

# Oscillation Property for the Eigenfunctions of Discrete Clamped Beam Equation and Its Applications

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**Abstract** In this article, we established the structure of all eigenvalues and the oscillation property of corresponding eigenfunctions for discrete clamped beam equation  $\Delta^4 u(k-2) = \lambda m(k)u(k)$ ,  $k \in [2, N+1]_{\mathbb{Z}}$ ,  $u(0) = \Delta u(0) = 0 = u(N+2) = \Delta u(N+2)$  with the weight function  $m : [2, N+1]_{\mathbb{Z}} \rightarrow (0, \infty)$ ,  $[2, N+1]_{\mathbb{Z}} = \{2, 3, \dots, N+1\}$ . As an application, we obtain the global structure of nodal solutions of the corresponding nonlinear problems based on the nonlinearity satisfying suitable growth conditions at zero and infinity.

**Keywords** eigenvalue; eigenfunctions; oscillation property; bifurcation point; nodal solutions

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## 1. Introduction

Nonlinear boundary value problems have important applications to physics, chemistry and biology. For example, the boundary value problems (shorthand BVPs)

$$\begin{cases} u^{(4)}(t) = m(t)f(u(t)), & t \in [0, 1], \\ u(0) = u'(0) = 0 = u(1) = u'(1) \end{cases} \quad (1.1)$$

arise in the study of elasticity and have definite physical meanings. The equation in (1.1) is often referred to as the beam equation, which describes the deflection of a beam under a certain force. The boundary condition in (1.1) means that the beam is clamped at both end points. The existence and multiplicity of positive solution for this problem (and its generalizations) have been widely investigated by many researchers [1–4]. However, there are little result in the literature for the positive solutions of the discrete analogue of (1.1).

It is well known that difference equations appear in numerous settings and forms, both as a fundamental tool in the discrete analogue of differential equation and as a useful model for several economical, medical and population problems [5]. Thus, the nonlinear fourth-order discrete BVPs have also been widely investigated by many researchers, such as [6–13]. In particular, the eigenvalues and eigenfunctions of linear BVPs play an important role on the existence of positive (nodal) solutions of nonlinear BVPs.

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Motivated above, we will devote this paper to establishing the structure of all eigenvalues and oscillation property of the corresponding eigenfunctions for the linear eigenvalue problem

$$\begin{cases} \Delta^4 u(k-2) = \lambda m(k)u(k), & k \in [2, N+1]_{\mathbb{Z}}, \\ u(0) = \Delta u(0) = 0 = u(N+2) = \Delta u(N+2), \end{cases} \tag{1.2}$$

where  $\lambda > 0$  is a parameter, the forward difference operator  $\Delta$  is defined as  $\Delta u(k) = u(k+1) - u(k)$ ,  $m : [2, N+1]_{\mathbb{Z}} \rightarrow (0, \infty)$  is a weight function and  $[2, N+1]_{\mathbb{Z}} := \{2, 3, \dots, N, N+1\}$  with  $N \geq 4$  is an integer. The property oscillation matrix and some novel techniques developed in [9,10] for the BVPs of the fourth order difference equation will be adopted to the new situation for such a task. Based on it and bifurcation theory, we obtain the global structure of nodal solutions for the discrete analogue of (1.1) as follows

$$\begin{cases} \Delta^4 u(k-2) = \lambda m(k)f(u(k)), & k \in [2, N+1]_{\mathbb{Z}}, \\ u(0) = \Delta u(0) = 0 = u(N+2) = \Delta u(N+2), \end{cases} \tag{1.3}$$

where  $m : [2, N+1]_{\mathbb{Z}} \rightarrow (0, \infty)$  and  $f \in C(\mathbb{R}, \mathbb{R})$  with  $f(s)s > 0, s \neq 0$ .

The rest of the paper is organized as follows. In Section 2, we give the structure of eigenvalues and eigenfunctions for linear problem (1.2), and as application, in Section 3, we discuss global structure of nodal solutions of (1.3).

## 2. The structure of eigenvalues and eigenfunctions

In this section, we denote by  $\mathbf{x}^*$  the conjugate transpose of a vector  $\mathbf{x}$  and  $\mathbf{x}^T$  the transpose of a vector  $\mathbf{x}$ . A Hermitian matrix  $A$  is said to be positive semidefinite if  $\mathbf{x}^* A \mathbf{x} \geq 0$  for any  $\mathbf{x}$ . It is said to be positive definite if  $\mathbf{x}^* A \mathbf{x} > 0$  for any nonzero  $\mathbf{x}$ . In what follows we will write  $X \geq Y$  if  $X$  and  $Y$  are Hermitian matrices of order  $n$  and  $X - Y$  is positive semidefinite.

A matrix  $A = (a_{ik})_{n \times n}$  will be called totally non-negative (or respectively totally positive) if all its minors of any order are non-negative (or respectively positive):

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} \geq 0 \text{ (resp., } > 0) \text{ for } \left( 1 \leq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ k_1 < k_2 < \cdots < k_p \end{matrix} \leq n, p = 1, 2, \dots, n \right).$$

Obviously, every element of totally positive matrix  $A$  is positive [14, 15].

From [14, P. 74], we can obtain the simplest properties of totally non-negative matrices.

- (1) A product of two totally non-negative matrices is totally non-negative.
- (2) A product of a totally positive matrix by a nonsingular totally non-negative matrix is a totally positive matrix.

**Definition 2.1** ([14, P. 76]) *A matrix  $A = (a_{ik})_{n \times n}$  is called oscillatory if  $A$  is totally non-negative and there exists an integer  $q > 0$  such that  $A^q$  is totally positive.*

The problem (1.2) is equivalent to the linear system

$$(-D + \lambda M)u = 0, \tag{2.1}$$

where  $M = \text{diag}(m(2), m(3), \dots, m(N), m(N+1))$ ,  $u = (u(2), u(3), \dots, u(N), u(N+1))^T$ , and

$D$  is a banded  $N \times N$  matrix given by

$$D = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -4 & 6 \end{pmatrix}. \tag{2.2}$$

Obviously, there is a one-to-one corresponding between the solution

$$\mathbf{u} = (u(2), u(3), \dots, u(N), u(N + 1)) \tag{2.3}$$

of the problem (2.1) and the solution  $(u(0), u(1), u(2), \dots, u(N + 1), u(N + 2), u(N + 3))^T$  of the problem (1.2). In that sense, these two problems are equivalent. We will not distinguish one from the other, denote by  $\mathbf{u}$  either one of these two vectors in the remainder of this paper, and say that problems (1.2) and (2.1) are equivalent.

By the similar argument as [9], we can directly obtain the following results.

**Lemma 2.2**  $D$  is positive definite and  $D^{-1}$  is a totally positive matrix.

**Lemma 2.3** If  $\lambda$  is an eigenvalue of the problem (1.2) and  $\mathbf{u} = (u(2), u(3), \dots, u(N), u(N + 1))^T$  is a corresponding eigenvector, then

- (i)  $\mathbf{u}^* M \mathbf{u} > 0$ ;
- (ii)  $\lambda$  is real and positive;
- (iii) If  $\mu \neq \lambda$  is an eigenvalue of the problem (1.2) and  $\mathbf{v} = (v(2), v(3), \dots, v(N), v(N + 1))^T$  is a corresponding eigenvector, then  $\mathbf{v}^* M \mathbf{u} = 0$ .

**Lemma 2.4** ([15, Vol. 2, P. 105]) A totally non-negative matrix  $A = (a_{ik})_{n \times n}$  is oscillatory if and only if (i)  $A$  is non-singular ( $|A| > 0$ ); (ii) All the elements of  $A$  in the principal diagonal and the first super-diagonals and sub-diagonals are different from zero ( $a_{ik} > 0$  for  $|i - k| \leq 1$ ).

Let us consider a vector  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ , we shall count the number of variations of sign in the sequence of coordinates  $x_1, x_2, \dots, x_n$  of  $\mathbf{x}$ , attributing arbitrary signs to the zero coordinates (if any such exist). Depending on what signs we give to the zero coordinates the number of variations of sign will vary within certain limits. The maximal and the minimal number of variations of sign so obtained will be denoted by  $S_{\mathbf{x}}^+$  and  $S_{\mathbf{x}}^-$ , respectively. If  $S_{\mathbf{x}}^- = S_{\mathbf{x}}^+$ , we shall speak of the exact number of sign changes and denote it by  $S_{\mathbf{x}}$ . Obviously,  $S_{\mathbf{x}}^- = S_{\mathbf{x}}^+$  if and only if (1)  $x_1 \cdot x_n \neq 0$  and (2)  $x_i = 0$  ( $1 < i < n$ ) always implies that  $x_{i-1}x_{i+1} < 0$ .

Let  $\mathbf{u} = (u(2), u(3), \dots, u(N + 1))$  be a nontrivial solution of (2.1). We say that  $\mathbf{u}$  has a generalized zero at  $k_0 \in [2, N + 1]_{\mathbb{Z}}$  in case either  $u(k_0) = 0$  or there exists an integer  $j$  with  $1 \leq j \leq k_0 - 2$  such that

$$(-1)^j u(k_0 - j)u(k_0) > 0, \text{ and } u(k) = 0 \text{ for } k_0 - j < k < k_0 \text{ (if } j > 1\text{)}.$$

We say  $k_0$  is a simple zero point if  $u(k_0 - 1)u(k_0) \leq 0$  and  $u(k_0 - 1)u(k_0 + 1) < 0$ . If  $\mathbf{u}$  has only simple zero points on  $[2, N + 1]_{\mathbb{Z}}$ , the solution  $\mathbf{u}$  is called a nodal solution of (2.1). Clearly, the changing-sign number of nodal solution depends on the numbers of simple zero points in  $[2, N + 1]_{\mathbb{Z}}$ . In the following text, the zero point we mean that simple zero point.

**Lemma 2.5** ([14, P. 87, Theorem 6], [15, Vol. 2, P. 105, Theorem 13]) (1) *An oscillatory matrix  $A = (a_{ik})_{n \times n}$  always has  $n$  distinct positive characteristic values*

$$\mu_1 > \mu_2 > \cdots > \mu_n > 0.$$

(2) *The characteristic vector  $\mathbf{w}^1 = (w_{11}, w_{21}, \dots, w_{n1})$  of  $A$  that belongs to the largest characteristic value  $\mu_1$  has only non-zero coordinates of like sign; the characteristic vector  $\mathbf{w}^2 = (w_{12}, w_{22}, \dots, w_{n2})$  that belongs to the second largest characteristic value  $\mu_2$  has exactly one variation of sign in its coordinates; more generally, the characteristic vector  $\mathbf{w}^k = (w_{1k}, w_{2k}, \dots, w_{nk})$  that belongs to the characteristic value  $\mu_k$  has exactly  $k - 1$  variation of sign ( $k = 1, 2, \dots, n$ ).*

(3) *For arbitrary real numbers  $c_g, c_{g+1}, \dots, c_h$  ( $1 \leq g \leq h \leq n$ ;  $\sum_{k=g}^h c_k^2 > 0$ ) the number of variations of sign in the coordinates of the vector*

$$\mathbf{w} = \sum_{k=g}^h c_k \mathbf{w}^k$$

*lies between  $g - 1$  and  $h - 1$ .*

**Lemma 2.6** *Suppose that  $\mathbf{u} = (u(0), u(1), u(2), \dots, u(N + 2), u(N + 3))^T$  is a nonzero solution to (1.2). Then  $u(2) \neq 0$  and  $u(N + 1) \neq 0$ .*

**Proof** Suppose on the contrary that  $u(2) = 0$ . Then it is easily seen from the initial conditions in (1.2) that

$$\Delta u(1) = u(2) - u(1) = 0, \quad \Delta^2 u(0) = \Delta u(1) - \Delta u(0) = 0. \tag{2.4}$$

We claim that  $u(3) = 0$ . Suppose on the contrary that  $u(3) \neq 0$ . For simplicity, we recall the vector  $\mathbf{u}$  so that  $u(3) = 1$ . Therefore, it is seen from (1.2) and (2.4) that

$$\Delta u(2) = u(3) - u(2) = 1, \quad \Delta^2 u(1) = \Delta u(2) - \Delta u(1) = 1, \quad \Delta^3 u(0) = \Delta^2 u(1) - \Delta^2 u(0) = 1, \tag{2.5}$$

which further implies that

$$\begin{aligned} \Delta^4 u(0) &= \lambda m(2)u(2) \geq 0, \\ \Delta^3 u(1) &= \Delta^3 u(0) + \Delta^4 u(0) \geq 1 + 0 \geq 1, \\ \Delta^2 u(2) &= \Delta^2 u(1) + \Delta^3 u(1) \geq 1 + 1 \geq 1, \\ \Delta u(3) &= \Delta u(2) + \Delta^2 u(2) \geq 1 + 1 \geq 1, \\ u(4) &= u(3) + \Delta u(3) \geq 1 + 1 \geq 1. \end{aligned} \tag{2.6}$$

By a similar argument, it follows that

$$\Delta^4 u(1) = \lambda m(3)u(3) \geq 0, \quad \Delta^3 u(2) \geq 1, \quad \Delta^2 u(3) \geq 1, \quad \Delta u(4) \geq 1, \quad u(5) \geq 1. \tag{2.7}$$

Continuing this procedure, we will get that

$$\begin{aligned} \Delta^4 u(N-1) &= \lambda m(N+1)u(N+1) \geq 0, \\ \Delta^3 u(N) &\geq 1, \Delta^2 u(N+1) \geq 1, \Delta u(N+2) \geq 1, u(N+3) \geq 1, \end{aligned} \tag{2.8}$$

which contradicts the boundary condition  $\Delta u(N+2) = 0$ . Therefore, we have  $u(0) = u(1) = u(2) = u(3) = 0$ . The difference equation (1.2) can be rewritten as

$$u(k+2) = 4u(k+1) + (\lambda m(k) - 6)u(k) + 4u(k-1) - u(k-2), \quad k \in [2, N+1]_{\mathbb{Z}}. \tag{2.9}$$

Clearly,  $u(k) = 0$  for all  $k$  can be deduced recursively from the initial conditions  $u(0) = u(1) = u(2) = u(3) = 0$ . This contradicts the assumption  $\mathbf{u} \neq 0$ . Therefore,  $u(2) \neq 0$ . By a similar argument, it follows that  $u(N+1) \neq 0$ .  $\square$

**Lemma 2.7** Suppose that  $\mathbf{u} = (u(0), u(1), \dots, u(N+2), u(N+3))^T$  and  $\mathbf{v} = (v(0), v(1), \dots, v(N+2), v(N+3))^T$  are nonzero solutions to (1.2) for a fixed  $\lambda$ . Then  $\mathbf{u}$  and  $\mathbf{v}$  are linearly dependent.

**Proof** It is easy to see from Lemma 2.6 that  $u(2) \neq 0$  and  $v(2) \neq 0$ . Define  $\mathbf{y} = v(2)\mathbf{u} - u(2)\mathbf{v}$ . Then  $\mathbf{y}$  is a solution of (1.2) with  $y(2) = 0$ . Therefore, in view of Lemma 2.6,  $\mathbf{y}$  is a trivial solution, that is  $\mathbf{y} = v(2)\mathbf{u} - u(2)\mathbf{v} = 0$ , leading to the desired result.  $\square$

Based on Lemmas 2.5–2.7, it follows that

**Theorem 2.8** Let  $m(i) > 0, i = 2, 3, \dots, N+1$ . Then there are  $N$  distinct real eigenvalues  $\lambda_k (k = 1, 2, \dots, N)$  of the problem (1.2) satisfying

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_N, \tag{2.10}$$

and the eigenfunction corresponding to  $\lambda_k$  changes  $k-1$  sign in  $[2, N+1]_{\mathbb{Z}}$ .

**Proof** Clearly, (2.1) can be rewritten as

$$D^{-1}Mu = \frac{1}{\lambda}u. \tag{2.11}$$

Since  $D^{-1}$  is a totally positive matrix, diagonal matrix  $M$  is a nonsingular totally non-negative matrix, it yields that  $D^{-1}M$  is a totally positive matrix. This together with Definition 2.1 concludes that  $D^{-1}M$  is an oscillatory matrix. From Lemma 2.5,  $D^{-1}M$  has  $N$  distinct positive characteristic values

$$\frac{1}{\lambda_1} > \frac{1}{\lambda_2} > \dots > \frac{1}{\lambda_N} > 0,$$

which means that (2.11) holds. Moreover, the characteristic vector  $\mathbf{u}^k = (u^k(2), u^k(3), \dots, u^k(N+1))$  that belongs to the eigenvalue  $\frac{1}{\lambda_k}$  has exactly  $k-1$  variation of sign ( $k = 1, 2, \dots, N$ ). That is,

$$D\mathbf{u}^k - \lambda_k M\mathbf{u}^k = 0$$

and the eigenfunction  $\mathbf{u}^k$  corresponding to  $\lambda_k$  has exactly  $k-1$  simple zeros. Therefore, the problem (1.2) has  $N$  distinct positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_N,$$

and the eigenfunction  $\varphi^k = (0, 0, u^k(2), u^k(3), \dots, u^k(N + 1), 0, 0)$  changes  $k - 1$  sign in  $[2, N + 1]_{\mathbb{Z}}$ .  $\square$

By the similar argument of [9, Theorem 2.9], we give the eigenvalue comparisons of problem (1.2) with the weight function  $m(k)$  changes.

Let us consider the following eigenvalue problems

$$\begin{cases} \Delta^4 u(k - 2) = \lambda a(k)u(k), & k \in [2, N + 1]_{\mathbb{Z}}, \\ u(0) = \Delta u(0) = 0 = u(N + 2) = \Delta u(N + 2), \end{cases} \tag{2.12}$$

and

$$\begin{cases} \Delta^4 v(k - 2) = \lambda b(k)v(k), & k \in [2, N + 1]_{\mathbb{Z}}, \\ v(0) = \Delta v(0) = 0 = v(N + 2) = \Delta v(N + 2), \end{cases} \tag{2.13}$$

where  $a, b : [2, N + 1]_{\mathbb{Z}} \rightarrow (0, \infty)$  with  $a(k) \geq b(k)$  for  $k \in [2, N + 1]_{\mathbb{Z}}$ .

**Theorem 2.9** Assume  $a, b : [2, N + 1]_{\mathbb{Z}} \rightarrow (0, \infty)$  with  $a(k) \geq b(k)$  for  $k \in [2, N + 1]_{\mathbb{Z}}$ . Let  $\{\lambda_1 < \lambda_2 < \dots < \lambda_N\}$  be the set of all eigenvalues of (2.12) and  $\{\mu_1 < \mu_2 < \dots < \mu_N\}$  be the set of all eigenvalues of (2.13). Then  $\lambda_i \leq \mu_i$  for  $1 \leq i \leq N$ .

### 3. Applications

In this section, we consider the global structure of nodal solutions of (1.3), that is, we consider the nonlinear BVPs as follows

$$\begin{cases} \Delta^4 u(i - 2) = \lambda m(i)f(u(i)), & i \in [2, N + 1]_{\mathbb{Z}}, \\ u(0) = \Delta u(0) = u(N + 2) = \Delta u(N + 2) = 0, \end{cases} \tag{3.1}$$

where  $\lambda > 0$  is a parameter,  $N \geq 4$  is an integer,  $m : [2, N + 1]_{\mathbb{Z}} \rightarrow (0, \infty)$ ,  $f \in C(\mathbb{R}, \mathbb{R})$  is continuous and  $f(s)s > 0$  with  $s \neq 0$ . The problem (3.1) can be viewed as the discrete analogue of the clamped beam equation (1.1).

By a nodal solution of (3.1) we mean a pair  $(\lambda, u)$ , where  $\lambda > 0$  and  $u$  is a nontrivial solution of (3.1) which has simple zeros in  $[2, N + 1]_{\mathbb{Z}}$ . Based on Theorem 2.8 and bifurcation theory [16], we establish the global structure of nodal solutions of (3.1) under the following conditions

(A)  $m : [2, N + 1]_{\mathbb{Z}} \rightarrow (0, \infty)$ ,  $f : [2, N + 1]_{\mathbb{Z}} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f(s)s > 0$  with  $s \neq 0$ .

For convenience, set

$$f_0 = \lim_{|u| \rightarrow 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{|u| \rightarrow \infty} \frac{f(u)}{u}. \tag{3.2}$$

Let  $E = \{u : [0, N + 3]_{\mathbb{Z}} \rightarrow \mathbb{R} \mid u(0) = \Delta u(0) = u(N + 2) = \Delta u(N + 2) = 0\}$ . Then  $E$  is Banach space with the norm  $\|u\| = \max_{i \in [0, N + 3]_{\mathbb{Z}}} |u(i)|$ . In addition,  $E$  is isomorphic to  $\mathbb{R}^N$ . Thus,  $u \in E$  can be rewritten the vector form  $(u(0), u(1), u(2), \dots, u(N + 1), u(N + 2), u(N + 3))$ . This together with (2.1) implies that the vector  $\mathbf{u} = \{u(2), u(3), \dots, u(N + 1)\}$  is one-to-one corresponding vector for  $u$ . Therefore, we do not distinguish these two forms. Let  $S_k^+$  denote the set of functions in  $E$  which have exactly  $k - 1$  interior nodal zeros in  $[2, N + 1]_{\mathbb{Z}}$  and are positive at  $i = 2$ , and set  $S_k^- = -S_k^+$  and  $S_k = S_k^+ \cup S_k^-$ . It is easy to see that  $S_k^+$  and  $S_k^-$  are disjoint

and open in  $E$ , here  $k \in \{1, 2, \dots, N\}$ . Let  $\mathcal{S}$  be the closure of the set of nontrivial solutions pairs of problem (3.1) in  $\mathbb{R} \times E$ .

**Theorem 3.1** Assume that (A) and  $f_0 \in (0, \infty)$  hold.

(i) If  $f_\infty \in (0, \infty)$ , then for each  $1 \leq k \leq N$ , there are two unbounded component  $\mathcal{C}_k^+$  and  $\mathcal{C}_k^-$ , of  $\mathcal{S}$  bifurcating from  $(\frac{\lambda_k}{f_0}, 0)$ , such that  $\mathcal{C}_k^\pm \setminus \{(\frac{\lambda_k}{f_0}, 0)\} \subseteq [0, \infty) \times S_k^\pm$ .

(ii) If  $f_\infty = \infty$ , then for each  $1 \leq k \leq N$ , there are two unbounded component  $\mathcal{C}_k^+$  and  $\mathcal{C}_k^-$ , of  $\mathcal{S}$  bifurcating from  $(\frac{\lambda_k}{f_0}, 0)$ , such that  $\mathcal{C}_k^\pm \setminus \{(\frac{\lambda_k}{f_0}, 0)\} \subseteq [0, \infty) \times S_k^\pm$  and  $\text{Proj}_{\mathbb{R}} \mathcal{C}_k^\pm = (0, \frac{\lambda_k}{f_0})$ .

(iii) If  $f_\infty = 0$ , then for each  $1 \leq k \leq N$ , there are two unbounded components  $\mathcal{C}_k^+$  and  $\mathcal{C}_k^-$ , of  $\mathcal{S}$  bifurcating from  $(\frac{\lambda_k}{f_0}, 0)$ , such that  $\mathcal{C}_k^\pm \setminus \{(\frac{\lambda_k}{f_0}, 0)\} \subseteq [0, \infty) \times S_k^\pm$  and  $\text{Proj}_{\mathbb{R}} \mathcal{C}_k^\pm = (\frac{\lambda_k}{f_0}, \infty)$ .

**Corollary 3.2** Assume that (A) and  $f_0 \in (0, \infty)$  hold.

(i) If  $f_\infty \in (0, \infty)$ , then either for any  $\lambda \in (\frac{\lambda_k}{f_0}, \frac{\lambda_k}{f_\infty})$  with  $f_\infty < f_0$ , problem (3.1) has at least two nodal solutions  $\mathbf{u}_k^\pm \in S_k^\pm$ , or for any  $\lambda \in (\frac{\lambda_k}{f_\infty}, \frac{\lambda_k}{f_0})$  with  $f_0 < f_\infty$ , (3.1) has at least two nodal solutions  $\mathbf{u}_k^\pm \in S_k^\pm$  which change its sign  $k - 1$  in  $[2, N + 1]_{\mathbb{Z}}$ .

(ii) If  $f_\infty = \infty$ , then for any  $\lambda \in (0, \frac{\lambda_k}{f_0})$ , problem (1.1) has at least two nodal solutions  $\mathbf{u}_k^\pm \in S_k^\pm$ , which change its sign  $k - 1$  in  $[2, N + 1]_{\mathbb{Z}}$ .

(iii) If  $f_\infty = 0$ , then for any  $\lambda \in (\frac{\lambda_k}{f_0}, \infty)$ , problem (1.1) has at least two nodal solutions  $\mathbf{u}_k^\pm \in S_k^\pm$ , which change its sign  $k - 1$  in  $[2, N + 1]_{\mathbb{Z}}$ .

First, we give some preliminaries.

**Lemma 3.3** Let  $h : [2, N + 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ . Then the linear boundary value problem

$$\begin{aligned} \Delta^4 u(i - 2) &= h(i), \quad i \in [2, N + 1]_{\mathbb{Z}}, \\ u(0) = u(N + 2) &= \Delta u(0) = \Delta u(N + 2) = 0 \end{aligned} \tag{3.3}$$

has a solution

$$u(i) = \sum_{s=2}^{N+1} G(i, s)h(s), \quad i \in [0, N + 3]_{\mathbb{Z}}, \tag{3.4}$$

where

$$G(i, s) = \begin{cases} \frac{s(s-1)(N+2-i)(N+3-i)(3(N+1)(i-1) - (N+1)(s-2) - 2i(s-2))}{6(N+1)(N+2)(N+3)}, & s \leq i, \\ \frac{i(i-1)(N+2-s)(N+3-s)(3(N+1)(s-1) - (N+1)(i-2) - 2s(i-2))}{6(N+1)(N+2)(N+3)}, & i \leq s. \end{cases} \tag{3.5}$$

**Proof** By a simple summing computation and  $u(0) = \Delta u(0) = 0$ , we conclude that

$$u(i) = \Delta^2 u(0) \frac{(i-1)i}{2} + \frac{i(i-1)(i-2)}{6} \Delta^3 u(0) + \sum_{s=2}^{i-1} \frac{(i-s)(i-s-1)(i-s+1)}{6} h(s).$$

This together with  $u(N + 2) = \Delta u(N + 2) = 0$  leads to

$$\begin{aligned}
 u(i) &= \sum_{s=2}^{N+1} \frac{(N + 2 - s)(N + 3 - s)i(i - 1)[3(N + 1)(s - 1) - (N + 1)(i - 2) - 2s(i - 2)]}{6(N + 1)(N + 2)(N + 3)} h(s) + \\
 &\quad \sum_{s=2}^{i-1} \frac{(i - s)(i - s - 1)(i - s + 1)}{6(N + 1)(N + 2)(N + 3)} h(s) \\
 &= \sum_{s=i}^{N+1} \frac{(N + 2 - s)(N + 3 - s)i(i - 1)[3(N + 1)(s - 1) - (N + 1)(i - 2) - 2s(i - 2)]}{6(N + 1)(N + 2)(N + 3)} h(s) + \\
 &\quad \sum_{s=2}^{i-1} \frac{(N + 2 - i)(N + 3 - i)s(s - 1)[3(N + 1)(i - 1) - (N + 1)(s - 2) - 2i(s - 2)]}{6(N + 1)(N + 2)(N + 3)} h(s).
 \end{aligned}$$

Therefore, (3.4) holds.  $\square$

By a simple computation, it follows that  $G(i, s)$  satisfies

$$\begin{aligned}
 G(i, s) &\leq \Phi(s), & \text{for } s, i \in [1, N + 2]_{\mathbb{Z}}, \\
 G(i, s) &\geq c(i)\Phi(s), & \text{for } s, i \in [1, N + 2]_{\mathbb{Z}},
 \end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
 \Phi(s) &= \begin{cases} \frac{(N + 3 - s)(N + 2 - s)s(s - 1)^2}{2(N + 2)(N + 3)}, & 1 \leq i \leq s \leq N + 2, \\ \frac{(N + 3 - s)^2(N + 2 - s)s(s - 1)}{2(N + 1)(N + 3)}, & 1 \leq s \leq i \leq N + 2, \end{cases} \\
 c(i) &= \begin{cases} \frac{(N + 3 - i)(N + 2 - i)(i - 1)}{3(N + 2)N}, & 1 \leq s \leq i \leq N + 2, \\ \frac{(N + 3 - i)i(i - 1)}{(N + 3)(N + 1)^2}, & 1 \leq i \leq s \leq N + 2. \end{cases}
 \end{aligned} \tag{3.7}$$

Moreover, we have

$$G(i, s) \geq \sigma\Phi(s), \text{ for } s \in [1, N + 2]_{\mathbb{Z}}, i \in [2, N + 1]_{\mathbb{Z}}, \tag{3.8}$$

here  $\sigma = \frac{2}{(N+1)(N+2)}$ .

For any  $r > 0$ , let  $B_r = \{u \in E \mid \|u\| < r\}$  and  $\partial B_r = \{u \in E \mid \|u\| = r\}$ . We denote by  $\theta$  the zero element of  $E$ . Define  $P$  be a cone in  $E$  by

$$P = \{u \in E \mid u(k) \geq 0, \min_{k \in [2, N+1]_{\mathbb{Z}}} u(k) \geq \sigma\|u\|\}.$$

It is well known that (3.1) is equivalent to

$$u(i) = \lambda \sum_{s=2}^{N+1} G(i, s)m(s)f(u(s)) =: (Au)(i), \quad i \in [1, N + 2]_{\mathbb{Z}}, \tag{3.9}$$

where  $G(i, s)$  is defined as (3.5).

Define the operator  $L, F : E \rightarrow E$  respectively, by

$$Lu(i) := \sum_{s=2}^{N+1} G(i, s)m(s)u(s), \quad u \in E, i \in [1, N + 2]_{\mathbb{Z}}, \tag{3.10}$$

$$Fu(i) := f(u(i)), \quad u \in E, \quad i \in [2, N + 1]_{\mathbb{Z}}.$$

So  $A = \lambda L \circ F$ .

**Lemma 3.4**  $A(P) \subset P$  and  $A : P \rightarrow P$  is completely continuous.

**Proof** For any  $u \in P$ , it follows that

$$\begin{aligned} \|Au\| &= \max_{i \in [1, N+2]_{\mathbb{Z}}} \lambda \sum_{s=2}^{N+1} G(i, s)m(s)f(u(s)) \leq \lambda \sum_{s=2}^{N+1} \Phi(s)m(s)f(u(s)), \\ \min_{i \in [2, N+1]_{\mathbb{Z}}} Au(i) &= \min_{i \in [2, N+1]_{\mathbb{Z}}} \lambda \sum_{s=2}^{N+1} G(i, s)m(s)f(u(s)) \geq \sigma \lambda \sum_{s=2}^{N+1} \Phi(s)m(s)f(u(s)) \geq \sigma \|Au\|. \end{aligned}$$

So  $A(P) \subset P$ . Since  $E$  is a finite dimension space and  $f$  is continuous, it is easy to prove  $A : P \rightarrow P$  is completely continuous.  $\square$

It follows from Lemma 3.3 that  $u = \{u(i)\}_{i=0}^{N+3}$  is a solution of the problem (3.1) if and only if  $u = \{u(i)\}_{i=0}^{N+3} \in E$  is a fixed point of the operator  $A$ .

Let us consider the following problem

$$\Delta^4 u(i - 2) = \lambda m(i)f_0 u(i) + \lambda m(i)\xi(u(i)), \quad u(0) = \Delta u(0) = u(N + 2) = \Delta u(N + 2) = 0 \quad (3.11)$$

as a bifurcation problem from the trivial solution  $u \equiv 0$ , here  $\xi \in C(\mathbb{R}, \mathbb{R})$  satisfying  $\lim_{u \rightarrow 0} \frac{\xi(u)}{u} = 0$ . Clearly, (3.11) can be converted to the following operator equation

$$u(i) = \lambda f_0 Lu(i) + \lambda \sum_{s=2}^{N+1} G(i, s)m(s)\xi(u(s)).$$

It follows from Theorem 2.8 that the eigenvalue  $\lambda_k$  ( $k \in \{1, 2, \dots, N\}$ ) of the linear operator  $L$  is isolated and has geometric multiplicity 1, this together with Lemma 3.4 yields that the operator  $\Phi_\lambda u := u - \lambda f_0 Lu$  is completely continuous and the Brouwer degree  $\text{deg}(\Phi_\lambda, B_r, 0)$  is well defined (see [17]) for arbitrary  $B_r$  and  $\lambda \neq \frac{\lambda_k}{f_0}$ .

**Lemma 3.5** For any  $r > 0$ , we have

$$\text{deg}(\Phi_\lambda, B_r, 0) = \begin{cases} 1, & 0 \leq \lambda < \frac{\lambda_1}{f_0}, \\ (-1)^k, & \lambda \in (\frac{\lambda_k}{f_0}, \frac{\lambda_{k+1}}{f_0}), \quad k \in \{1, 2, \dots, N - 1\}. \end{cases}$$

**Proof** Since  $L$  is a linear compact operator, it follows from [17, Theorem 8.10] that

$$\text{deg}(\Phi_\lambda, B_r, 0) = (-1)^{m(\lambda)},$$

where  $m(\lambda)$  is the sum of algebraic multiplicity of the eigenvalue  $\lambda$  of

$$u(i) = \lambda f_0 \sum_{s=2}^{N+1} G(i, s)m(s)u(s)$$

satisfying  $\lambda^{-1} \frac{\lambda_k}{f_0} < 1$ . If  $\lambda < \frac{\lambda_1}{f_0}$ , then there are no such  $\lambda$  at all, hence

$$\text{deg}(\Phi_\lambda, B_r, 0) = (-1)^{m(\lambda)} = (-1)^0 = 1.$$

If  $\lambda \in (\frac{\lambda_k}{f_0}, \frac{\lambda_{k+1}}{f_0})$  for some  $k \in \{1, 2, \dots, N-1\}$ , then  $\lambda^{-1} \frac{\lambda_j}{f_0} < 1$ ,  $j \in \{1, 2, \dots, k\}$  and  $\lambda^{-1} \frac{\lambda_{k+1}}{f_0} > 1$ . This together with Theorem 2.8 concludes that  $\deg(\Phi_\lambda, B_r, 0) = (-1)^{m(\lambda)} = (-1)^k$ .  $\square$

**Proposition 3.6** *Assume that (A) holds and  $f_0 \in (0, \infty)$ . Then for each  $(\frac{\lambda_k}{f_0}, 0)$ ,  $k = 1, 2, \dots, N$ , there are two unbounded continua  $C_k^+$  and  $C_k^-$  of solutions to the problem (3.1) bifurcated from  $(\frac{\lambda_k}{f_0}, 0)$  with  $C_k^\nu \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^\nu)$ ,  $\nu \in \{+, -\}$ .*

**Proof** Lemmas 3.4 and 3.5 together with the results of unilateral global bifurcation [16] and [18, Theorem 1] for (3.11) can be stated as follows: for each integer  $k \in \{1, 2, \dots, N\}$ , there are two distinct continua,  $C_k^\pm \subset [0, \infty) \times E$  of solutions to (3.11) bifurcated from  $(\frac{\lambda_k}{f_0}, 0)$  such that either they are unbounded, or  $C_k^+ \cap C_k^- \neq \{(\frac{\lambda_k}{f_0}, 0)\}$ .

By a similar argument of [16, Sections 6.4-6.5], it is not difficult to verify that there is a neighborhood  $\mathbb{O}_k$  of  $(\frac{\lambda_k}{f_0}, 0)$  such that

$$C_k^\nu \cap \mathbb{O}_k \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^\nu) \text{ or } C_k^\nu \cap \mathbb{O}_k \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^{-\nu}).$$

Without loss of generality, we assume that

$$C_k^\nu \cap \mathbb{O}_k \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^\nu).$$

We shall show  $C_k^\nu \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^\nu)$ . Suppose on the contrary that  $C_k^\nu \not\subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^\nu)$ . Then there exist  $(\lambda, u) \in C_k^\nu \cap (\mathbb{R} \times \partial S_k^\nu)$  with  $(\lambda, u) \neq (\frac{\lambda_k}{f_0}, 0)$  and  $(\lambda_n, u_n) \in C_k^\nu \cap (\mathbb{R} \times S_k^\nu)$  with  $(\lambda_n, u_n) \rightarrow (\lambda, u)$  as  $n \rightarrow \infty$ . Since  $u \in \partial S_k^\nu$ , it's easy to see that  $u \equiv 0$ . Let  $v_n = \frac{u_n}{\|u_n\|}$ , then there exists a subsequence and relabelling if necessary,  $v_n$ , such that  $v_n \rightarrow v_0$  as  $n \rightarrow \infty$ , here  $\|v_0\| = 1$  and  $v_0$  satisfies

$$\Delta^4 v_0(i-2) = \lambda m(i) f_0 u(i), \quad v_0(0) = \Delta v_0(0) = v_0(N+2) = \Delta v_0(N+2) = 0.$$

Hence, we have  $\lambda = \frac{\lambda_j}{f_0}$  for some  $j \neq k$  and  $u \in S_j$ , which indicates that  $u_n \in S_j$  for sufficiently large  $n$ . This yields a contradiction. Thus,

$$C_k^+ \cap C_k^- = \{(\frac{\lambda_k}{f_0}, 0)\}.$$

Therefore,  $C_k^+ \cap C_k^- \neq \{(\frac{\lambda_k}{f_0}, 0)\}$  is not true, which means that  $C_k^\pm \setminus \{(\frac{\lambda_k}{f_0}, 0)\}$  to infinity in  $\mathcal{S}$ .  $\square$

**Lemma 3.7** *Assume that  $m : [2, N+1]_{\mathbb{Z}} \rightarrow (0, \infty)$ , and  $g_n : [2, N+1]_{\mathbb{Z}} \rightarrow (0, \infty)$  with  $g_n(i) \geq \rho > 0$ ,  $i \in [2, N+1]_{\mathbb{Z}}$  and  $\rho$  is a constant. If there exists a sequence  $(\mu_n, u_n)$  satisfying*

$$\begin{aligned} \Delta^4 u_n(i-2) &= \mu_n m(i) g_n(i) u_n(i), \quad i \in [2, N+1]_{\mathbb{Z}}, \\ u_n(0) &= \Delta u_n(0) = u_n(N+2) = \Delta u_n(N+2) = 0 \end{aligned} \tag{3.12}$$

such that for any given subinterval of  $I \subset [2, N+1]_{\mathbb{Z}}$  and sufficiently large enough  $n \in \mathbb{N}$ ,

$$u_n(i) > 0, \quad i \in I \text{ or } u_n(i) < 0, \quad i \in I,$$

then there is a positive constant  $M_0 > 0$ , such that

$$|\mu_n| \leq M_0, \quad \forall n \in \mathbb{N}.$$

**Proof** We only show the case  $u_n(i) > 0$ ,  $i \in I$  for  $n \in \mathbb{N}$  sufficiently large enough. The other case can be true by the similar argument.

Without loss of generality, let for  $n \in \mathbb{N}$ ,  $u_n(i) > 0$ ,  $i \in I$ . Set  $[\alpha_n, \beta_n]_{\mathbb{Z}} \subseteq [2, N + 1]_{\mathbb{Z}}$  is a subinterval satisfying (i)  $I \subset [\alpha_n, \beta_n]_{\mathbb{Z}}$ , (ii)  $\alpha_n, \beta_n$  are zero points of  $u_n$ , and (iii)  $u_n(i) > 0$ ,  $i \in [\alpha_n, \beta_n]_{\mathbb{Z}}$ . Then it follows from Lemmas 3.3 and 3.4 that for any  $i \in [2, N + 1]_{\mathbb{Z}}$ ,

$$u_n(i) = \mu_n \sum_{s=2}^{N+1} G(i, s)m(s)g_n(s)u_n(s) \geq \frac{2\mu_n}{(N + 1)(N + 2)} \sum_{s=2}^{N+1} G(i, s)m(s)g_n(s)\|u_n\|.$$

Moreover, for  $i \in [\alpha_n, \beta_n]_{\mathbb{Z}}$ ,

$$\begin{aligned} u_n(i) &\geq \frac{2}{(N + 1)(N + 2)} \mu_n \sum_{s=2}^{N+1} G(i, s)m(s)g_n(s)\|u_n\| \\ &\geq \frac{2}{(N + 1)(N + 2)} \mu_n \sum_{s=\alpha_n}^{\beta_n} G(i, s)m(s)\rho \max_{s \in [\alpha_n, \beta_n]_{\mathbb{Z}}} |u_n(s)| \\ &\geq \frac{4\rho}{(N + 1)^2(N + 2)^2} \mu_n \sum_{s=\alpha_n}^{\beta_n} \Phi(s)m(s) \max_{s \in [\alpha_n, \beta_n]_{\mathbb{Z}}} |u_n(s)| \end{aligned}$$

Therefore,  $|\mu_n| \leq \frac{(N+1)^2(N+2)^2}{4\rho} [\sum_{s=\alpha_n}^{\beta_n} \Phi(s)m(s)]^{-1}$ .  $\square$

**Lemma 3.8** ([19, Theorem 1.2], [20]) *Let  $X$  be a normal space and let  $\{C_n\}$  be a sequence of unbounded connected subsets of  $X$ . Suppose that*

- (i) *There exists  $z^* \in \lim_{n \rightarrow +\infty} C_n$  with  $\|z^*\| = +\infty$ ;*
- (ii) *There exists a homeomorphism  $T : X \rightarrow X$  such that  $\|T(z^*)\| < +\infty$  and  $\{T(C_n)\}$  be a sequence of unbounded connected subsets of  $X$ ;*
- (iii) *For every  $R > 0$ ,  $(\cup_{n=1}^{\infty} T(C_n)) \cap \bar{B}_R$  is a relatively compact set of  $X$ .*

*Then  $D := \limsup_{n \rightarrow +\infty} C_n$  is unbounded closed connected.*

**Proof of Theorem 3.1** Let us consider the bifurcation problem (3.11). From Proposition 3.6, for each  $(\frac{\lambda_k}{f_0}, 0)$ ,  $k = 1, 2, \dots, N$ , there are two unbounded continua  $\mathcal{C}_k^{\pm}$  of solutions to the problem (3.1) bifurcated from  $(\frac{\lambda_k}{f_0}, 0)$  with  $\mathcal{C}_k^{\nu} \subseteq \{(\frac{\lambda_k}{f_0}, 0)\} \cup (\mathbb{R} \times S_k^{\nu})$ ,  $\nu \in \{+, -\}$ .

- (i) If  $f_{\infty} \in (0, \infty)$ , let us consider

$$\Delta^4 u(i - 2) = \lambda m(i) f_{\infty} u(i) + \lambda m(i) \zeta(u(i)), \quad u(0) = \Delta u(0) = u(N + 2) = \Delta u(N + 2) = 0 \quad (3.13)$$

as a bifurcation problem from infinity to (3.1), here  $\zeta \in C(\mathbb{R}, \mathbb{R})$  satisfying  $\lim_{u \rightarrow \infty} \frac{\zeta(u)}{u} = 0$ . Then (3.13) can be converted to the following operator equation

$$u(i) = \lambda f_{\infty} \sum_{s=2}^{N+1} G(i, s)m(s)u(s) + \lambda \sum_{s=2}^{N+1} G(i, s)m(s)\zeta(u(s)) := \lambda f_{\infty} Lu + H(\lambda, u),$$

here  $H(\lambda, u) := \lambda \sum_{s=2}^{N+1} G(i, s)m(s)\zeta(u(s))$ , by a similar argument of Lemma 3.4, it easy to see  $H$  is completely continuous. Let  $\tilde{\zeta}(u) := \max_{0 \leq |s| \leq u} |\zeta(s)|$ . Then  $\tilde{\zeta}$  is nondecreasing with respect to  $u$  and

$$\lim_{|s| \rightarrow \infty} \frac{\tilde{\zeta}(s)}{s} = 0. \quad (3.14)$$

It follows from (3.14) that

$$\left| \frac{\zeta(u)}{u} \right| \leq \frac{\tilde{\zeta}(|u|)}{|u|} \leq \frac{\tilde{\zeta}(\|u\|)}{\|u\|} \rightarrow 0 \text{ as } \|u\| \rightarrow \infty. \tag{3.15}$$

We shall show that the unbounded continua  $\mathcal{C}_k^\nu$  joins  $(\frac{\lambda_k}{f_0}, 0)$  to  $(\frac{\lambda_k}{f_\infty}, \infty)$ . Let  $(\mu_n, u_n) \in \mathcal{C}_k^\nu$  satisfy  $\mu_n + \|u_n\| \rightarrow \infty$ . Notice that  $\mu_n > 0, n \in \mathbb{N}$ . Since  $(0, 0)$  is the only solution of (3.11) for  $\lambda = 0$  and  $\mathcal{C}_k^\nu \cap (\{0\} \times E) = \emptyset$ .

Case 1.  $\frac{\lambda_k}{f_\infty} < \lambda < \frac{\lambda_k}{f_0}$ .

In this case, we show that  $(\frac{\lambda_k}{f_\infty}, \frac{\lambda_k}{f_0}) \subseteq \{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{C}_k^\nu\}$ .

First, we show that if there exists a constant number  $M > 0$  such that  $0 < \mu_n \leq M$ , then  $\mathcal{C}_k^\nu$  joins  $(\frac{\lambda_k}{f_0}, 0)$  to  $(\frac{\lambda_k}{f_\infty}, \infty)$ . In this case, it follows that  $\|u_n\| \rightarrow \infty$ . Since  $u_n$  satisfies (3.13) with  $\lambda = \mu_n$ , we divide (3.13) by  $\|u_n\|$  and set  $v_n = u_n/\|u_n\|$ , then

$$\Delta^4 v_n(i-2) = \mu_n m(i) [f_\infty v_n(i) + \frac{\zeta(u_n(i))}{\|u_n\|}], \quad v_n(0) = \Delta v_n(0) = v_n(N+2) = \Delta v_n(N+2) = 0.$$

Since  $v_n$  is bounded in  $E$ , choosing a subsequence and relabelling if necessary, it yields that  $v_n \rightarrow v_0$  as  $n \rightarrow \infty$  for some  $v_0 \in E$  with  $\|v_0\| = 1$ . From the definition of  $\tilde{\zeta}$  and (3.14), (3.15), we have that

$$\lim_{n \rightarrow \infty} \frac{|\zeta(u_n)|}{\|u_n\|} = 0.$$

Therefore,  $v_0(i) = \mu_0 \sum_{s=2}^{N+1} G(i, s) m(s) f_\infty v_0(s)$ , where  $\mu_0 = \lim_{n \rightarrow \infty} \mu_n$ . Again choosing a subsequence and relabelling if necessary gives

$$\Delta^4 v_0(i-2) = \mu_0 m(i) f_\infty v_0(i), \quad v_0(0) = \Delta v_0(0) = v_0(N+2) = \Delta v_0(N+2) = 0. \tag{3.16}$$

We claim that  $v_0 \in S_k^\nu$ .

Suppose on the contrary that  $v_0 \notin S_k^\nu$ . Since  $v_0 \neq 0$  is a solution of (3.16), all zeros of  $v_0$  in  $[2, N+1]$  are simple. It follows that  $v_0 \in S_m^\nu$  for some  $m \in \{1, 2, \dots, N\}$  with  $m \neq k$  and  $\iota \in \{+, -\}$ . By the openness of  $S_m^\nu$ , there exists a neighborhood  $U(v_0, r)$  such that  $U(v_0, r) \subset S_m^\nu$ , which together with  $v_n \rightarrow v_0$  implies that there exists  $n_0 \in \{1, 2, \dots, N\}$  such that  $v_n \in S_m^\nu, n \geq n_0$ . However, this contradicts the fact that  $v_n \in S_k^\nu$ . Therefore,  $v_0 \in S_k^\nu$ .

From Theorem 2.8, we have that  $\mu_0 f_\infty = \lambda_k$ , i.e.,  $\mu_0 = \frac{\lambda_k}{f_\infty}$ . Hence,  $\mathcal{C}_k^\nu$  joins  $(\frac{\lambda_k}{f_0}, 0)$  to  $(\frac{\lambda_k}{f_\infty}, \infty)$ .

Next, we show that there exists a constant number  $M > 0$  such that  $0 < \mu_n \leq M$  for all  $n$ .

Suppose, to the contrary, there is no such  $M$ , then choosing a subsequence and relabelling if necessary, it follows that

$$\lim_{n \rightarrow \infty} \mu_n = \infty.$$

Let  $1 = \tau(0, n) < \tau(1, n) < \dots < \tau(k, n) = N + 2$  denote the zeros of  $u_n$ . Then there exists a subsequence  $\{\tau(1, n_m)\} \subseteq \{\tau(1, n)\}$  such that  $\lim_{m \rightarrow \infty} \tau(1, n_m) := \tau(1, \infty)$ . Clearly,  $\lim_{m \rightarrow \infty} \tau(0, n_m) := \tau(0, \infty)$ . We claim that

$$\tau(1, \infty) - \tau(0, \infty) = 0. \tag{3.17}$$

Suppose on the contrary that  $\tau(0, \infty) < \tau(1, \infty)$ . Define a function  $p : [0, \infty) \rightarrow \mathbb{R}$  as follows

$$p(u) = \begin{cases} \frac{f(u)}{u}, & u \neq 0, \\ f_0, & u = 0. \end{cases}$$

Then there exist two positive numbers  $\rho_1$  and  $\rho_2$ , such that

$$\rho_1 \leq \frac{f(u)}{u} \leq \rho_2, \quad \forall u \geq 0.$$

This together with the fact  $\lim_{n \rightarrow \infty} \mu_{n_m} = \infty$  concludes that there exists a closed interval  $I_1 \subset (\tau(0, \infty), \tau(1, \infty))$  such that

$$\lim_{m \rightarrow \infty} \mu_{n_m} p(u_{n_m}(i)) = \infty, \quad \text{for } i \in I_1.$$

However, since  $u_{n_m}$  satisfies

$$\begin{aligned} \Delta^4 u_{n_m}(i-2) &= \mu_{n_m} m(i) p(u_{n_m}(i)) u_{n_m}(i), \\ u_{n_m}(0) &= \Delta u_{n_m}(0) = u_{n_m}(N+2) = \Delta u_{n_m}(N+2) = 0, \end{aligned}$$

it follows from Lemma 3.7 that for all  $n$  large enough,  $u_{n_m}$  must change sign on  $I_1$ , which contradicts the fact that for all  $m$  sufficiently large,  $I_1 \subset (\tau(0, \infty), \tau(1, \infty))$  and  $\nu u_{n_m}(i) > 0$ ,  $i \in (\tau(0, \infty), \tau(1, \infty))$ . Therefore, (3.17) holds. Next, we work with  $\{(\tau(1, n_m), \tau(2, n_m))\}$ . Obviously, there is a subsequence  $\{\tau(2, n_{m_j})\} \subseteq \{\tau(2, n_m)\}$  such that  $\lim_{j \rightarrow \infty} \tau(2, n_{m_j}) := \tau(2, \infty)$ . Clearly,  $\lim_{j \rightarrow \infty} \tau(1, n_{m_j}) := \tau(1, \infty)$ . By the same argument and comparison results of Theorem 2.9, we show that  $\tau(2, \infty) - \tau(1, \infty) = 0$ . Similarly, we can show that for each  $l \in \{2, \dots, k-1\}$ ,

$$\tau(l+1, \infty) - \tau(l, \infty) = 0.$$

Taking a subsequence and relabelling it if necessary as  $(\mu_n, u_n)$ , we get that for each  $l \in \{2, \dots, k-1\}$ ,

$$\tau(l+1, n) - \tau(l, n) = 0.$$

However, it is impossible since  $1 = \tau(k, n) - \tau(0, n) = \sum_{l=0}^{k-1} (\tau(l+1, n) - \tau(l, n))$  for all  $n$ . Therefore,  $|\mu_n| \leq M$  for some constant  $M > 0$ , independent of  $n \in \mathbb{N}$ .

Case 2.  $\frac{\lambda_k}{f_0} < \lambda < \frac{\lambda_k}{f_\infty}$ .

In this case, suppose  $(\mu_n, u_n) \in \mathcal{C}_k^v$  with  $\mu_n + \|u_n\| \rightarrow \infty$ . If  $\lim_{n \rightarrow \infty} \mu_n = \infty$ , then  $(\frac{\lambda_k}{f_0}, \frac{\lambda_k}{f_\infty}) \subseteq \{\lambda \in (0, \infty) \mid (\lambda, u) \in \mathcal{C}_k^v\}$ . If there exists  $M > 0$ , such that for all  $n \in \mathbb{N}$ ,  $\mu_n \in (0, M]$ . By a similar argument of Case 1, after taking a subsequence and relabelling if necessary  $(\mu_n, u_n)$ , we have that  $(\mu_n, u_n) \rightarrow (\frac{\lambda_k}{f_\infty}, \infty)$ ,  $n \rightarrow \infty$ . Therefore,  $\mathcal{C}_k^v$  joins  $(\frac{\lambda_k}{f_0}, 0)$  to  $(\frac{\lambda_k}{f_\infty}, \infty)$ .

(ii) If  $f_\infty = \infty$ , define auxiliary function

$$\tilde{f}_n(s) = \begin{cases} f(s), & s \in [-n, n]; \\ \frac{ns-f(n)}{n}(s-n) + f(n), & s \in (n, 2n); \\ \frac{-ns+f(-n)}{n}(n+s) + f(-n), & s \in (-2n, -n); \\ ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \end{cases}$$

then we consider the following problem

$$\Delta^4 u(i-2) = \lambda m(i) \tilde{f}_n(u(i)), \quad u(0) = \Delta u(0) = u(N+2) = \Delta u(N+2) = 0. \quad (3.18)$$

Then  $\lim_{n \rightarrow +\infty} \tilde{f}_n(s) = f(s)$  and

$$\lim_{s \rightarrow 0} \frac{\tilde{f}_n(s)}{s} = f_0, \quad \lim_{s \rightarrow +\infty} \frac{\tilde{f}_n(s)}{s} = n.$$

Applying the conclusion of (i), there are two unbounded continua  $\mathcal{C}_k^{n,\pm}$  of solutions to the problem (3.18) bifurcated from  $(\frac{\lambda_k}{f_0}, 0)$  and join  $(\frac{\lambda_k}{f_0}, 0)$  to  $(\frac{\lambda_k}{n}, \infty)$ .

Set  $z_* = (0, \infty)$ . Then it is easy to see that  $z_* = \liminf_{n \rightarrow +\infty} \mathcal{C}_k^{n,\nu}$  with  $\|z_*\|_{\mathbb{R} \times E} = +\infty$ . Let  $T : \mathbb{R} \times E \rightarrow \mathbb{R} \times E$  be a mapping defined by

$$T(\lambda, u) = \begin{cases} (\lambda, \frac{u}{\|u\|^2}), & 0 < \|u\| < \infty; \\ (\lambda, 0), & \|u\| = +\infty; \\ (\lambda, \infty), & \|u\| = 0. \end{cases}$$

It is not difficult to verify that  $T$  is homeomorphism and  $\|T(z_*)\|_{\mathbb{R} \times E} = 0$ . For every  $R > 0$ , the compactness of operator  $L$  implies that  $(\cup_{n=1}^{+\infty} T(\mathcal{C}_k^{n,\nu})) \cap \bar{B}_R$  is a relatively compact set of  $\mathbb{R} \times E$ . Therefore, the assumption conditions (i)–(iii) in Lemma 3.8 hold, which implies that  $\mathcal{C}_k^\nu = \limsup_{n \rightarrow +\infty} \mathcal{C}_k^{n,\nu}$  is unbounded closed connected such that  $z_* \in \mathcal{C}_k^\nu$ . Since  $(\frac{\lambda_k}{n}, \infty) \subset \text{Proj}_{\mathbb{R}} \mathcal{C}_k^{n,\nu}$ , we obtain that

$$\text{Proj}_{\mathbb{R}} \mathcal{C}_k^\nu = (0, \frac{\lambda_k}{f_0}).$$

(iii) If  $f_\infty = 0$ , define auxiliary function

$$\hat{f}_n(s) = \begin{cases} f(s), & s \in [-n, n]; \\ \frac{\frac{1}{n}s - f(n)}{n}(s - n) + f(n), & s \in (n, 2n); \\ -\frac{\frac{1}{n}s + f(-n)}{n}(n + s) + f(-n), & s \in (-2n, -n); \\ \frac{1}{n}s, & s \in (-\infty, -2n] \cup [2n, +\infty), \end{cases}$$

then we consider the following problem

$$\Delta^4 u(i - 2) = \lambda m(i) \hat{f}_n(u(i)), \quad u(0) = \Delta u(0) = u(N + 2) = \Delta u(N + 2) = 0. \tag{3.19}$$

Then  $\lim_{n \rightarrow +\infty} \hat{f}_n(s) = f(s)$  and

$$\lim_{s \rightarrow 0} \frac{\hat{f}_n(s)}{s} = f_0, \quad \lim_{s \rightarrow +\infty} \frac{\hat{f}_n(s)}{s} = \frac{1}{n}.$$

By the conclusion of (i), there are two unbounded continua  $\mathcal{C}_{k,n}^\pm$  of solutions to the problem (3.19) bifurcated from  $(\frac{\lambda_k}{f_0}, 0)$  and joining  $(\frac{\lambda_k}{f_0}, 0)$  to  $(n\lambda_k, \infty)$ .

Set  $z^* = (\infty, \infty)$ . It is easy to see that  $z^* = \liminf_{n \rightarrow +\infty} \mathcal{C}_{k,n}^\nu$  with  $\|z^*\|_{\mathbb{R} \times E} = +\infty$ . Using a similar argument of the proof of (ii) and applying Lemma 3.8 yields that  $\mathcal{C}_k^\nu = \limsup_{n \rightarrow +\infty} \mathcal{C}_{k,n}^\nu$  is unbounded closed connected such that  $z^* \in \mathcal{C}_k^\nu$ . Since  $(n\lambda_k, \infty) \subset \text{Proj}_{\mathbb{R}} \mathcal{C}_{k,n}^\nu$ , we obtain that

$$\text{Proj}_{\mathbb{R}} \mathcal{C}_k^\nu = (\frac{\lambda_k}{f_0}, \infty). \quad \square$$

**Proof of Corollary 3.2** The conclusions are directly desired from Theorem 3.1, we omit it.  $\square$

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## References

- [1] A. CABADA, R. R. ENGUIÇA. *Positive solutions of fourth order problems with clamped beam boundary conditions*. *Nonlinear Anal.*, 2011, **74**(10): 3112–3122.
- [2] Ruyun MA, Haiyan WANG, M. ELSANOSI. *Spectrum of a linear fourth-order differential operator and its applications*. *Math. Nachr.*, 2013, **286**(17-18): 1805–1819.
- [3] Minghe PEI, S. K. CHANG. *Monotone iterative technique and symmetric positive solutions for a fourth order boundary value problem*. *Math. Comput. Modelling*, 2010, **51**(9-10): 1260–1267.
- [4] J. WEBB, G. INFANTE, D. FRANCO. *Positive solutions of nonlinear fourth-order boundary-value problems with local and nonlocal boundary conditions*. *P. Roy. Soc. Edinb. A.*, 2008, **138**(2): 427–446.
- [5] W. G. KELLEY, A. C. PETERSON. *Difference Equations, in: An Introduction with Applications*. Harcourt/Academic Press, San Diego, CA, 2001.
- [6] J. R. GRAEF, S. HEIDARKHANI, Lingju KONG, et al. *Existence of solutions to a discrete fourth order boundary value problem*. *J. Difference Equ. Appl.*, 2018, **24**(6): 849–858.
- [7] Zhimin HE, Jianshe YU. *On the existence of positive solutions of fourth-order difference equations*. *Appl. Math. Comput.*, 2005, **161**(1): 139–148.
- [8] S. HEIDARKHANI, G. A. AFROUZI, A. SALARI, et al. *Discrete fourth-order boundary value problems with four parameters*. *Appl. Math. Comput.*, 2019, **346**: 167–182.
- [9] Jun JI, Bo YANG. *Eigenvalue comparisons for boundary value problems of the discrete beam equation*. *Adv. Difference Equ.* 2006, Art. ID 81025, 9 pp.
- [10] Jun JI, Bo YANG. *Spectral properties of a boundary value problem for the discrete beam equation*. *J. Appl. Math. Comput.*, 2017, **54**(1-2): 95–108.
- [11] Xia LIU, Tao ZHOU, Haiping SHI. *Existence and multiple solutions to a discrete fourth order boundary value problem*. *Adv. Difference Equ.* 2018, Paper No. 427, 13 pp.
- [12] Ruyun MA, Chenghua GAO. *Bifurcation of positive solutions of a nonlinear discrete fourth-order boundary value problem*. *Z. Angew. Math. Phys.*, 2013, **64**(3): 493–506.
- [13] Ruyun MA, Youji XU. *Existence of positive solution for nonlinear fourth-order difference equations*. *Comput. Math. Appl.*, 2010, **59**(12): 3770–3777.
- [14] F. R. GANTMACHER, M. G. KREIN. *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*. AMS Chelsea Publishing, Providence, RI, 2002.
- [15] F. R. GANTMACHER. *The Theory of Matrices*. Vols. 1, 2. Translated by K. A. Hirsch Chelsea Publishing Co., New York 1959 Vol. 1, x+374 pp. Vol. 2, ix+276 pp.
- [16] J. LÓPEZ-GÓMEZ, C. MORA-CORRAL. *Algebraic Multiplicity of Eigenvalues of Linear Operators*. Birkhäuser Verlag, Basel, 2007.
- [17] K. DEIMLING. *Nonlinear Functional Analysis*. Springer-Verlag, Berlin, 1985.
- [18] Guwei DAI, Zhaosheng FENG. *Unilateral global bifurcation for eigenvalue problems with homogeneous operator*. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2019, **29**(6): 1950084, 15 pp.
- [19] Guwei DAI. *Bifurcation and one-sign solutions of the  $p$ -Laplacian involving a nonlinearity with zeros*. *Discrete Contin. Dyn. Syst.*, 2016, **36**(10): 5323–5345.
- [20] Ruyun MA, Yulian AN. *Global structure of positive solutions for superlinear second order  $m$ -point boundary value problems*. *Topol. Methods Nonlinear Anal.*, 2009, **34**(2): 279–290.