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On Some Generalized Countably Compact Spaces II

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Abstract This paper is a continuation of [Erguang YANG. On some generalized countably compact spaces. J. Math. Res. Appl., 2019, **39**(5): 540–550] in which characterizations of some generalized countably compact spaces such as quasi-γ spaces, quasi-Nagata spaces, wN-spaces and wM-spaces in terms of real-valued functions were obtained. In this paper, we shall present some other forms of characterizations of the spaces mentioned above.

Keywords real-valued functions; semi-continuity; g-functions; quasi-γ spaces; $w\gamma$-spaces; quasi-Nagata spaces; wN-spaces; $M^{\#}$-spaces; wM-spaces

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1. Introduction

Throughout, a space always means a Hausdorff topological space.

Let X be a space. Denote by $\mathcal{C}_X$ ($\mathcal{S}_X$) the family of all compact (sequentially compact) subsets of X. $\mathcal{F}_0(X)$ is the family of all decreasing sequences of closed subsets of X with empty intersection. $\mathbb{N}$ is the set of all positive integers and $\langle x_n \rangle$ denotes a sequence.

A real-valued function f on a space X is called upper semi-continuous [1] if for any real number r, $\{x \in X : f(x) < r\}$ is open. We write $U(X)$ for the set of all upper semi-continuous functions from X into the interval $[0,1]$.

A g-function for a space (X,τ) is a map $g : \mathbb{N} \times X \to \tau$ such that for each $x \in X$ and $n \in \mathbb{N}$, $x \in g(n+1,x) \subset g(n,x)$. For a subset $A \subset X$, let $g(n,A) = \cup\{g(n,x) : x \in A\}$.

Consider the following conditions:

(q) If $x_n \in g(n,x)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(quasi-γ) If $x_n \in g(n,y_n)$ for each $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

($w\gamma$) If $y_n \in g(n,x)$ and $x_n \in g(n,y_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(β) If $x \in g(n,x_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

($w\sigma$) If $x \in g(n,y_n)$ and $y_n \in g(n,x_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(quasi-Nagata) If $y_n \in g(n,x_n)$ for each $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

($k\beta$) For each $K \in \mathcal{C}_X$, if $K \cap g(n,x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(wcc) If $\langle y_n \rangle$ has a cluster point and $y_n \in g(n,x_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

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($w\Delta$) If $x, x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(wN) If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

A space that has a g-function satisfying condition (q) ((quasi-γ), ($w\gamma$), (β), ($w\sigma$), (quasi-Nagata), ($k\beta$), (wcc), (wN)) is called a q-space [2] (quasi-γ space [3], $w\gamma$-space [4], β-space [5], $w\sigma$-space [6], quasi-Nagata space [7], $k\beta$-space [8], wcc-space [9], wN-space [4]). $w\Delta$-spaces [10] can be characterized by a g -function satisfying ($w\Delta$). The g-function satisfying condition (q) is called a q-function. The others are defined analogously.

A space X is called an $M^{\#}$-space [11] if there exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of closure preserving closed covers of X such that if $x_n \in st(x, \mathcal{F}_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

A space X is called a wM-space [12] if there exists a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of open covers of X such that if $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

Since every sequence in a countably compact space has a cluster point, all spaces listed above can be seen as generalizations of countably compact space. The relationships between these spaces are as follows.

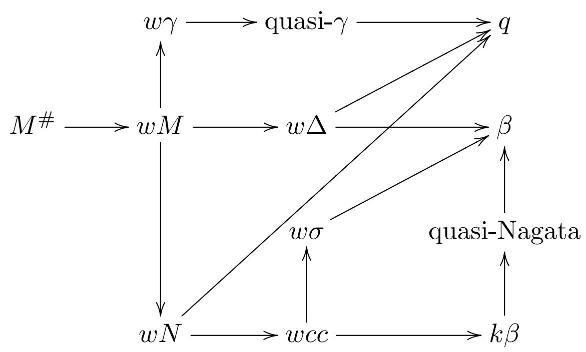

Diagram 1 The relationships between the spaces listed above

In [13], characterizations of some generalized countably compact spaces such as quasi-γ s-paces, quasi-Nagata spaces and wM-spaces in terms of real-valued functions were obtained. For example X is a quasi-Nagata space if and only if for each $F \in \tau^c$, one can assign a function $f_F \in U(X)$ such that (1) $F \subset f_F^{-1}(0)$, (2) if $F_1 \subset F_2$, then $f_{F_1} \geq f_{F_2}$ and (3) for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $S \in \mathcal{S}_X$, there is $m \in \mathbb{N}$ such that $\inf\{f_{F_m}(x) : x \in S\} > 0$.

In this paper, we shall continue with the study on the characterizations of some generalized countably compact spaces with real-valued functions. Some other forms of characterizations of the corresponding spaces are presented. The main results took the forms: X is a P-space if and only if for each point x of X, one can assign a function $f_x \in U(X)$ satisfying the corresponding conditions formulated with convergence of sequences. The proof of these results is essentially a process of turning real-valued functions and g-functions into each others. The following notations will be applied to shorten the expressions of the corresponding results.

Let $\mathcal{A}$ be a family of subsets of a space X, $\mathcal{F}$ a family of real-valued functions on X and $f : \mathcal{A} \to \mathcal{F}$. For $A \in \mathcal{A}$, we write f_A instead of $f(A)$. For a singleton $\{x\}$, we write f_x for $f_{\{x\}}$.

For $A \subset X$, denote $f_x[A] = \inf\{f_x(y) : y \in A\}$. Let $\langle x_n \rangle$, $\langle y_n \rangle$ be two sequences in X. Consider the following conditions.

(e) $f_x(x) = 0$.

(q_f) If $f_x(x_n) \to 0$, then $\langle x_n \rangle$ has a cluster point.

$(q\gamma_f)$ If $f_{y_n}(x_n) \to 0$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

$(w\gamma_f)$ If $\max\{f_x(y_n), f_{y_n}(x_n)\} \to 0$, then $\langle x_n \rangle$ has a cluster point.

(β_f) If $f_{x_n}(x) \to 0$, then $\langle x_n \rangle$ has a cluster point.

$(w\sigma_f)$ If $\max\{f_{y_n}(x), f_{x_n}(y_n)\} \to 0$, then $\langle x_n \rangle$ has a cluster point.

(qN_f) If $f_{x_n}(y_n) \to 0$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

$(k\beta_f)$ If $K \in \mathcal{C}_X$ and $f_{x_n}[K] \to 0$, then $\langle x_n \rangle$ has a cluster point.

$(w\Delta_f)$ If $\max\{f_{y_n}(x), f_{y_n}(x_n)\} \to 0$, then $\langle x_n \rangle$ has a cluster point.

(wN_f) If $\max\{f_x(y_n), f_{x_n}(y_n)\} \to 0$, then $\langle x_n \rangle$ has a cluster point.

2. Main results

In this section, we present characterizations of some generalized countably compact spaces listed in Section 1. The following lemma will be frequently applied to the proof of the corresponding results.

Lemma 2.1 ([14]) *Let g be a g-function for X. For each $x \in X$, let*

$$f_x = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,x)}.$$

Then $f_x \in U(X)$ satisfies (e) and for each $y \in X$, $f_x(y) \le \frac{1}{2^m}$ if and only if $y \in g(m, x)$.

Theorem 2.2 *For a space X, the following are equivalent.*

(a) X is a quasi-γ space.

(b) For each $S \in \mathcal{S}_X$, there exists $f_S \in U(X)$ such that (1) $S \subset f_S^{-1}(0)$; (2) if $S_1 \subset S_2$, then $f_{S_1} \ge f_{S_2}$; (3) if $f_S(x_n) \to 0$, then $\langle x_n \rangle$ has a cluster point.

(c) For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e) and $(q\gamma_f)$.

Proof (a) $\Rightarrow$ (b). Let g be the quasi-γ function for X and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1. For each $S \in \mathcal{S}_X$, let $f_S = \inf\{f_x : x \in S\}$. Then $f_S \in U(X)$ and (2) holds. If $y \in S$, then $f_S(y) = \inf\{f_x(y) : x \in S\} = 0$.

Suppose that $f_S(x_n) \to 0$. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $f_S(x_{n_k}) < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. By the definition of f_S, there exists a sequence $\langle y_k \rangle$ in S such that $f_{y_k}(x_{n_k}) < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Since S is sequentially compact, there exists a convergent subsequence $\langle y_{k_j} \rangle$ of $\langle y_k \rangle$. Then $f_{y_{k_j}}(x_{n_{k_j}}) < \frac{1}{2^{k_j}} \le \frac{1}{2^j}$ for each $j \in \mathbb{N}$. By Lemma 2.1, $x_{n_{k_j}} \in g(j, y_{k_j})$ for each $j \in \mathbb{N}$. Thus $\langle x_{n_{k_j}} \rangle$ has a cluster point which is also a cluster point of $\langle x_n \rangle$.

(b) $\Rightarrow$ (c). Assume (b). Then for each $x \in X$, there exists $f_x \in U(X)$ such that $f_x(x) = 0$. Suppose that $y_n \to x$ and $f_{y_n}(x_n) \to 0$. Let $S = \{y_n : n \in \mathbb{N}\} \cup \{x\}$. Then $S \in \mathcal{S}_X$. By (2) of (b), $f_S(x_n) \le f_{y_n}(x_n)$ for each $n \in \mathbb{N}$. From $f_{y_n}(x_n) \to 0$ it follows that $f_S(x_n) \to 0$. By (3) of (b), $\langle x_n \rangle$ has a cluster point.

(c) $\Rightarrow$ (a). For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$. Then g is a g-function for X. Suppose that $\langle y_n \rangle$ converges and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Then $f_{y_n}(x_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$ and thus $f_{y_n}(x_n) \to 0$. By $(q\gamma_f)$, $\langle x_n \rangle$ has a cluster point. Therefore, X is a quasi-γ space. $\square$

Theorem 2.3 *For a space X, the following are equivalent.*

(*a*) *X is a $w\gamma$-space.*

(*b*) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (†)*: for each $\langle F_n \rangle \in \mathcal{F}_0(X)$, $\inf\{\max\{f_x(x_n), f_{x_n}[F_n]\} : n \in \mathbb{N}\} > 0$.*

(*c*) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* $(w\gamma_f)$.

(*d*) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (‡)*: if $\langle y_n \rangle$ has a cluster point and $f_{y_n}(x_n) \to 0$, then $\langle x_n \rangle$ has a cluster point.*

Proof (a) $\Rightarrow$ (b). Let g be the $w\gamma$-function for X. For each $x \in X$, define $f_x \in U(X)$ as that in Lemma 2.1.

Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and assume that $\inf\{\max\{f_x(x_n), f_{x_n}[F_n]\} : n \in \mathbb{N}\} = 0$. Then there exist subsequences $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ and $\langle F_{n_k} \rangle$ of $\langle F_n \rangle$ such that $\max\{f_x(x_{n_k}), f_{x_{n_k}}[F_{n_k}]\} < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Then there exists $y_k \in F_{n_k}$ such that $f_{x_{n_k}}(y_k) < \frac{1}{2^k}$. By Lemma 2.1, $x_{n_k} \in g(k,x)$ and $y_k \in g(k, x_{n_k})$ for each $k \in \mathbb{N}$. Thus $\langle y_k \rangle$ has a cluster point. It follows that $\bigcap_{k\in\mathbb{N}} F_{n_k} \neq \emptyset$ and thus $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$, a contradiction.

(b) $\Rightarrow$ (c). Assume (b). Suppose that $\max\{f_x(y_n), f_{y_n}(x_n)\} \to 0$ and let $F_n = \overline{\{x_m : m \geq n\}}$ for each $n \in \mathbb{N}$. If $\langle x_n \rangle$ has no cluster point, then $\langle F_n \rangle \in \mathcal{F}_0(X)$. Since $f_{y_n}[F_n] \leq f_{y_n}(x_n)$ for each $n \in \mathbb{N}$, we have that $\max\{f_x(y_n), f_{y_n}[F_n]\} \to 0$ and thus $\inf\{\max\{f_x(y_n), f_{y_n}[F_n]\} : n \in \mathbb{N}\} = 0$, a contradiction to (†).

(c) $\Rightarrow$ (d). Assume (c). Suppose that x is a cluster point of $\langle y_n \rangle$ and $f_{y_n}(x_n) \to 0$. For each $k \in \mathbb{N}$, $\{y \in X : f_x(y) < \frac{1}{k}\}$ is an open neighborhood of x and thus there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $f_x(y_{n_k}) < \frac{1}{k}$ for each $k \in \mathbb{N}$. This implies that $f_x(y_{n_k}) \to 0$ and thus $\max\{f_x(y_{n_k}), f_{y_{n_k}}(x_{n_k})\} \to 0$. By $(w\gamma_f)$, $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(d) $\Rightarrow$ (a). Define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Suppose that $y_n \in g(n,x)$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$. Then $f_x(y_n) \to 0$ and $f_{y_n}(x_n) \to 0$. Since $f_x(y_n) \to 0$, by (‡), $\langle y_n \rangle$ has a cluster point. Now, since $f_{y_n}(x_n) \to 0$, $\langle x_n \rangle$ has a cluster point by (‡). Thus X is a $w\gamma$-space. $\square$

Analogously, we can prove the following result for q-spaces.

Proposition 2.4 *X is a q-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (q_f).

Theorem 2.5 *For a space X, the following are equivalent.*

(*a*) *X is a quasi-Nagata space.*

(*b*) *There exists a g-function g for X such that for each $S \in \mathcal{S}_X$, if $S \cap g(n, x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.*

(*c*) *For each* $x \in X$, *there exists* $f_x \in U(X)$ *satisfying* (*e*) *and* ($\sharp$): *if* $S \in \mathcal{S}_X$ *and* $f_{x_n}[S] \to 0$, *then* $\langle x_n \rangle$ *has a cluster point.*

(*d*) *For each* $x \in X$, *there exists* $f_x \in U(X)$ *satisfying* (*e*) *and* (qN_f).

Proof (a) $\Rightarrow$ (b). Let g be the quasi-Nagata function for X and $S \in \mathcal{S}_X$. Suppose that $y_n \in S \cap g(n, x_n)$ for each $n \in \mathbb{N}$. Then there exists a convergent subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$. Since $y_{n_k} \in g(k, x_{n_k})$ for each $k \in \mathbb{N}$, $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(b) $\Rightarrow$ (c). Let g be the g-function in (b) and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1.

Suppose that $S \in \mathcal{S}_X$ and $f_{x_n}[S] \to 0$. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $f_{x_{n_k}}[S] < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Thus there exists $\langle y_k \rangle$ in S such that $f_{x_{n_k}}(y_k) < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. By Lemma 2.1, $y_k \in g(k, x_{n_k})$ and hence $S \cap g(k, x_{n_k}) \neq \emptyset$ for each $k \in \mathbb{N}$. By (b), $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(c) $\Rightarrow$ (d). Suppose that $y_n \to x$ and $f_{x_n}(y_n) \to 0$. Let $S = \{y_n : n \in \mathbb{N}\} \cup \{x\}$. Then $S \in \mathcal{S}_X$ and $f_{x_n}[S] \leq f_{x_n}(y_n)$ for each $n \in \mathbb{N}$. From $f_{x_n}(y_n) \to 0$ it follows that $f_{x_n}[S] \to 0$. By ($\sharp$), $\langle x_n \rangle$ has a cluster point.

(d) $\Rightarrow$ (a). Similar to the proof of (c) $\Rightarrow$ (a) of Theorem 2.2. $\square$

Theorem 2.6 *X is a $k\beta$-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* ($k\beta_f$).

Proof Necessity is similar to the proof of (b) $\Rightarrow$ (c) of Theorem 2.5.

Conversely, define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $K \in \mathcal{C}_X$ and $y_n \in K \cap g(n, x_n)$ for each $n \in \mathbb{N}$. Then $f_{x_n}[K] \leq f_{x_n}(y_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$ which implies that $f_{x_n}[K] \to 0$. By ($k\beta_f$), $\langle x_n \rangle$ has a cluster point. Thus X is a $k\beta$-space. $\square$

For β-spaces, we have the following.

Proposition 2.7 *X is a β-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (β_f).

Theorem 2.8 *X is a wcc-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (wcc_f): *if $\langle y_n \rangle$ has a cluster point and $f_{x_n}(y_n) < \frac{1}{2^n}$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.*

Proof Let g be the *wcc*-function for X and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1.

Suppose that $\langle y_n \rangle$ has a cluster point x and $f_{x_n}(y_n) < \frac{1}{2^n}$ for each $n \in \mathbb{N}$. By Lemma 2.1, $y_n \in g(n, x_n)$ for each $n \in \mathbb{N}$. Thus $\langle x_n \rangle$ has a cluster point.

Conversely, define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{2^n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Suppose that $\langle y_n \rangle$ has a cluster point and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$. Then $f_{x_n}(y_n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. By (wcc_f), $\langle x_n \rangle$ has a cluster point. Thus X is a *wcc*-space. $\square$

Theorem 2.9 *X is a $w\sigma$-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* ($w\sigma_f$).

Proof Let g be the $w\sigma$-function for X and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1.

Suppose that $\max\{f_{y_n}(x), f_{x_n}(y_n)\} \to 0$. Then there exist subsequences $\langle x_{n_k}\rangle$ of $\langle x_n\rangle$ and $\langle y_{n_k}\rangle$ of $\langle y_n\rangle$ such that $\max\{f_{y_{n_k}}(x), f_{x_{n_k}}(y_{n_k})\} < \frac{1}{2^k}$. By Lemma 2.1, $x \in g(k, y_{n_k})$ and $y_{n_k} \in g(k, x_{n_k})$ for each $k \in \mathbb{N}$. Thus $\langle x_{n_k}\rangle$ has a cluster point and so does $\langle x_n\rangle$.

Conversely, define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Suppose that $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$. Then $\max\{f_{y_n}(x), f_{x_n}(y_n)\} \to 0$. By ($w\sigma_f$), $\langle x_n\rangle$ has a cluster point. Thus X is a $w\sigma$-space. □

Similarly, we can prove the following.

Theorem 2.10 *X is a $w\Delta$-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* ($w\Delta_f$).

In the sequel, the following notation will be used: ($\flat$) if $\langle y_n\rangle$ has a cluster point and $f_{x_n}(y_n) \to 0$, then $\langle x_n\rangle$ has a cluster point.

Theorem 2.11 *For a space X, the following are equivalent.*

(*a*) *X is a wN-space.*

(*b*) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*), (q_f) *and* ($\flat$).

(*c*) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying* (*e*) *and* (wN_f).

Proof (a) $\Rightarrow$ (b). Let g be the wN-function for X and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1.

If $f_x(x_n) \to 0$, then there exists a subsequence $\langle x_{n_k}\rangle$ of $\langle x_n\rangle$ such that $f_x(x_{n_k}) < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. By Lemma 2.1, $x_{n_k} \in g(k,x)$ and thus $g(k,x) \cap g(k, x_{n_k}) \neq \emptyset$ for each $k \in \mathbb{N}$. Thus $\langle x_{n_k}\rangle$ has a cluster point and so does $\langle x_n\rangle$.

Suppose that $\langle y_n\rangle$ has a cluster point x and $f_{x_n}(y_n) \to 0$. Then there exists a subsequence $\langle y_{n_k}\rangle$ of $\langle y_n\rangle$ such that $y_{n_k} \in g(k,x)$ for each $k \in \mathbb{N}$. Since $f_{x_{n_k}}(y_{n_k}) \to 0$, there exists subsequences $\langle x_{n_{k_j}}\rangle$ of $\langle x_{n_k}\rangle$ and $\langle y_{n_{k_j}}\rangle$ of $\langle y_{n_k}\rangle$ such that $f_{x_{n_{k_j}}}(y_{n_{k_j}}) < \frac{1}{2^j}$ for each $j \in \mathbb{N}$. By Lemma 2.1, $y_{n_{k_j}} \in g(j, x_{n_{k_j}})$ for each $j \in \mathbb{N}$. Since $y_{n_{k_j}} \in g(j,x)$, we have that $g(j,x) \cap g(j, x_{n_{k_j}}) \neq \emptyset$. Thus $\langle x_{n_{k_j}}\rangle$ has a cluster point and so does $\langle x_n\rangle$.

(b) $\Rightarrow$ (c). Suppose that $\max\{f_x(y_n), f_{x_n}(y_n)\} \to 0$. Then $f_x(y_n) \to 0$ and $f_{x_n}(y_n) \to 0$. By (q_f), $\langle y_n\rangle$ has a cluster point. Then by ($\flat$), $\langle x_n\rangle$ has a cluster point.

(c) $\Rightarrow$ (a). Similar to the proof of the sufficiency of Theorem 2.9. □

Lemma 2.12 ([13]) *X is an $M^{\#}$-space if and only if there exists a g-function g for X such that* (1) *g is a wN-function;* (2) *if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$.*

In the following two theorems, we shall use the notation: (U) for each $x, y, z \in X$, $f_x(z) \leq \max\{f_x(y), f_y(z)\}$.

Theorem 2.13 *For a space X, the following are equivalent.*

(a) *X is an $M^{\#}$-space.*

(b) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), (q_f), ($\flat$) and (U).*

(c) *For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), (wN_f) and (U).*

Proof (a) $\Rightarrow$ (b). Let g be the g-function in Lemma 2.12 and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1. (q_f) and ($\flat$) has been shown in the proof of (a) $\Rightarrow$ (b) of Theorem 2.11. By (2) of Lemma 2.12, we can show that f_x satisfies (U) (see [13]).

(b) $\Rightarrow$ (c). Similar to the proof of (b) $\Rightarrow$ (c) of Theorem 2.11.

(c) $\Rightarrow$ (a). Define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. With a similar argument to the proof of (c) $\Rightarrow$ (a) of Theorem 2.11, we can show that g is a wN-function. From (U) it follows that if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$. By Lemma 2.12, X is an $M^{\#}$-space. □

For the definition of a $\Sigma^{\#}$-space [15]. It was shown in [16] that X is a $\Sigma^{\#}$-space if and only if there exists a g-function g for X such that (1) g is a β-function; (2) if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$. Thus we have the following.

Theorem 2.14 *X is a $\Sigma^{\#}$-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), (β_f) and (U).*

A g-function g for a space X is called symmetric if for each $x, y \in X$ and $n \in \mathbb{N}$, $y \in g(n,x)$ if and only if $x \in g(n,y)$.

Lemma 2.15 *For a space X, the following are equivalent.*

(a) *X is a wM-space.*

(b) *There exists a symmetric g-function g for X satisfying ($w\gamma$).*

(c) *There exists a g-function g for X satisfying ($w\gamma$) and (wcc).*

Proof (a) $\Rightarrow$ (b). Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a sequence of open covers for a wM-space X and $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in \mathbb{N}$. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = st(x, \mathcal{G}_n)$. Then g is the required g-function for X.

(b) $\Rightarrow$ (c). Suppose that $\langle y_n \rangle$ has a cluster point x and $y_n \in g(n, x_n)$ for each $n \in \mathbb{N}$. Then there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in g(k,x)$ for each $k \in \mathbb{N}$. Since $y_{n_k} \in g(n_k, x_{n_k}) \subset g(k, x_{n_k})$ and g is symmetric, $x_{n_k} \in g(k, y_{n_k})$. Now, by ($w\gamma$), $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(c) $\Rightarrow$ (a). For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{g(n,x) : x \in X\}$. Then $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a sequence of open covers of X.

Suppose that $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$. Then there exist $y_n, z_n, w_n \in X$ such that $x \in g(n, z_n)$, $w_n \in g(n, y_n) \cap g(n, z_n)$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Since $x \in g(n, z_n)$ for each $n \in \mathbb{N}$, by (wcc), $\langle z_n \rangle$ has a cluster point p. Then there is a subsequence $\langle z_{n_k} \rangle$ of $\langle z_n \rangle$ such that $z_{n_k} \in g(k,p)$ for all $k \in \mathbb{N}$. Since $w_{n_k} \in g(k, z_{n_k})$, by ($w\gamma$), $\langle w_{n_k} \rangle$ has a cluster point. Now, since $w_{n_k} \in g(k, y_{n_k})$, by ($wcc$), $\langle y_{n_k} \rangle$ has a cluster point q. Then there is a subsequence $\langle y_{n_{k_j}} \rangle$ of $\langle y_{n_k} \rangle$ such that $y_{n_{k_j}} \in g(j,q)$ for each $j \in \mathbb{N}$. Since $x_{n_{k_j}} \in g(n_{k_j}, y_{n_{k_j}}) \subset g(j, y_{n_{k_j}})$ for each

$j \in \mathbb{N}$, by (wcc), $\langle x_{n_{k_j}} \rangle$ has a cluster point and so does $\langle x_n \rangle$. Therefore X is a wM-space. □

In the following theorem, the following notations are used.

(S) For each $x, y \in X$, $f_x(y) = f_y(x)$.

(wM_f) If $\max\{f_x(y_n), f_{y_n}(z_n), f_{x_n}(z_n)\} \to 0$, then $\langle x_n \rangle$ has a cluster point.

Theorem 2.16 *For a space X, the following are equivalent.*

(a) X is wM-space.

(b) For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), $(w\gamma_f)$ and (S).

(c) For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), $(w\gamma_f)$ and $(\flat)$.

(d) For each $x \in X$, there exists $f_x \in U(X)$ satisfying (e) and (wM_f).

Proof (a) $\Rightarrow$ (b). Let g be the g-function in Lemma 2.15 (b) and define $f_x \in U(X)$ for each $x \in X$ as that in Lemma 2.1. Since g is symmetric, f_x satisfies (S).

To show $(w\gamma_f)$, suppose that $\max\{f_x(y_n), f_{y_n}(x_n)\} \to 0$. Then there exist subsequences $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$, $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $\max\{f_x(y_{n_k}), f_{y_{n_k}}(x_{n_k})\} < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. By Lemma 2.1, $y_{n_k} \in g(k, x)$ and $x_{n_k} \in g(k, y_{n_k})$ for each $k \in \mathbb{N}$. By Lemma 2.15 (b), $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(b) $\Rightarrow$ (c). We only need to show $(\flat)$. Suppose that $\langle y_n \rangle$ has a cluster point x and $f_{x_n}(y_n) \to 0$. For each $n \in \mathbb{N}$, $U_n = \{y \in X : f_x(y) < \frac{1}{n}\}$ is an open neighborhood of x. Since x is a cluster point of $\langle y_n \rangle$, there is a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in U_k$ for each $k \in \mathbb{N}$. It follows that $f_x(y_{n_k}) < \frac{1}{k}$ and thus $f_x(y_{n_k}) \to 0$. Since $f_{x_{n_k}}(y_{n_k}) \to 0$, by (S), we have that $f_{y_{n_k}}(x_{n_k}) \to 0$. Thus $\max\{f_x(y_{n_k}), f_{y_{n_k}}(x_{n_k})\} \to 0$. By $(w\gamma_f)$, $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

(c) $\Rightarrow$ (d). Suppose that $\max\{f_x(y_n), f_{y_n}(z_n), f_{x_n}(z_n)\} \to 0$. By $(w\gamma_f)$, $\langle z_n \rangle$ has a cluster point. Now, by $(\flat)$, $\langle x_n \rangle$ has a cluster point.

(d) $\Rightarrow$ (a). Define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. We show that g satisfies $(w\gamma)$ and (wcc).

Suppose that $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Then $\max\{f_x(y_n), f_{y_n}(x_n)\} \to 0$. Let $z_n = x_n$ for each $n \in \mathbb{N}$. Then $\max\{f_x(y_n), f_{y_n}(z_n), f_{x_n}(z_n)\} \to 0$. By (wM_f), $\langle x_n \rangle$ has a cluster point.

Now suppose that $\langle y_n \rangle$ has a cluster point x and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in g(k, x)$ for each $k \in \mathbb{N}$ and thus $f_x(y_{n_k}) \to 0$. Since $y_{n_k} \in g(k, x_{n_k})$ for each $k \in \mathbb{N}$, we have that $f_{x_{n_k}}(y_{n_k}) \to 0$. Let $z_k = y_{n_k}$ for each $k \in \mathbb{N}$. Then $\max\{f_x(y_{n_k}), f_{y_{n_k}}(z_k), f_{x_{n_k}}(z_k)\} \to 0$. By (wM_f), $\langle x_{n_k} \rangle$ has a cluster point and so does $\langle x_n \rangle$.

By Lemma 2.15 (c), X is a wM-space. □

Consider the following condition imposed on f_x: $(*)$ for each $A \subset X$ and $x \in \overline{A}$, there exists a sequence $\langle x_n \rangle$ in A such that $f_{x_n}(x) \to 0$. It was shown that [14] if $f_x \in U(X)$ satisfies (e) and (S), then it also satisfies $(*)$.

Theorem 2.17 *X is a wM-space if and only if for each $x \in X$, there exists $f_x \in U(X)$*

satisfying (e), $(*)$ *and* $(w\sigma_f)$.

Proof Let X be a wM-space. By Theorem 2.16 (b), for each $x \in X$, there exists $f_x \in U(X)$ satisfying (e), (S) and $(w\gamma_f)$. Then f_x also satisfies $(*)$. Since f_x satisfies (S) and $(w\gamma_f)$, it also satisfies $(w\sigma_f)$.

Conversely, for each $x \in X$ and $n \in \mathbb{N}$, let $V_n(x) = \{y \in X : f_x(y) < \frac{1}{n}\} \setminus \overline{\{y \in X : f_y(x) \geq \frac{1}{n}\}}$. Then $\{V_n(x) : n \in \mathbb{N}\}$ is a decreasing sequence of open neighborhoods of x [14]. For each $x \in X$, define $g_x \in U(X)$ as that in Lemma 2.1 (replacing $g(n,x)$ with $V_n(x)$). Then for each $n \in \mathbb{N}$, $g_x(y) \leq \frac{1}{2^n}$ implies that $y \in V_n(x)$, i.e., $f_x(y) < \frac{1}{n}$ and $f_y(x) < \frac{1}{n}$. We show that g_x satisfies (wM_f).

Suppose that $\max\{g_x(y_n), g_{y_n}(z_n), g_{x_n}(z_n)\} \to 0$. Then there exist subsequences $\langle x_{n_k}\rangle$ of $\langle x_n\rangle$, $\langle y_{n_k}\rangle$ of $\langle y_n\rangle$ and $\langle z_{n_k}\rangle$ of $\langle z_n\rangle$ such that $\max\{g_x(y_{n_k}), g_{y_{n_k}}(z_{n_k}), g_{x_{n_k}}(z_{n_k})\} < \frac{1}{2^k}$ and thus $\max\{f_{y_{n_k}}(x), f_{z_{n_k}}(y_{n_k}), f_{x_{n_k}}(z_{n_k})\} < \frac{1}{k}$ for each $k \in \mathbb{N}$. Then $\max\{f_{y_{n_k}}(x), f_{z_{n_k}}(y_{n_k})\} \to 0$. Since f_x satisfies $(w\sigma)$, $\langle z_{n_k}\rangle$ has a cluster point p. Then there exists a subsequence $\langle z_{n_{k_j}}\rangle$ of $\langle z_{n_k}\rangle$ such that $z_{n_{k_j}} \in V_j(p)$ and thus $f_{z_{n_{k_j}}}(p) < \frac{1}{j}$ for each $j \in \mathbb{N}$. Since also $f_{x_{n_{k_j}}}(z_{n_{k_j}}) < \frac{1}{j}$ for each $j \in \mathbb{N}$, we have that $\max\{f_{z_{n_{k_j}}}(p), f_{x_{n_{k_j}}}(z_{n_{k_j}})\} \to 0$. By $(w\sigma)$, $\langle x_{n_{k_j}}\rangle$ has a cluster point which is also a cluster point of $\langle x_n\rangle$. This implies that g_x satisfies (wM_f). By Theorem 2.16 (d), X is a wM-space. $\square$

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