^\infty(\mathbb{D})\}.$$

It is well-known that $C^1(\bar{\mathbb{D}})$ is the subspace of $L^{\infty,1}(\mathbb{D})$ and it is indeed a norm-closed algebra with respect to the norm

$$\|\varphi\|_*=\max_{z\in\bar{\mathbb{D}}}\max\{|\varphi|,|\frac{\partial\varphi}{\partial z}|,|\frac{\partial\varphi}{\partial\bar{z}}|\}.$$

For $\varphi\in L^{\infty,1}(\mathbb{D})$, the Toeplitz operator $T_\varphi:\mathcal{D}_\alpha\to\mathcal{D}_\alpha$ is denoted by

$$T_\varphi f(z)=P_\alpha(\varphi f)(z)=\int_{\mathbb{D}}\frac{\partial(f\varphi)}{\partial w}\overline{\frac{\partial K_z^\alpha(w)}{\partial w}}\mathrm{d}A_\alpha(w),\quad\forall f\in\mathcal{D}_\alpha.$$

Respectively, the Hankel operator $H_\varphi:\mathcal{D}_\alpha\to\mathcal{D}_\alpha^\perp$, with symbol $\varphi\in L^{\infty,1}(\mathbb{D})$, is defined as

$$H_\varphi f=(I-P_\alpha)(\varphi f),\quad\forall f\in\mathcal{D}_\alpha.$$

Accordingly, the Toeplitz operator on the weighted Bergman space A_α^2 is given as

$$\mathcal{T}_\psi g(z)=\mathcal{P}(\psi g)(z)=\int_{\mathbb{D}}\psi(w)g(w)\overline{\mathbb{K}_z^\alpha(w)}\mathrm{d}A_\alpha(w)$$

for $\psi \in L^\infty(\mathbb{D}), g \in A^2_\alpha$, where $\mathcal{P}$ is the orthogonal projection from $L^2(\mathrm{d}A_\alpha)$ to A^2_α and $\mathbb{K}^\alpha_z(w)$ is the reproducing kernel in A^2_α.

Let $\mathcal{K}(H)$ denote the set of compact operators on Hilbert space H. The essential norm of an operator T is defined by

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(H)\},$$

where $\|\cdot\|$ is the operator norm of operator space. This means the essential norm is the distance to the space of compact operators.

At the beginning, on the Hardy space $H^2(\mathbb{D})$, a theorem of Nehari stated in [1] that

$$\|H_f\|_e = \mathrm{dist}(f, H^\infty + C(\partial\mathbb{D}))$$

and Axler etc. obtained in [2] the following beautiful result:

Theorem 1.1 *Let H_f be a noncompact Hankel operator on Hardy space $H^2(\mathbb{D})$. Then there exist infinitely many different compact Hankel operators H_ϕ such that $\|H_f\|_e = \|H_f - H_\phi\|$.*

In other words, for a noncompact Hankel operator H_f with symbol $f \in L^\infty(\mathbb{D})$ on Hardy space $H^2(\mathbb{D})$, its distance to the space of compact operators is realized by infinitely many compact Hankel operators.

For Toeplitz operators on the weighted Bergman space A^2_α, Li [3] showed the analogous result:

Theorem 1.2 *Let $f \in L^\infty(\mathbb{D})$, and T_f be the associated noncompact Toeplitz operator on A^2_α. Then there exist infinitely many different compact Toeplitz operators T_ϕ with symbol ϕ in $C_0(\bar{\mathbb{D}})$ such that $\|T_f\|_e = \|T_f - T_\phi\|$.*

Similarly, Li also obtained in [3] the analogous result about Hankel operator on A^2_α. Motivated by the ideas of Li, we mainly consider the essential norm of Toeplitz operator on the Dirichlet-type space $\mathcal{D}_\alpha$ of unit disk in this note. Just as you see in [3], our results will be also based on the following theorem. See [2] for more details.

Theorem 1.3 *Let H_1 and H_2 be two Hilbert spaces and $T : H_1 \to H_2$ a noncompact bounded operator. Let $\{T_n\}_{n\geq 1}$ be a sequence of compact operators from H_1 to H_2 such that $T_n \to T$ and $T_n^* \to T^*$ in the strong operator topology. Then there exist sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of nonnegative real numbers such that $\sum_{n\geq 1} a_n = \sum_{n\geq 1} b_n = 1$ and $\|T - K_1\| = \|T - K_2\| = \|T\|_e$, where $K_1 = \sum_{n\geq 1} a_n T_n$ and $K_2 = \sum_{n\geq 1} b_n T_n$; Moreover, $K_1 \neq K_2$.*

In the sequel, we show that, on the Dirichlet-type space, the essential norm of a noncompact Toeplitz operator equals its distance to the set of compact Toeplitz operators and is realized by infinitely many compact Toeplitz operators.

2. Main results

From now on, we are going to consider the essential norm of Toeplitz operators on Dirichlet-type space $\mathcal{D}_\alpha$. The following theorem is the main theorem.

Theorem 2.1 *Let $f \in L^{\infty,1}(\mathbb{D})$ and T_f the associated noncompact Toeplitz operator on $\mathcal{D}_\alpha$.*

There exist infinitely many distinct compact Toeplitz operators T_φ *with symbol* $\varphi \in C_0^1(\overline{\mathbb{D}})$ *such that* $\|T_f\|_e = \|T_f - T_\varphi\|$.

In order to prove the above Theorem, we need some lemmas.

About Toeplitz operator on Dirichlet space $\mathcal{D}$, Cao had showed in [4] that

Lemma 2.2 *If* $\varphi \in C^1(\overline{\mathbb{D}})$, *then the Toeplitz operator* T_φ: $\mathcal{D} \to \mathcal{D}$ *is a compact operator if and only if* $\varphi|_{\partial\mathbb{D}} = 0$.

In [4] Cao also pointed out that it is possible to extend Lemma 2.2 from $\mathcal{D}$ to $\mathcal{D}_\alpha$, for the sake of completeness, we prove the following lemma.

Lemma 2.3 $\varphi \in C_0^1(\overline{\mathbb{D}})$ *if and only if the Toeplitz operator* T_φ: $\mathcal{D}_\alpha \to \mathcal{D}_\alpha$ *is compact.*

Proof If $\varphi \in C_0^1(\overline{\mathbb{D}})$, then $\varphi|_{\partial\mathbb{D}} = 0$, we have that $\mathcal{T}_{|\varphi|^2}$ is a compact operator on A_α^2. So for all $\{f_k\}_{k\geq 1} \subset A_\alpha^2$, $||f_k||_2 = 1$, $f_k \xrightarrow{w} 0$ $(k \to \infty)$, we have

$$\|\mathcal{T}_{|\varphi|^2} f_k\|_2 \to 0, \quad k \to \infty.$$

Suppose that T_φ is not a compact operator on $\mathcal{D}_\alpha$, then there exists $\{F_k\} \subset \mathcal{D}_\alpha$, $\|F_k\|_\alpha = 1$, $F_k \xrightarrow{w} 0$ $(k \to \infty)$, such that $\|\mathcal{T}_\varphi F_k\|_\alpha \nrightarrow 0$ $(k \to \infty)$, then $\|\varphi F_k\|_\alpha \nrightarrow 0$ $(k \to \infty)$. That is,

$$\int_{\mathbb{D}} \frac{\partial(\varphi F_k)}{\partial z} \overline{\frac{\partial(\varphi F_k)}{\partial z}} \mathrm{d}A_\alpha + \int_{\mathbb{D}} \frac{\partial(\varphi F_k)}{\partial \bar{z}} \overline{\frac{\partial(\varphi F_k)}{\partial \bar{z}}} \mathrm{d}A_\alpha \nrightarrow 0, \quad k \to \infty. \tag{2.1}$$

It is easy to calculate, by the property of derivative, that

$$\frac{\partial(\varphi F_k)}{\partial z} = \frac{\partial \varphi}{\partial z} F_k + \frac{\partial F_k}{\partial z} \varphi, \ \frac{\partial(\varphi F_k)}{\partial \bar{z}} = \frac{\partial \varphi}{\partial \bar{z}} F_k.$$

Therefore, the estimate of the second integral in (2.1) is as follows

$$\Big| \int_{\mathbb{D}} \frac{\partial(\varphi F_k)}{\partial \bar{z}} \overline{\frac{\partial(\varphi F_k)}{\partial \bar{z}}} \mathrm{d}A_\alpha \Big| \leq \|\varphi\|_*^2 \|F_k\|_2^2 \to 0, \quad k \to \infty,$$

where the embedding map $i : \mathcal{D}_\alpha \to A_\alpha^2$ is a compact operator, that is $\|F_k\|_2^2 \to 0$. On the other side, the first integral becomes

$$\begin{aligned} \int_{\mathbb{D}} \frac{\partial(\varphi F_k)}{\partial z} \overline{\frac{\partial(\varphi F_k)}{\partial z}} \mathrm{d}A_\alpha = & \int_{\mathbb{D}} \frac{\partial \varphi}{\partial z} F_k \overline{\frac{\partial \varphi}{\partial z} F_k} \mathrm{d}A_\alpha + \int_{\mathbb{D}} \frac{\partial \varphi}{\partial z} F_k \overline{\varphi F_k'} \mathrm{d}A_\alpha + \\ & \int_{\mathbb{D}} \varphi F_k' \overline{\frac{\partial \varphi}{\partial z} F_k} \mathrm{d}A_\alpha + \int_{\mathbb{D}} \varphi F_k' \overline{\varphi F_k'} \mathrm{d}A_\alpha. \end{aligned} \tag{2.2}$$

In views of the fact that $\|F_k'\|_2 = \|F_k\|_\alpha = 1$, $F_k' \xrightarrow{w} 0$ in A_α^2, from the last integral in (2.2) it follows that

$$\int_{\mathbb{D}} \varphi F_k' \overline{\varphi F_k'} \mathrm{d}A_\alpha = \int_{\mathbb{D}} |\varphi|^2 F_k' \overline{F_k'} \mathrm{d}A_\alpha = \int_{\mathbb{D}} \mathcal{T}_{|\varphi|^2} F_k' \overline{F_k'} \mathrm{d}A_\alpha \to 0, \quad k \to \infty.$$

It is trivial to estimate the first integral in (2.2) that

$$\Big| \int_{\mathbb{D}} \frac{\partial \varphi}{\partial z} F_k \overline{\frac{\partial \varphi}{\partial z} F_k} \mathrm{d}A_\alpha \Big| \leq \|\varphi\|_*^2 \|F_k\|_2^2,$$

and the second integral or the third integral in (2.2) is bounded as follows

$$\Big|\int_{\mathbb{D}} \frac{\partial \varphi}{\partial z} F_k \overline{\varphi F_k'} \mathrm{d}A_\alpha\Big| \leq \|\varphi\|_*^2 \|F_k\|_2 \|F_k'\|_2.$$

Combining all the analysis above, we have

$$\Big|\int_{\mathbb{D}} \frac{\partial(\varphi F_k)}{\partial z} \overline{\frac{\partial(\varphi F_k)}{\partial z}} \mathrm{d}A_\alpha\Big| \to 0, \quad k \to \infty,$$

that is contrary to $||\varphi F_k||_\alpha \nrightarrow 0$ $(k \to \infty)$. Then T_φ: $\mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is a compact operator.

Conversely, assume that T_φ is compact on $\mathcal{D}_\alpha$. We need to prove that $\varphi \in C_0^1(\overline{\mathbb{D}})$. Otherwise, $\mathcal{T}_{|\varphi|^2}$ is not compact on A_α^2, thus there is a sequence $\{f_k\}_{k\geq 1} \subset A_\alpha^2$, $f_k \xrightarrow{w} 0$, $\|f_k\|_2 = 1$ such that

$$\|\mathcal{T}_{|\varphi|^2} f_k\|_2 \nrightarrow 0, \quad k \to \infty,$$

that is

$$\int_{\mathbb{D}} |\varphi(z)|^2 |f_k(z)|^2 \mathrm{d}A_\alpha(z) \nrightarrow 0, \quad k \to \infty.$$

However T_φ is compact on $\mathcal{D}(\alpha)$, so for

$$F_k(z) = \int_0^z f_k(\xi) \mathrm{d}m(\xi),$$

$\|T_\varphi F_k\|_\alpha \to 0$ $(k \to \infty)$, furthermore, $\|T_\varphi^* T_\varphi F_k\|_\alpha \to 0$ $(k \to \infty)$. If we can claim that $T_\varphi^* - T_{\overline{\varphi}}$ is compact, we can know that $\|T_{|\varphi|^2} F_k\|_\alpha \to 0$, since $F_k \xrightarrow{w} 0$ and $\|F_k\|_\alpha = 1$. Therefore,

$$\int_{\mathbb{D}} \frac{\partial|\varphi|^2}{\partial z} F_k \overline{F_k} \mathrm{d}A_\alpha + \int_{\mathbb{D}} |\varphi|^2 F_k' \overline{F_k'} \mathrm{d}A_\alpha = \int_{\mathbb{D}} \frac{\partial(|\varphi|^2 F_k)}{\partial z} \overline{F_k} \mathrm{d}A_\alpha \to 0, \quad k \to \infty.$$

Unfortunately, from the analysis above,

$$\int_{\mathbb{D}} |\varphi|^2 F_k' \overline{F_k'} \mathrm{d}A_\alpha \nrightarrow 0, \quad k \to \infty.$$

And note that

$$\Big|\int_{\mathbb{D}} \frac{\partial|\varphi|^2}{\partial z} F_k \overline{F_k} \mathrm{d}A_\alpha\Big| \leq \||\varphi|^2\|_* \|F_k\|_\alpha \|f_k\|_\alpha \to 0, \quad k \to \infty.$$

This contradiction shows $\varphi \in C_0^1(\overline{\mathbb{D}})$.

Next we will prove the key step that $T_\varphi^* - T_{\overline{\varphi}}$ is compact. For any $f, g \in \mathcal{D}_\alpha$,

$$\begin{aligned}|\langle (T_\varphi^* - T_{\overline{\varphi}}) f, g\rangle_\alpha| &\leq \|\frac{\partial f}{\partial z}\|_2 \|\frac{\partial \varphi}{\partial z} g\|_2 + \|f\|_2 \|\frac{\partial f}{\partial z} \frac{\partial g}{\partial z}\|_2 \\ &\leq \|\varphi\|_* \|g\|_2 \|f\|_\alpha + \|\varphi\|_* \|f\|_2 \|g\|_\alpha.\end{aligned}$$

Hence for any $\{F_k\} \subset \mathcal{D}_\alpha$, $\|F_k\|_\alpha = 1, F_k \xrightarrow{w} 0$ $(k \to \infty)$,

$$\|(T_\varphi^* - T_{\overline{\varphi}}) F_k\|_\alpha^2 \leq \|\varphi\|_* \|(T_\varphi^* - T_{\overline{\varphi}}) F_k\|_2 + \|\varphi\|_* \|F_k\|_2 \|(T_\varphi^* - T_{\overline{\varphi}}) F_k\|_\alpha.$$

Lastly, if we note that $(T_\varphi^* - T_{\overline{\varphi}}) F_k \xrightarrow{w} 0$ $(k \to \infty)$ or $\|(T_\varphi^* - T_{\overline{\varphi}}) F_k\|_2 \to 0$ $(k \to \infty)$, we can obtain the result. And the proof is completed. □

In order to use Theorem 1.3 to prove Theorem 2.1, we need to establish all the conditions in the premise; i.e., there exists a sequence of functions $\varphi_n \in C_0^1(\overline{\mathbb{D}})$ such that the sequence of compact Toeplitz operators T_{φ_n} and $T_{\varphi_n}^*$, respectively, converge to T_f and T_f^* in the strong operator topology. We start by approximating $f \in L^\infty(\mathbb{D})$ by continuous functions.

Suppose δ is a positive smooth function on the complex plane such that

(1) δ is compactly supported and identically zero outside of $\mathbb{D}$,

(2) $\int_{\mathbb{C}} \delta(z)\mathrm{d}A_\alpha(z) = 1$,

(3) For $\varepsilon > 0$, $\lim_{\varepsilon\to 0} \delta_\varepsilon(z)$ is a Dirac delta function where $\delta_\varepsilon(z) = \frac{1}{\varepsilon^2}\delta(\frac{z}{\varepsilon})$,

(4) $\int_{|z|>\varepsilon} \delta_\varepsilon(z)\mathrm{d}A_\alpha(z) = 0$.

Then δ_ε is called a mollifier and $\int_{\mathbb{C}} \delta_\varepsilon(z)\mathrm{d}A_\alpha(z) = 1$.

For any nonzero $f \in L^\infty(\mathbb{D})$, we can extend it to the whole complex plane $\mathbb{C}$ by taking it to be zero outside of $\mathbb{D}$. For convenience, we will denote it by the same function f and so we can assume $f \in L^\infty_{\mathrm{loc}}(\mathbb{C})$; thus, $f \in L^1_{\mathrm{loc}}(\mathbb{C}, \mathrm{d}A_\alpha)$, since

$$\int_{\mathbb{C}} |f(z)|\mathrm{d}A_\alpha(z) = \int_{\mathbb{D}} |f(z)|\mathrm{d}A_\alpha(z) \le \|f\|_\infty < \infty.$$

We define the convolution

$$\delta_\varepsilon * f(z) = \int_{\mathbb{C}} \delta_\varepsilon(z-w)f(w)\mathrm{d}A_\alpha(w) = \int_{\mathbb{C}} \delta_\varepsilon(w)f(z-w)\mathrm{d}A_\alpha(w).$$

For each fixed $z \in \mathbb{D}$, the non-trivial domain of integration for $\int_{\mathbb{C}} \delta_\varepsilon(z-w)f(w)\mathrm{d}A_\alpha(z)$ is the disk centered at z and of radius ε. Note the convolution is still defined for $z \in \partial\mathbb{D}$, hence $\delta_\varepsilon * f$ is a mollification of f.

It is well known that

(1) $\delta_\varepsilon * f \in C^\infty(\mathbb{C}, \mathrm{d}A_\alpha)$,

(2) $\delta_\varepsilon * f \in L^2(\mathbb{C}, \mathrm{d}A_\alpha)$ and $\|\delta_\varepsilon * f - f\|_2 \to 0$ as $\varepsilon \to 0$.

The reader may refer to [5–7] for more information.

Note, even if the function $f \in L^\infty(\mathbb{D})$ is zero on the boundary $\partial\mathbb{D}$, the convolution $\delta_\varepsilon * f$ may not be identically zero on $\partial\mathbb{D}$. We will need to modify the convolution to make sure that does not happen.

For a sequence r_n such that $0 < r_n < 1$ and $z \in \mathbb{D}$, we define $f_{r_n} = f(z)$ if $|z| < r_n$ and $f_{r_n} = 0$ else. Under the same assumption, we have $\frac{\partial f_{r_n}}{\partial z} = \frac{\partial f}{\partial z}$ and $\frac{\partial f_{r_n}}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}}$, when $|z| < r_n$, while are zero for else. So by the definition of convolution, we can see that

$$\frac{\partial(\delta_\varepsilon * f_{r_n})}{\partial z} = \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial z},\ \frac{\partial(\delta_\varepsilon * f_{r_n})}{\partial \bar{z}} = \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial \bar{z}}.$$

We claim that for any $\varepsilon > 0$ the convolution $\delta_\varepsilon * f_{r_n}$ has the following properties which are from [3].

Lemma 2.4 *For any $f \in L^\infty(\mathbb{D})$, we have the following facts:*

(1) $\delta_\varepsilon * f_{r_n}(z)$ converges to $\delta_\varepsilon * f(z)$ pointwise as $r_n \to 1$;

(2) If $\mathrm{dist}(r_n\mathbb{D}, \partial\mathbb{D}) > \varepsilon$, the convolution $\delta_\varepsilon * f_{r_n}$ is equal to zero on $\partial\mathbb{D}$.

In the sequence, we will prove the main theorem.

Proof of Theorem 2.1 For any $f \in L^{\infty,1}(\mathbb{D})$, that is, $f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \in L^\infty(\mathbb{D})$, by Lemma 2.4, we obtain that a sequence of functions $\delta_\varepsilon * f_{r_n}, \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial z}, \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial \bar{z}}$ belong to $C_0(\overline{\mathbb{D}})$, what is more, $\delta_\varepsilon * f_{r_n} \in C^1_0(\overline{\mathbb{D}})$. By Lemma 2.3, it was established that $T_{\delta_\varepsilon * f_{r_n}}$ and $T^*_{\delta_\varepsilon * f_{r_n}}$ are both compact on $\mathcal{D}_\alpha$ for $0 < r_n < 1$.

First, we will show that $T_{\delta_\varepsilon * f_{r_n}}$ converges to T_f in the strong operator topology as $r_n \to 1$.

It is well-known that the subalgebra $H^{\infty,1}(\mathbb{D})$ is dense in $\mathcal{D}_\alpha$, i.e., $\forall \varepsilon_0 > 0$, for any $g \in \mathcal{D}_\alpha$, there exists a $g_1 \in H^{\infty,1}(\mathbb{D})$ such that $\|g - g_1\|_\alpha < \epsilon_0$, that is, $\|g' - g_1'\|_2 < \varepsilon_0$.

Since $f \in L^{\infty,1}(\mathbb{D})$, that is, $f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \in L^\infty(\mathbb{D})$. It is easy to see that

$$\max\{\|\delta_\varepsilon * f_{r_n}\|_\infty, \|\delta_\varepsilon * \frac{\partial f_{r_n}}{\partial z}\|_\infty, \|\delta_\varepsilon * \frac{\partial f_{r_n}}{\partial \bar{z}}\|_\infty\} \leq \|f\|_{\infty,1}.$$

And the fact that the orthogonal projection P_α is a bounded operator on $\mathcal{D}_\alpha$, we have

$$\begin{aligned}\|(T_f - T_{\delta_\varepsilon * f_{r_n}})g\|_\alpha \leq & \|(T_f - T_{\delta_\varepsilon * f_{r_n}})(g - g_1)\|_\alpha + \|(T_f - T_{\delta_\varepsilon * f_{r_n}})g_1\|_\alpha \\ = & \|(P_\alpha(f - \delta_\varepsilon * f_{r_n})(g - g_1)\|_\alpha + \|P_\alpha(f - \delta_\varepsilon * f_{r_n})g_1\|_\alpha \\ \leq & \|(f - \delta_\varepsilon * f_{r_n})(g - g_1)\|_\alpha + \|(f - \delta_\varepsilon * f_{r_n})g_1\|_\alpha. \end{aligned} \tag{2.3}$$

For the first part in (2.3), we can calculate that

$$\|(f - \delta_\varepsilon * f_{r_n})(g - g_1)\|_\alpha^2 = \|\frac{\partial}{\partial z}[(f - \delta_\varepsilon * f_{r_n})(g - g_1)]\|_2^2 + \|\frac{\partial}{\partial \bar{z}}[(f - \delta_\varepsilon * f_{r_n})(g - g_1)]\|_2^2. \tag{2.4}$$

An easy computation shows that the first norm in (2.4) is bounded as follows

$$\|\frac{\partial}{\partial z}[(f - \delta_\varepsilon * f_{r_n})(g - g_1)]\|_2^2 \leq \|\frac{\partial f}{\partial z} - \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial z}\|_\infty^2 \|g - g_1\|_2^2 \leq \|f\|_{\infty,1}^2 \|g - g_1\|_2^2.$$

While the other is achieved by the triangle inequality that

$$\begin{aligned}\|\frac{\partial}{\partial \bar{z}}[(f - \delta_\varepsilon * f_{r_n})(g - g_1)]\|_2^2 & \leq \|(\frac{\partial f}{\partial \bar{z}} - \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial \bar{z}})(g - g_1)\|_2^2 \\ & \leq \|\frac{\partial f}{\partial \bar{z}} - \delta_\varepsilon * \frac{\partial f_{r_n}}{\partial \bar{z}}|_\infty^2 \|g - g_1\|_2^2 \\ & \leq \|f\|_{\infty,1}^2 \|g - g_1\|_2^2. \end{aligned}$$

For the second part in (2.3), we can see that

$$\|(f - \delta_\varepsilon * f_{r_n})g_1\|_\alpha^2 \leq \|(f - \delta_\varepsilon * f)g_1\|_\alpha^2 + \|(\delta_\varepsilon * f - \delta_\varepsilon * f_{r_n})g_1\|_\alpha^2. \tag{2.5}$$

The first norm in (2.5) has the estimate

$$\|(f - \delta_\varepsilon * f)g_1\|_\alpha^2 \leq \|\frac{\partial f}{\partial z} - \delta_\varepsilon * \frac{\partial f}{\partial z}\|_2^2 \|g_1\|_\infty^2 + \|\frac{\partial f}{\partial \bar{z}} - \delta_\varepsilon * \frac{\partial f}{\partial \bar{z}}\|_2^2 \|g_1\|_\infty^2 + \|f - \delta_\varepsilon * f\|_2^2 \|g_1'\|_\infty^2.$$

When $f \in L^\infty(\mathbb{D})$, then $\|f - \delta_\varepsilon * f\|_2 \to 0$ as $\varepsilon \to 0$. Similarly,

$$\lim_{\varepsilon \to 0} \|\frac{\partial f}{\partial z} - \delta_\varepsilon * \frac{\partial f}{\partial z}\|_2 = 0, \ \lim_{\varepsilon \to 0} \|\frac{\partial f}{\partial \bar{z}} - \delta_\varepsilon * \frac{\partial f}{\partial \bar{z}}\|_2 = 0.$$

From Lemma 2.4, we see that $\delta_\varepsilon * f_{r_n} \to \delta_\varepsilon * f$ pointwise as $r_n \to 1$. For $g_1 \in H^{\infty,1}(\mathbb{D})$, we also have $(\delta_\varepsilon * f_{r_n})g_1 \to (\delta_\varepsilon * f)g_1$ pointwise as $r_n \to 1$. It is easy to see that $\|(\delta_\varepsilon * f)g_1\|_\alpha < \infty$. By the dominated convergence theorem $\|(\delta_\varepsilon * f - \delta_\varepsilon * f_{r_n})g_1\|_\alpha \to 0$ as $r_n \to 1$. Thus, $\|(T_f - T_{\delta_\varepsilon * f_{r_n}})g\|_\alpha \to 0$, as $r_n \to 1$ and ε, $\varepsilon_0 \to 0$. We have shown that $T_{\delta_\varepsilon * f_{r_n}}$ converges to T_f in the strong operator topology.

Similar to the previous argument, it is easy to see that $T^*_{\delta_\varepsilon * f_{r_n}}$ converges to T^*_f in the strong operator topology.

From Theorem 1.3, there exist two sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of nonnegative real numbers such that $\sum_{n\geq 1} a_n = \sum_{n\geq 1} b_n = 1$. Let

$$\varphi_1 = \sum_{n\geq 1} a_n \delta_\varepsilon * f_{r_n} \quad \text{and} \quad \varphi_2 = \sum_{n\geq 1} b_n \delta_\varepsilon * f_{r_n}.$$

Since we have known that $\delta_\varepsilon * f_{r_n}$, $\frac{\partial(\delta_\varepsilon * f_{r_n})}{\partial z}$ and $\frac{\partial(\delta_\varepsilon * f_{r_n})}{\partial \bar{z}}$ are continuous on $\overline{\mathbb{D}}$ and $\delta_\varepsilon * f_{r_n}|_{\partial\mathbb{D}} = 0$, then φ_1 and φ_2 are continuous on $\overline{\mathbb{D}}$ and equal to zero on $\partial\mathbb{D}$. This implies $T_{\delta_\varepsilon * f_{r_n}}$ is compact for $n \geq 1$ by Lemma 2.3. From the formula

$$T_{\varphi_1} = T_{\sum_{n\geq 1} a_n \delta_\varepsilon * f_{r_n}} = \sum_{n\geq 1} a_n T_{\delta_\varepsilon * f_{r_n}},$$

it follows that T_{φ_1} is compact, so is T_{φ_2}. Obviously, $T_{\varphi_1} \neq T_{\varphi_2}$.

The two distinct compact Toeplitz operators T_{φ_1} and T_{φ_2} satisfy $\|T_f\|_e = \|T_f - T_{\varphi_1}\| = \|T_f - T_{\varphi_2}\|$. Let $\varphi = t\varphi_1 + (1-t)\varphi_2$ for $t \in (0,1)$. Hence there exist infinitely many distinct compact Toeplitz operators T_φ with symbol $\varphi \in C_0^1(\overline{\mathbb{D}})$ such that $\|T_f\|_e = \|T_f - T_\varphi\|$. $\square$

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