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Algebraic Properties of Reduced Biquaternions

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Abstract In this paper, we study the algebraic properties of reduced biquaternions. With the aid of the real and complex matrix representations of reduced biquaternions, we introduce the concept of the Moore-Penrose inverse in reduced biquaternions. As applications, we solve the linear equations $ax = d$ and the quadratic equation $ax^2 + bx + c = 0$. By complex representation, we find the nth roots, the nth powers of a reduced biquaternion and obtain some properties of the matrix exponential of reduced biquaternion matrices.

Keywords reduced biquaternion; Moore-Penrose inverse; power function; root; exponential function

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1. Introduction

In 1843, Hamilton [1] discovered quaternions, which is a noncommutative algebra $\mathbb{H} = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_i \in \mathbb{R}, i = 0, 1, 2, 3\}$, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. There are many studies on geometric, algebraic physical meaning of Hamilton quaternions. In 1849, James Cockle [2] introduced the split quaternions $\mathbb{H}_s = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_i \in \mathbb{R}, i = 0, 1, 2, 3\}$, where $\mathbf{i}^2 = -\mathbf{j}^2 = -\mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{k} = -\mathbf{ji}$. The split quaternions are recently developing topic as they play an important role in a modern understanding of four dimensional physics [3, 4]. Complex time-discrete systems offer several advantages as realizing two complementary transfer functions with one system [5,6]. The treatment of complex discrete-time signals needs some complex number systems of high order. Such number systems are also referred to as hypercomplex numbers. Hamilton quaternions and split quaternions are two examples of hypercomplex numbers. However, quaternions turned out not to be well suited for the purpose of digital signal processing. Due to this situation, in 1990 Hans-Dieter Schutte and Jorg Wenzel introduced a new hypercomplex number, the reduced biquaternion [7], in digital signal processing.

There are many studies in the mathematical properties of hypercomplex number systems. For example, Huang and Cao [8, 9] obtained the quadratic formulas for quaternions and split quaternions. Özdemir etc. [3, 4] investigated the root, power and exponential functions of split

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quaternion or split quaternion matrix. After the first foundation of the theory of reduced biquaternions [7], there are numerous applications of reduced biquaternions in both mathematics and physics [10–13].

In this paper we mainly concentrate on algebraic properties of reduced biquaternions. In Section 2, by using the real and complex matrix representations of reduced biquaternions, we introduce the concept of the Moore-Penrose inverse. We will obtain some properties of the Moore-Penrose inverse and solve the linear equation $ax = d$. In Section 3, we find the nth roots and the nth powers of a reduced biquaternion. In Section 4, we solve the quadratic equation $ax^2 + bx + c = 0$. In Section 5, we obtain some properties of the matrix exponential of reduced biquaternion matrices.

2. The Moore-Penrose inverse of elements in $\mathbb{H}_r$

We recall that the reduced biquaternions [7] are elements of a 4-dimensional associative algebra which can be represented as

$$\mathbb{H}_r = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},\quad q_i \in \mathbb{R},\ i = 0, 1, 2, 3\},$$

where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis of $\mathbb{H}_r$ with the following multiplication rules:

$$\mathbf{i}^2 = -\mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = \mathbf{ji} = \mathbf{k},\ \mathbf{jk} = \mathbf{kj} = \mathbf{i},\ \mathbf{ki} = \mathbf{ik} = -\mathbf{j}. \tag{2.1}$$

Hence the reduced biquaternions is a commutative algebra.

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real numbers and complex numbers, respectively. We can view $\mathbb{R} = \operatorname{span}\{1\}$ and $\mathbb{C} = \operatorname{span}\{1, \mathbf{i}\}$. According to the multiplication rules, a reduced biquaternion can be written as

$$q = (q_0 + q_1\mathbf{i}) + (q_2 + q_3\mathbf{i})\mathbf{j} = c_1 + c_2\mathbf{j},\quad c_1, c_2 \in \mathbb{C}.$$

Hence $\mathbb{H}_r = \mathbb{C} + \mathbb{C}\mathbf{j}$.

For $q = (q_0 + q_1\mathbf{i}) + (q_2 + q_3\mathbf{i})\mathbf{j} = c_1 + c_2\mathbf{j}$, $c_1, c_2 \in \mathbb{C}$, let $q^* = q_0 + q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ be the conjugate of q and

$$N(q) = qq^* = q_0^2 - q_1^2 - q_2^2 + q_3^2 + 2(q_0q_1 - q_2q_3)\mathbf{i}; \tag{2.2}$$

let $\bar{q} = q_0 - q_1\mathbf{i} + q_2\mathbf{j} - q_3\mathbf{k}$ be the complex conjugate of q and

$$I(q) = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 + 2(q_0q_2 + q_1q_3)\mathbf{j}. \tag{2.3}$$

For a complex number $c_1 = q_0 + q_1\mathbf{i}$, the conventional conjugate $\bar{c_1} = q_0 - q_1\mathbf{i}$ is the same as its complex conjugate in $\mathbb{H}_r$. Using the complex representation, we have

$$q^* = c_1 - c_2\mathbf{j},\quad \bar{q} = \bar{c_1} + \bar{c_2}\mathbf{j}. \tag{2.4}$$

Unlike the Hamilton quaternion algebra, the reduced biquaternions contain nontrivial zero divisors. For example, $(1 + \mathbf{j})(1 - \mathbf{j}) = 0$. In such an algebra, it is interesting to search for its pseudo-inverse for zero divisions. This motivates us to the real and complex matrices representations of reduced biquaternions.

Using the multiplication rules, we have the following formulas:

$$qx = (q_0x_0 - q_1x_1 + q_2x_2 - q_3x_3) + (q_1x_0 + q_0x_1 + q_3x_2 + q_2x_3)\mathbf{i} + \\ (q_2x_0 - q_3x_1 + q_0x_2 - q_1x_3)\mathbf{j} + (q_3x_0 + q_2x_1 + q_1x_2 + q_0x_3)\mathbf{k}.$$

Denote $\overrightarrow{x} = (x_0, x_1, x_2, x_3)^T$ for $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}_r$, where T denotes the transpose of a matrix. Then the above equation can be reformulated as

$$\overrightarrow{qx} = R(q)\overrightarrow{x},$$

where

$$R(q) = \begin{pmatrix} q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \\ q_2 & -q_3 & q_0 & -q_1 \\ q_3 & q_2 & q_1 & q_0 \end{pmatrix}. \tag{2.5}$$

An alternative way is to consider its complex matrix representation. With a little abuse of notation, we also denote $\overrightarrow{x} = (t_1, t_2)^T$ as complex vector for $x = (x_0 + x_1\mathbf{i}) + (x_2 + x_3\mathbf{i})\mathbf{j} = t_1 + t_2\mathbf{j} \in \mathbb{H}_r$. Let $q = (q_0 + q_1\mathbf{i}) + (q_2 + q_3\mathbf{i})\mathbf{j} = c_1 + c_2\mathbf{j}, c_1, c_2 \in \mathbb{C}$. Using the multiplication rules, $qx = c_1t_1 + c_2t_2 + (c_2t_1 + c_1t_2)\mathbf{j}$. Hence

$$\overrightarrow{qx} = C(q)\overrightarrow{x},$$

where

$$C(q) = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix}. \tag{2.6}$$

Proposition 2.1 *The two mappings of $R : \mathbb{H}_r \to \mathbb{R}^{4\times4}$, $C : \mathbb{H}_r \to \mathbb{C}^{2\times2}$ given by (2.5) and (2.6) are isomorphic.*

Proof It is obvious that $\mathrm{Ker}(R) = \mathrm{Ker}(C) = 1$. Since $pqx = (pq)x = p(qx)$, we have that $\overrightarrow{pqx} = R(pq)\overrightarrow{x} = R(p)\overrightarrow{qx} = R(p)R(q)\overrightarrow{x}$. The arbitrary of x implies $R(pq) = R(p)R(q)$. Similarly, we have $C(pq) = C(p)C(q)$. □

We can verify the following proposition directly.

Proposition 2.2 *Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $R(q)$, $C(q)$ be given by (2.5) and (2.6). Then*

(i) The eigenvalues of $C(q)$ are $(q_0-q_2)+(q_1-q_3)\mathbf{i}, (q_0+q_2)+(q_1+q_3)\mathbf{i}$ and $\det(C(q)) = N(q)$.

(ii) The eigenvalues of $R(q)$ are $(q_0 - q_2) \pm (q_3 - q_1)\mathbf{i}, (q_0 + q_2) \pm (q_3 + q_1)\mathbf{i}$ and

$$\begin{aligned} \det(R(q)) &= (q_0^2 - q_1^2 - q_2^2 + q_3^2)^2 + (2q_0q_1 - 2q_2q_3)^2 \\ &= (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 - (2q_0q_2 + 2q_1q_3)^2 \\ &= N(I(q)) = I(N(q)). \end{aligned}$$

We recall that the Moore-Penrose inverse of a complex matrix A is the unique complex matrix X satisfying the following equations:

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA,$$

where * is the conjugate of a matrix. We denote the Moore-Penrose inverse of A by $A^\dagger$. The following lemma is well known.

Lemma 2.3 *Let E_n be the identity matrix of order n and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Then the linear equation $Ax = b$ has a solution if and only if $AA^\dagger b = b$, furthermore the general solution is*

$$x = A^\dagger b + (E_n - A^\dagger A)y, \quad \forall y \in \mathbb{R}^n.$$

For $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, $q_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, if $\det(R(q)) = (q_0^2 - q_1^2 - q_2^2 + q_3^2)^2 + (2q_0q_1 - 2q_2q_3)^2 \neq 0$, then $N(q) \neq 0$ and $pq = qp = 1$ for $p = \frac{q*}{N(q)}$. That is, p is the inverse of q and denoted by

$$q^{-1} = \frac{q^*}{N(q)}, \quad N(q) \neq 0. \tag{2.7}$$

The above formula implies that the set of zero divisions of reduced biquaternions is

$$Z(\mathbb{H}_r) = \{q \in \mathbb{H}_r : N(q) = 0\}. \tag{2.8}$$

Proposition 2.4 *By definition, we have $Z(\mathbb{H}_r) = \{q \in \mathbb{H}_r : q = c_1 \pm c_1\mathbf{j}, c_1 = q_0 + q_1\mathbf{i} \in \mathbb{C}\}$.*

Proof Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in Z(\mathbb{H}_r)$. Then

$$q_0^2 - q_1^2 - q_2^2 + q_3^2 = 0, \quad q_0q_1 - q_2q_3 = 0.$$

Solving the above equations with unknowns q_2 and q_3, we get two solutions $q_2 = q_0$, $q_3 = q_1$ or $q_2 = -q_0$, $q_3 = -q_1$. Hence we have $q = q_0 + q_1\mathbf{i} + q_0\mathbf{j} + q_1\mathbf{k}$ or $q = q_0 + q_1\mathbf{i} - q_0\mathbf{j} - q_1\mathbf{k}$. $\square$

By the definition of Moore-Penrose inverse in matrix algebra, we can verify the following proposition.

Proposition 2.5 *For $q = q_0 + q_1\mathbf{i} + q_0\mathbf{j} + q_1\mathbf{k} = c_1 + c_1\mathbf{j} \in Z(\mathbb{H}_r)$, we have*

$$R(q)^\dagger = \frac{1}{4(q_0^2 + q_1^2)} \begin{pmatrix} q_0 & q_1 & q_0 & q_1 \\ -q_1 & q_0 & -q_1 & q_0 \\ q_0 & q_1 & q_0 & q_1 \\ -q_1 & q_0 & -q_1 & q_0 \end{pmatrix}$$

and

$$C(q)^\dagger = \frac{1}{4|c_1|^2} \begin{pmatrix} \overline{c_1} & \overline{c_1} \\ \overline{c_1} & \overline{c_1} \end{pmatrix} = \frac{1}{4c_1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};$$

for $q = q_0 + q_1\mathbf{i} - q_0\mathbf{j} - q_1\mathbf{k} \in Z(\mathbb{H}_r) = c_1 - c_1\mathbf{j}$, we have

$$R(q)^\dagger = \frac{1}{4(q_0^2 + q_1^2)} \begin{pmatrix} q_0 & q_1 & -q_0 & -q_1 \\ -q_1 & q_0 & q_1 & -q_0 \\ -q_0 & -q_1 & q_0 & q_1 \\ q_1 & -q_0 & -q_1 & q_0 \end{pmatrix}$$

and

$$C(q)^\dagger = \frac{1}{4|c_1|^2} \begin{pmatrix} \overline{c_1} & -\overline{c_1} \\ -\overline{c_1} & \overline{c_1} \end{pmatrix} = \frac{1}{4c_1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The above two propositions, together with (2.7), give rise to the following definition.

Definition 2.6 *The Moore-Penrose inverse of $q = c_1 + c_2\mathbf{j} \in \mathbb{H}_r$ with $c_1, c_2 \in \mathbb{C}$ is defined to be*

$$q^\dagger = \begin{cases} 0, & \text{if } q=0; \\ \frac{q^*}{N(q)}, & \text{if } c_1 \pm c_2 \neq 0; \\ \frac{\bar{q}}{4|c_1|^2}, & \text{if } c_1 + c_2 = 0 \text{ or } c_1 - c_2 = 0. \end{cases}$$

The essence of the above definition is to realize the following proposition.

Proposition 2.7 *Let $q \in \mathbb{H}_r$. Then $R(q^\dagger) = R(q)^\dagger$, $C(q^\dagger) = C(q)^\dagger$.*

By the above proposition and the fact that R and C are isomorphisms, we have the following identities:

Proposition 2.8 *Let $q \in \mathbb{H}_r$. Then*

$$C(q)C(q^\dagger)C(q) = C(q), \quad C(q^\dagger)C(q)C(q^\dagger) = C(q^\dagger);$$

$$C(q)C(q^\dagger) = (C(q)C(q^\dagger))^*, \quad C(q^\dagger)C(q) = (C(q^\dagger)C(q))^*;$$

$$R(q)R(q^\dagger)R(q) = R(q), \quad R(q^\dagger)R(q)R(q^\dagger) = R(q^\dagger);$$

$$R(q)R(q^\dagger) = (R(q)R(q^\dagger))^*, \quad R(q^\dagger)R(q) = (R(q^\dagger)R(q))^*.$$

By Definition 2.6, we have the following identities:

$$qq^\dagger q = q, \ q^\dagger qq^\dagger = q^\dagger, \ qq^\dagger = q^\dagger q = \frac{1}{2}(1+\mathbf{j}), \quad \forall\, q = c_1(1+\mathbf{j}) \in Z(\mathbb{H}_r) - \{0\} \tag{2.9}$$

and

$$qq^\dagger q = q, \ q^\dagger qq^\dagger = q^\dagger, \ qq^\dagger = q^\dagger q = \frac{1}{2}(1-\mathbf{j}), \quad \forall\, q = c_1(1-\mathbf{j}) \in Z(\mathbb{H}_r) - \{0\}. \tag{2.10}$$

It is obvious that $x = a^{-1}d$ is the unique solution of $ax = d$. For a is not invertible, we have the following theorem.

Theorem 2.9 *Let $a = c_1 + c_2\mathbf{j} \in Z(\mathbb{H}_r) - \{0\}$. Then the equation $ax = d$ is solvable if and only if*

$$\left(\frac{1}{2} + \frac{\bar{c_1}c_2 + \bar{c_2}c_1}{4|c_1|^2}\mathbf{j}\right)d = d, \tag{2.11}$$

in which case all solutions are given by

$$x = \frac{\bar{c_1} + \bar{c_2}\mathbf{j}}{4|c_1|^2}d + \left(\frac{1}{2} - \frac{\bar{c_1}c_2 + \bar{c_2}c_1}{4|c_1|^2}\mathbf{j}\right)y, \quad \forall\, y \in \mathbb{H}_r. \tag{2.12}$$

Proof It is obvious that $ax = d$ is equivalent to $R(a)\overrightarrow{x} = \overrightarrow{d}$. By Lemma 2.3, $ax = d$ is solvable if and only if

$$R(a)R(a)^\dagger \overrightarrow{d} = \overrightarrow{d}.$$

Returning to reduced biquaternion form by Proposition 2.8, we have $aa^\dagger d = d$. By Definition 2.6, we can rewrite $aa^\dagger d = d$ as (2.11). By Lemma 2.3, the general solution is

$$\overrightarrow{x} = R(a)^\dagger \overrightarrow{d} + (E_4 - R(a)^\dagger R(a))\overrightarrow{y}, \quad \forall\, y \in \mathbb{H}_r.$$

Hence the general solution can be expressed as

$$x = a^{\dagger} d + (1 - a^{\dagger}) y, \quad \forall\, y \in \mathbb{H}_r.$$

That is (2.12). This concludes the proof. □

In fact, since $a \in Z(\mathbb{H}_r) - \{0\}$ has only two types $a = c_1 + c_1\mathbf{j}$ or $a = c_1 - c_1\mathbf{j}$. Thus we only need to consider $(1+\mathbf{j})x = \frac{d}{c_1}$ or $(1-\mathbf{j})x = \frac{d}{c_1}$. Obviously, $(1+\mathbf{j})x = \frac{d}{c_1}$ is solvable if and only if $(1+\mathbf{j})d = 2d$ and the general solution is $x = \frac{d}{4c_1}(1+\mathbf{j}) + \frac{1}{2}(1-\mathbf{j})y$, $\forall\, y \in \mathbb{H}_r$. Similarly, $(1-\mathbf{j})x = \frac{d}{c_1}$ is solvable if and only if $(1-\mathbf{j})d = 2d$ and the general solution is $x = \frac{d}{4c_1}(1-\mathbf{j}) + \frac{1}{2}(1+\mathbf{j})y$, $\forall\, y \in \mathbb{H}_r$. Theorem 2.9 contains both cases in one formula.

3. The nth power and root functions in $\mathbb{H}_r$

By $\mathbf{j}^2 = 1$, we can express an nth power of reduced biquaternion $q = c_1 + c_2\mathbf{j}$ as follows:

(i) If n is an even number,

$$q^n = \sum_{k=0}^{n/2} \begin{pmatrix} n \\ 2k \end{pmatrix} c_1^{n-2k} c_2^{2k} + \Big[\sum_{k=0}^{n/2-1} \begin{pmatrix} n \\ 2k+1 \end{pmatrix} c_1^{n-2k-1} c_2^{2k+1}\Big]\mathbf{j};$$

(ii) If n is an odd number,

$$q^n = \sum_{k=0}^{\frac{n-1}{2}} \begin{pmatrix} n \\ 2k \end{pmatrix} c_1^{n-2k} c_2^{2k} + \Big[\sum_{k=0}^{\frac{n-1}{2}} \begin{pmatrix} n \\ 2k+1 \end{pmatrix} c_1^{n-2k-1} c_2^{2k+1}\Big]\mathbf{j}.$$

The above two formulas seem very complicated. This motivate us to focus the following two elements in $Z(\mathbb{H}_r)$ to investigate the algebraic properties of reduced biquaternions. Let

$$u_1 = \frac{1+\mathbf{j}}{2}, \quad u_2 = \frac{1-\mathbf{j}}{2}. \tag{3.1}$$

The two numbers are very important in reduced biquaternions. We list some algebraic properties of them:

$$u_1^n = u_1^{n-1} = \cdots = u_1, \quad u_2^n = u_2^{n-1} = \cdots = u_2; \tag{3.2}$$

$$u_1 u_2 = 0, \quad u_1 + u_2 = 1; \tag{3.3}$$

$$u_1 q = (c_1 + c_2)u_1, \ u_2 q = (c_1 - c_2)u_2, \quad \forall\, q = c_1 + c_2\mathbf{j}. \tag{3.4}$$

It is obvious that

$$Z(\mathbb{H}_r) = \{q \in \mathbb{H}_r : q = u_1 y \text{ or } q = u_2 y, \quad \forall\, y \in \mathbb{H}_r\}.$$

By Theorem 2.9, the solution set of $u_1 x = 0$ is $\{x = u_2 y, \forall\, y \in \mathbb{H}_r\}$ and the solution set of $u_2 x = 0$ is $\{x = u_1 y, \forall\, y \in \mathbb{H}_r\}$.

For $q = c_1 + c_2\mathbf{j}$, by (3.4) we can decompose it by two elements in $Z(\mathbb{H}_r)$ as follows:

$$q = (c_1 + c_2\mathbf{j})(u_1 + u_2) = (c_1 + c_2)u_1 + (c_1 - c_2)u_2. \tag{3.5}$$

The above decomposition is crucial in what follows.

Theorem 3.1 *Let $q = c_1 + c_2\mathbf{j}$, $c_1, c_2 \in \mathbb{C}$. Then*

$$q^n = \frac{(c_1+c_2)^n + (c_1-c_2)^n}{2} + \frac{(c_1+c_2)^n - (c_1-c_2)^n}{2}\mathbf{j}.$$

Proof By (3.3) and (3.4), we have

$$\begin{aligned} q^n &= [(c_1+c_2)u_1 + (c_1-c_2)u_2]^n = (c_1+c_2)^n u_1 + (c_1-c_2)^n u_2 \\ &= \frac{(c_1+c_2)^n + (c_1-c_2)^n}{2} + \frac{(c_1+c_2)^n - (c_1-c_2)^n}{2}\mathbf{j}. \quad \square \end{aligned} \tag{3.6}$$

We recall that when $z \in \mathbb{C} - \{0\}$, the nth roots of $z = re^{\mathbf{i}\theta}$ with $0 \le \theta < 2\pi$, $r > 0$ in complex number field are

$$(\sqrt[n]{z})_k = r^{1/n}(\cos\frac{\theta + 2k\pi}{n} + \sin\frac{\theta + 2k\pi}{n}), \quad k = 0, 1, \ldots, n-1. \tag{3.7}$$

We mainly relay on the above formulas to treat the equations over $\mathbb{H}_r$.

Definition 3.2 *Let $a \in \mathbb{H}_r$. A reduced biquaternion q satisfying the equation $q^n = a$ is called an nth root of a.*

We will use the formula (3.7) to represent the nth root of reduced biquaternions.

Theorem 3.3 *Let $a = z_1 + z_2\mathbf{j} \in \mathbb{H}_r$ with $z_1, z_2 \in \mathbb{C}$. Then the equation $q^n = a$ has roots as follows:*

(i) If $a = 0$, then $q = 0$;

(ii) If $a = z_1 + z_1\mathbf{j}$, $z_1 \neq 0$, then $q = 2^{\frac{1-n}{n}}(\sqrt[n]{z_1})_k(1+\mathbf{j})$, $k = 0, 1, \ldots, n-1$;

(iii) If $a = z_1 - z_1\mathbf{j}$, $z_1 \neq 0$, then $q = 2^{\frac{1-n}{n}}(\sqrt[n]{z_1})_k(1-\mathbf{j})$, $k = 0, 1, \ldots, n-1$;

(iv) If $a = z_1 + z_2\mathbf{j} \in \mathbb{H}_r - Z(\mathbb{H}_r)$, then

$$q = \frac{(\sqrt[n]{z_1+z_2})_j + (\sqrt[n]{z_1-z_2})_k}{2} + \frac{(\sqrt[n]{z_1+z_2})_j - (\sqrt[n]{z_1-z_2})_k}{2}\mathbf{j}, \quad j, k = 0, 1, \ldots, n-1.$$

Proof Let $q = c_1 + c_2\mathbf{j}$, $c_1, c_2 \in \mathbb{C}$ and $q^n = a$. It follows from Theorem 3.1 that

$$(c_1+c_2)^n = z_1 + z_2, \quad (c_1-c_2)^n = z_1 - z_2. \tag{3.8}$$

If $a = z_1 + z_2\mathbf{j} \in Z(\mathbb{H}_r)$, then there are three cases: $a = 0, a = z_1 + z_1\mathbf{j}$ or $a = z_1 - z_1\mathbf{j}$, $z_1 \neq 0$. Obviously, if $a = 0$, then $q = 0$. If $a = z_1 + z_1\mathbf{j}$, then

$$c_2 = c_1 \text{ and } c_1 = 2^{\frac{1-n}{n}}(\sqrt[n]{z_1})_k, \quad k = 0, 1, \ldots, n-1.$$

If $a = z_1 - z_1\mathbf{j}$, then

$$c_2 = -c_1 \text{ and } c_1 = 2^{\frac{1-n}{n}}(\sqrt[n]{z_1})_k, \quad k = 0, 1, \ldots, n-1.$$

If $a = z_1 + z_2\mathbf{j} \in \mathbb{H}_r - Z(\mathbb{H}_r)$, then by (3.8) we have

$$c_1 = \frac{(\sqrt[n]{z_1+z_2})_j + (\sqrt[n]{z_1-z_2})_k}{2}, \quad c_2 = \frac{(\sqrt[n]{z_1+z_2})_j - (\sqrt[n]{z_1-z_2})_k}{2}, \quad j, k = 0, 1, \ldots, n-1.$$

These facts conclude the proof. $\square$

4. Quadratic equation $ax^2 + bx + c = 0$

In this section we consider the quadratic equation $ax^2+bx+c=0$ with $a\neq 0$.

Theorem 4.1 *Let $a=a_1+a_2\mathbf{j}\in\mathbb{H}_r-Z(\mathbb{H}_r)$, $b=b_1+b_2\mathbf{j}$, $c=c_1+c_2\mathbf{j}$, $a_i,b_i,c_i,\in\mathbb{C}$, $i=1,2$. Then the equation $ax^2+bx+c=0$ has roots as follows:*

(i) If $b^2-4ac=0$, then $x=-\frac{b}{2a}$;

(ii) If $b^2-4ac=z_1+z_1\mathbf{j}$, $0\neq z_1\in\mathbb{C}$, then

$$x=\frac{(\sqrt[2]{z_1})_k(1+\mathbf{j})}{2\sqrt{2}a}-\frac{b}{2a},\quad k=0,1;$$

(iii) If $b^2-4ac=z_1-z_1\mathbf{j}$, $0\neq z_1\in\mathbb{C}$, then

$$x=\frac{(\sqrt[2]{z_1})_k(1-\mathbf{j})}{2\sqrt{2}a}-\frac{b}{2a},\quad k=0,1;$$

(iv) If $b^2-4ac=z_1+z_2\mathbf{j}\in\mathbb{H}_r-Z(\mathbb{H}_r)$, $z_1,z_2\in\mathbb{C}$, then

$$x=\frac{1}{2a}[\frac{(\sqrt[2]{z_1+z_2})_j+(\sqrt[2]{z_1-z_2})_k}{2}+\frac{(\sqrt[2]{z_1+z_2})_j-(\sqrt[2]{z_1-z_2})_k}{2}\mathbf{j}]-\frac{b}{2a},\quad j,k=0,1.$$

Proof If $a\in\mathbb{H}_r-Z(\mathbb{H}_r)$, then $ax^2+bx+c=0$ is equivalent to $[2a(x+\frac{b}{2a})]^2=b^2-4ac$. By Theorem 3.3, we obtain the solution(s) of the above equation. □

Theorem 4.2 *Let $a=a_1+a_1\mathbf{j}\neq 0$, $b=b_1+b_2\mathbf{j}$, $c=c_1+c_2\mathbf{j}$, $b_i,c_i\in\mathbb{C}$, $i=1,2$, $a_1\in\mathbb{C}$. Consider the equation*

$$ax^2+bx+c=0. \tag{4.1}$$

(i) If $b=b_1+b_1\mathbf{j}$, then (4.1) is solvable if and only if $c=c_1+c_1\mathbf{j}$. In this case

$$x=x_1(1-\mathbf{j})+\Big[\Big(\sqrt[2]{\frac{(b_1+b_2)^2-8a_1(c_1+c_2)}{16a_1^2}}\Big)_k-\frac{b_1+b_2}{4a_1}\Big]\mathbf{j},\quad k=0,1,\ \forall x_1\in\mathbb{C}.$$

(ii) If $b=b_1+b_2\mathbf{j}$ with $b_1\neq b_2$, then (4.1) is solvable and the solutions are $x=x_1+x_2\mathbf{j}$, where

$$x_1=\Big(\Big(\sqrt[2]{\frac{(b_1+b_2)^2-8a_1(c_1+c_2)}{16a_1^2}}\Big)_k-\frac{b_1+b_2}{4a_1}+\frac{c_2-c_1}{b_1-b_2}\Big)/2,$$

$$x_2=\Big(\Big(\sqrt[2]{\frac{(b_1+b_2)^2-8a_1(c_1+c_2)}{16a_1^2}}\Big)_k-\frac{b_1+b_2}{4a_1}-\frac{c_2-c_1}{b_1-b_2}\Big)/2.$$

Proof Let $x=x_1+x_2\mathbf{j}$, $x_i\in\mathbb{C}$. Since $a=a_1+a_1\mathbf{j}\neq 0$, (4.1) can be reformulated as

$$(1+\mathbf{j})x^2+a_1^{-1}bx+a_1^{-1}c=0.$$

Hence, by complex representation and (3.1), (4.1) is equivalent to

$$(x_1+x_2)^2+\frac{b_1}{a_1}x_1+\frac{b_2}{a_1}x_2+\frac{c_1}{a_1}=0, \tag{4.2}$$

$$(x_1+x_2)^2+\frac{b_2}{a_1}x_1+\frac{b_1}{a_1}x_2+\frac{c_2}{a_1}=0. \tag{4.3}$$

By addition and subtraction of the above equations, we get

$$2(x_1+x_2)^2+\frac{b_1+b_2}{a_1}(x_1+x_2)+\frac{c_1+c_2}{a_1}=0, \tag{4.4}$$

$$(b_1 - b_2)(x_1 - x_2) + c_1 - c_2 = 0. \tag{4.5}$$

By (4.4) we have

$$x_1 + x_2 = \Big(\sqrt[2]{\frac{(b_1+b_2)^2 - 8a_1(c_1+c_2)}{16a_1^2}}\Big)_k - \frac{b_1+b_2}{4a_1}, \quad k = 0, 1. \tag{4.6}$$

If $b = b_1 + b_1\mathbf{j}$ then (4.5) is consistent if and only if $c_1 = c_2$. In this case, by (4.6), the general solutions can be formulated as

$$x = x_1 + x_2\mathbf{j} = x_1(1-\mathbf{j}) + \Big[\Big(\sqrt[2]{\frac{(b_1+b_2)^2 - 8a_1(c_1+c_2)}{16a_1^2}}\Big)_k - \frac{b_1+b_2}{4a_1}\Big]\mathbf{j}.$$

In this case we have infinity solutions. This proves Case 1.

If $b = b_1 + b_2\mathbf{j}$ with $b_1 \neq b_2$ then

$$x_1 - x_2 = \frac{c_2 - c_1}{b_1 - b_2}. \tag{4.7}$$

(4.6) and (4.7) conclude the proof of Case 2. □

Theorem 4.3 *Let* $a = a_1 - a_1\mathbf{j} \neq 0$, $b = b_1 + b_2\mathbf{j}$, $c = c_1 + c_2\mathbf{j}$, $b_i, c_i \in \mathbb{C}$, $i = 1, 2$. *Consider the equation*

$$ax^2 + bx + c = 0. \tag{4.8}$$

(i) If $b = b_1 - b_1\mathbf{j}$, *then (4.8) is solvable if and only if* $c = c_1 - c_1\mathbf{j}$. *In this case*

$$x = x_1(1+\mathbf{j}) - \Big[\Big(\sqrt[2]{\frac{(b_1-b_2)^2 - 8a_1(c_1-c_2)}{16a_1^2}}\Big)_k - \frac{b_1-b_2}{4a_1}\Big]\mathbf{j}, \quad k = 0, 1, \ \forall x_1 \in \mathbb{C}.$$

(ii) If $b = b_1 + b_2\mathbf{j}$ *with* $b_2 \neq -b_1$, *(4.8) is solvable and the solutions are*

$$x = x_1 + x_2\mathbf{j},$$

where

$$x_1 = \Big(\Big(\sqrt[2]{\frac{(b_1-b_2)^2 - 8a_1(c_1-c_2)}{16a_1^2}}\Big)_k - \frac{b_1-b_2}{4a_1} - \frac{c_1+c_2}{b_1+b_2}\Big)/2,$$

$$x_2 = \Big(-\Big(\sqrt[2]{\frac{(b_1-b_2)^2 - 8a_1(c_1-c_2)}{16a_1^2}}\Big)_k + \frac{b_1-b_2}{4a_1} - \frac{c_1+c_2}{b_1+b_2}\Big)/2.$$

Proof Let $x = x_1 + x_2\mathbf{j}$, $x_1, x_2 \in \mathbb{C}$. By complex representation and (3.4), (4.8) is equivalent to

$$(x_1 - x_2)^2 + \frac{b_1}{a_1}x_1 + \frac{b_2}{a_1}x_2 + \frac{c_1}{a_1} = 0, \tag{4.9}$$

$$-(x_1 - x_2)^2 + \frac{b_2}{a_1}x_1 + \frac{b_1}{a_1}x_2 + \frac{c_2}{a_1} = 0. \tag{4.10}$$

By addition and subtraction of the above equations, we get

$$2(x_1 - x_2)^2 + \frac{b_1 - b_2}{a_1}(x_1 - x_2) + \frac{c_1 - c_2}{a_1} = 0, \tag{4.11}$$

$$(b_1 + b_2)(x_1 + x_2) + c_1 + c_2 = 0. \tag{4.12}$$

By (4.11) we have

$$x_1 - x_2 = \Big(\sqrt[2]{\frac{(b_1-b_2)^2 - 8a_1(c_1-c_2)}{16a_1^2}}\Big)_k - \frac{b_1-b_2}{4a_1}, \quad k=0,1. \tag{4.13}$$

If $b = b_1 - b_1\mathbf{j}$, then (4.12) is consistent if and only if $c_2 = -c_1$. In this case, by (4.13), the general solutions can be formulated as

$$x = x_1 + x_2\mathbf{j} = x_1(1+\mathbf{j}) - \Big[\Big(\sqrt[2]{\frac{(b_1-b_2)^2 - 8a_1(c_1-c_2)}{16a_1^2}}\Big)_k - \frac{b_1-b_2}{4a_1}\Big]\mathbf{j}.$$

In this case we have infinity solutions. This proves Case 1.

If $b = b_1 + b_2\mathbf{j}$ with $b_2 \neq -b_1$ then

$$x_1 + x_2 = -\frac{c_2+c_1}{b_1+b_2}. \tag{4.14}$$

By (4.13) and (4.14), we conclude the proof of Case 2. □

5. The exponential function of reduced biquaternionic matrices

We denote the set of $n\times n$ reduced biquaternion matrices with $M_n(\mathbb{H}_r)$. It is a ring with unity with ordinary matrix addition and multiplication. We define the exponential function of reduced biquaternionic matrices as

$$e^A = E_n + A + \cdots + \frac{A^k}{k!} + \cdots = \sum_{k=0}^{\infty}\frac{A^k}{k!}, \quad \forall\, A \in M_n(\mathbb{H}_r).$$

Any matrix $A \in M_n(\mathbb{H}_r)$ can be decomposed by

$$\begin{aligned} &A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = C_1 + C_2\mathbf{j},\ A_i \in M_n(\mathbb{R}),\\ &C_1 = A_0 + A_1\mathbf{i},\ C_2 = A_2 + A_3\mathbf{i} \in M_n(\mathbb{C}). \end{aligned} \tag{5.1}$$

As results of the above decompositions, we define

$$\chi_R(A) = \begin{pmatrix} A_0 & -A_1 & A_2 & -A_3\\ A_1 & A_0 & A_3 & A_2\\ A_2 & -A_3 & A_0 & -A_1\\ A_3 & A_2 & A_1 & A_0 \end{pmatrix} \in M_{4n}(\mathbb{R}),\ \chi_C(A) = \begin{pmatrix} C_1 & C_2\\ C_2 & C_1 \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

Proposition 5.1 *Let $A, B \in M_n(\mathbb{H}_r)$. Then*

$$\chi_R(A+B) = \chi_R(A) + \chi_R(B), \quad \chi_C(A+B) = \chi_C(A) + \chi_C(B);$$

$$\chi_R(AB) = \chi_R(A)\chi_R(B), \quad \chi_C(AB) = \chi_C(A)\chi_C(B).$$

Proof Let $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $B = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Obviously,

$$\chi_R(A+B) = \chi_R(A) + \chi_R(B),\ \chi_C(A+B) = \chi_C(A) + \chi_C(B).$$

Since

$$AB = (A_0B_0 - A_1B_1 + A_2B_2 - A_3B_3) + (A_1B_0 + A_0B_1 + A_3B_2 + A_2B_3)\mathbf{i} +$$

$$(A_2B_0 - A_3B_1 + A_0B_2 - A_1B_3)\mathbf{j} + (A_3B_0 + A_2B_1 + A_1B_2 + A_0B_3)\mathbf{k},$$

we have $\chi_R(AB) = \chi_R(A)\chi_R(B)$. Similarly, we have $\chi_C(AB) = \chi_C(A)\chi_C(B)$. □

Proposition 5.2 *Let $A \in M_n(\mathbb{H}_r)$. Then*

$$e^{\chi_R(A)} = \chi_R(e^A), \quad e^{\chi_C(A)} = \chi_C(e^A).$$

Proof By Proposition 5.1, we get

$$e^{\chi_R(A)} = E_{4n} + \chi_R(A) + \cdots + \frac{\chi_R(A)^k}{k!} + \cdots = \chi_R(E_n + A + \cdots + \frac{A^k}{k!} + \cdots).$$

Hence $e^{\chi_R(A)} = \chi_R(e^A)$. Similarly, we have $e^{\chi_C(A)} = \chi_C(e^A)$. □

Proposition 5.3 *For $A, B \in M_n(\mathbb{H}_r)$, if $AB = BA$, then we have $e^{A+B} = e^Ae^B$.*

Proof Noting that $\chi_C(AB) = \chi_C(BA)$, we have

$$e^{\chi_C(A+B)} = e^{\chi_C(A)+\chi_C(B)} = e^{\chi_C(A)}e^{\chi_C(B)} = \chi_C(e^A)\chi_C(e^B).$$

Hence we have $\chi_C(e^{A+B}) = \chi_C(e^Ae^B)$. This implies that $e^{A+B} = e^Ae^B$. □

Let $A = C_1 + C_2\mathbf{j}, C_1, C_2 \in M_n(\mathbb{C})$. By (3.4) and (3.5) we have

$$u_1A = (C_1 + C_2)u_1, \quad u_2A = (C_1 - C_2)u_2 \tag{5.2}$$

and

$$A = (C_1 + C_2\mathbf{j})(u_1 + u_2) = (C_1 + C_2)u_1 + (C_1 - C_2)u_2. \tag{5.3}$$

Proposition 5.4 *Let $A = C_1 + C_2\mathbf{j}$, $C_1, C_2 \in M_n(\mathbb{C})$. Then*

$$A^n = \frac{(C_1 + C_2)^n + (C_1 - C_2)^n}{2} + \frac{(C_1 + C_2)^n - (C_1 - C_2)^n}{2}\mathbf{j}.$$

Proof By (5.3), we have

$$\begin{aligned} A^n =& [(C_1 + C_2)u_1 + (C_1 - C_2)u_2]^n = (C_1 + C_2)^n u_1 + (C_1 - C_2)^n u_2 \\ =& \frac{(C_1 + C_2)^n + (C_1 - C_2)^n}{2} + \frac{(C_1 + C_2)^n - (C_1 - C_2)^n}{2}\mathbf{j}. \ \square \end{aligned}$$

Theorem 5.5 *Let $A = C_1 + C_2\mathbf{j}$, $C_1, C_2 \in M_n(\mathbb{C})$. Then*

$$e^A = e^{(C_1+C_2)}u_1 + e^{(C_1-C_2)}u_2 = \frac{e^{(C_1+C_2)} + e^{(C_1-C_2)}}{2} + \frac{e^{(C_1+C_2)} - e^{(C_1-C_2)}}{2}\mathbf{j}.$$

Proof It follows from $A = (C_1 + C_2)u_1 + (C_1 - C_2)u_2$ and Proposition 5.4 that

$$\begin{aligned} e^A =& E_n(u_1 + u_2) + [(C_1 + C_2)u_1 + (C_1 - C_2)u_2] + \cdots + \frac{[(C_1 + C_2)u_1 + (C_1 - C_2)u_2]^k}{k!} + \cdots \\ =& E_n(u_1 + u_2) + [(C_1 + C_2)u_1 + (C_1 - C_2)u_2] + \cdots + \frac{(C_1 + C_2)^k u_1 + (C_1 - C_2)^k u_2}{k!} + \cdots \\ =& e^{(C_1+C_2)}u_1 + e^{(C_1-C_2)}u_2 = \frac{e^{(C_1+C_2)} + e^{(C_1-C_2)}}{2} + \frac{e^{(C_1+C_2)} - e^{(C_1-C_2)}}{2}\mathbf{j}. \ \square \end{aligned}$$

We provide several examples as follows.

Let

$$A=\begin{pmatrix} 1+\mathbf{i}+2\mathbf{j}-\mathbf{k} & 0 \\ \mathbf{j} & 2+\mathbf{i}+\mathbf{j}-\mathbf{k} \end{pmatrix}=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}=C_1+C_2\mathbf{j}.$$

Then

$$A_0=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ A_1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_2=\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},\ A_3=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C_1=\begin{pmatrix} 1+\mathbf{i} & 0 \\ 0 & 2+\mathbf{i} \end{pmatrix},\ C_2=\begin{pmatrix} 2-\mathbf{i} & 0 \\ 1 & 1-\mathbf{i} \end{pmatrix}.$$

Obviously,

$$\chi_C(A)=\begin{pmatrix} 1+\mathbf{i} & 0 & 2-\mathbf{i} & 0 \\ 0 & 2+\mathbf{i} & 1 & 1-\mathbf{i} \\ 2-\mathbf{i} & 0 & 1+\mathbf{i} & 0 \\ 1 & 1-\mathbf{i} & 0 & 2+\mathbf{i} \end{pmatrix}$$

and

$$\chi_R(A)=\begin{pmatrix} 1 & 0 & -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Let

$$T_1=C_1+C_2=\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix},\quad T_2=C_1-C_2=\begin{pmatrix} -1+2\mathbf{i} & 0 \\ -1 & 1+2\mathbf{i} \end{pmatrix}.$$

Then $A=T_1u_1+T_2u_2$ and $A^n=\frac{(T_1^n+T_2^n)}{2}+\frac{(T_1^n-T_2^n)}{2}\mathbf{j}$. Especially,

$$A^2=\begin{pmatrix} 3-2\mathbf{i} & 0 \\ 3-2\mathbf{i} & 3+2\mathbf{i} \end{pmatrix}+\begin{pmatrix} 6+2\mathbf{i} & 0 \\ 3+2\mathbf{i} & 6-2\mathbf{i} \end{pmatrix}\mathbf{j}.$$

$$\begin{aligned} e^A&=e^{T_1}u_1+e^{T_2}u_2\\ &=\begin{pmatrix} 9.9662+0.1673\mathbf{i} & 1 \\ 1.5431 & 9.4772+1.2359\mathbf{i} \end{pmatrix}+\begin{pmatrix} 10.1193-0.1673\mathbf{i} & 0 \\ 1.1752 & 10.6084-1.2359\mathbf{i} \end{pmatrix}\mathbf{j}\\ &=\begin{pmatrix} 9.9662+0.1673\mathbf{i}+10.1193\mathbf{j}-0.1673\mathbf{k} & 1 \\ 1.5431+1.1752\mathbf{j} & 9.4772+1.2359\mathbf{i}+10.6084\mathbf{j}-1.2359\mathbf{k} \end{pmatrix}. \end{aligned}$$

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