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Asymptotic Behavior of Solutions to a Logistic Chemotaxis System with Singular Sensitivity

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Abstract In this paper, we study the asymptotic behavior of solutions to the parabolic-elliptic chemotaxis system with singular sensitivity and logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(\frac{u}{v}\nabla v) + ru - \mu u^k, & x\in\Omega,\ t>0,\\ 0=\Delta v - v + u, & x\in\Omega,\ t>0\end{cases}$$

in a smooth bounded domain $\Omega\subset\mathbb{R}^n$ $(n>2)$ with the non-flux boundary, where χ, r, $\mu>0$, $k\ge 2$. It is proved that the global bounded classical solution will exponentially converge to $((\frac{r}{\mu})^{\frac{1}{k-1}}, (\frac{r}{\mu})^{\frac{1}{k-1}})$ as $t\to\infty$ if r is suitably large.

Keywords asymptotic behavior; chemotaxis; singular sensitivity; logistic source

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1. Introduction

In recent years, many papers are concerned with the variant of the classical Keller-Segel chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\chi_0(v)\nabla v) + f(u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \tau v(x,0)=\tau v_0(x), & x\in\Omega,\end{cases} \tag{1.1}$$

which describes the cells move towards the high concentration of a chemical signal produced by the cells themselves, where Ω is a smooth bounded domain in $\mathbb{R}^n\,(n\ge 1)$ with smooth boundary $\partial\Omega$ and ν represents the outward normal vector of $\partial\Omega$, $u(x,t)$ and $v(x,t)$ stand for cell density, the concentration of a chemical signal produced by the cells themselves, respectively. The nonlinear cross-diffusive term $-\nabla\cdot(u\chi_0(v)\nabla v)$ with the sensitivity function $\chi_0(v)$ reflects the chemotactic movement, while the inhomogeneity $f(u)$ represents the cell kinetic mechanism. Since the pioneering work of Keller and Segel, many efforts have been devoted to the properties of solutions to the system (1.1). We refer the reader to [1–13] and references therein.

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The sensitivity function $\chi_0(v) = \frac{\chi}{v}$ with $\chi > 0$ was first proposed in [14] based on the Weber-Fechner law of stimulus perception. When $\tau = 0$ and $f(u) = 0$, it was shown in [15] that all radial solutions are globally bounded if $n = 2$ with $\chi > 0$, or $n \geq 3$ with $\chi < \frac{2}{n-2}$, and there exist radial blow-up solutions if $\chi > \frac{2N}{N-2}, N \geq 3$. It has been proved that the system (1.1) has a unique global bounded classical solution if $n \geq 2$ with $\chi < \frac{2}{n}$ (see [5]). When $\tau = 1$ and $f(u) = 0$, it was shown that the system (1.1) has global bounded classical solutions for $n \geq 2$ and $0 < \chi < \sqrt{\frac{n}{2}}$ (see [2]), while $0 < \chi < \sqrt{\frac{n+2}{3n-4}}$ with $n \geq 2$, then there exists a global weak solution to (1.1) (see [8,16]). The stabilization in the logarithmic Keller-Segel system was considered in paper [9]. More results with singular sensitivity for chemotaxis systems have been obtained [3,6,7].

Now we focus on the chemotaxis system (1.1) with sensitivity function $\chi_0(v) = \frac{\chi}{v}, \tau = 1$ and $f(u) = ru - \mu u^k$ as the cell kinetic term. The logistic source $f(u)$ exerts a certain growth-inhibiting influence. If $n, k = 2$, there exists a unique global bounded classical solution [10], whenever

$$r > \begin{cases} \frac{\chi^2}{4}, & \chi \leq 2, \\ \chi - 1, & \chi > 2. \end{cases} \tag{1.2}$$

In addition, the authors have proved for $n \geq 2$, $k > \frac{3(n+2)}{n+4}$ that the system (1.1) possesses a global bounded solution provided χ is suitably small relative to r and k (see [11]). Meanwhile, the global bounded solution exponentially converges to the steady state [13].

When $\chi_0(v) = \frac{\chi}{v}$, $\tau = 0$ and $f(u) = ru - \mu u^k$, the system (1.1) becomes the following parabolic-elliptic chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^k, & x \in \Omega,\, t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\chi, r, \mu > 0$ and $k \geq 2$. Recently, the authors [4,12] have studied global existence and boundedness of classical solutions to the system (1.3) with $n \geq 2$, $k \geq \frac{3n-2}{n}$ and r, $\chi > 0$ satisfying (1.2). Under $n = 2$ and $k = 2$, Cao et al. [17] obtained that the global bounded solution exponentially converges to the steady state when initial data u_0 and r are suitably large relative to χ and Ω.

In the present work, we will study the large time behavior of solutions to the system (1.3). The main result of this paper reads as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ $(n > 2)$ be a bounded domain with smooth boundary, $k \geq \frac{3n-2}{n}$, r, $\chi > 0$ and satisfy (1.2). Suppose that r is suitably large and the initial data satisfy*

$$u_0(x) \in C^0(\bar{\Omega}),\ u_0(x) \geq 0 \ \text{ and } \ u_0 \not\equiv 0,\ x \in \bar{\Omega},$$

then for the global bounded classical solution (u, v) of (1.3), there exist constants $C, \lambda > 0$ such that

$$\| u(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}} \|_{L^\infty(\Omega)} + \| v(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}} \|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \ \text{ as } \ t \to \infty. \tag{1.4}$$

This paper is organized as follows. In Section 2, we mainly give preliminary estimates for the parabolic-elliptic chemotaxis system (1.3). Then we prove the main results of this paper in Section 3.

2. Preliminaries

In this section, we first recall the global boundedness of classical solutions to (1.3) (see [4,12]).

Lemma 2.1 *Under the assumptions in Theorem 1.1, the system (1.3) has a global bounded classical solution satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all} \quad t > 0, \tag{2.1}$$

where C is a constant independent of t.

Next we give the unform-in-time lower bound of v in Ω with the conditions of Lemma 2.1, which can be obtained in the proof in Lemma 2.4 of [12].

Lemma 2.2 *Let the conditions in Lemma 1.1 hold. There exists $\delta > 0$ such that*

$$v(x,t) \ge \delta := \rho(\frac{1}{\mu})^{\frac{1}{k-1}}$$

with some constant $\rho > 0$ for all $(x,t) \in \Omega \times (0,\infty)$.

Now we give the regularity of u and v (see [17,18]).

Lemma 2.3 *Let (u,v) be a global bounded classical solution of (1.3). Then there exist $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} \le C \quad \text{for all} \quad t \ge 1.$$

We utilize the well-known Neumann heat semigroup $\{e^{t\Delta}\}_{t\ge0}$ estimates in $\Omega \subset \mathbb{R}^n$ to show the $W^{1,\infty}$-estimates of u.

Lemma 2.4 *Let (u,v) be a global bounded classical solution of (1.3). Then there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \tag{2.2}$$

for all $t > 1$.

Proof In order to prove that there exists $C > 0$ such that $\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$ for all $t > 1$. First, for all $T > 2$, let

$$M(T) := \sup_{t\in(2,T)} \|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}.$$

Because ∇u is continuous on $\Omega \times [0,T]$, we infer that $M(T)$ is finite. According to (2.1), we only need to prove that there exists a positive constant c_1 such that $M(T) \le c_1$ for all $T > 2$. Due to the variation-of-constants formula, for all $t \in (2,T)$, we can conclude that

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \le \|\nabla e^{\Delta} u(\cdot,t-1)\|_{L^\infty(\Omega)} + \chi \int_{t-1}^{t} \|\nabla e^{(t-s)\Delta} \nabla \cdot (\frac{u}{v}\nabla v)\|_{L^\infty(\Omega)} \mathrm{d}s +$$

$$\int_{t-1}^{t}\|\nabla e^{(t-s)\Delta}(ru-\mu u^k)\|_{L^\infty(\Omega)}\mathrm{d}s := I_1+I_2+I_3. \tag{2.3}$$

With semigroup estimates [19, 20] and Lemma 2.1, we obtain

$$I_1=\|\nabla e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)}\le c_1\|u(\cdot,t-1)\|_{L^\infty(\Omega)}\le c_2 \tag{2.4}$$

and

$$\begin{aligned} I_3&=\int_{t-1}^{t}\|\nabla e^{(t-s)\Delta}(ru-\mu u^k)\|_{L^\infty(\Omega)}\mathrm{d}s\\ &\le c_3\int_{t-1}^{t}[1+(t-s)^{-\frac12}]e^{-\lambda_1(t-s)}\mathrm{d}s\\ &=c_3\int_0^1(1+\tau^{-\frac12})e^{-\lambda_1\tau}\mathrm{d}\tau\le c_4 \end{aligned}\tag{2.5}$$

as well as

$$\begin{aligned} I_2=&\chi\int_{t-1}^{t}\|\nabla e^{(t-s)\Delta}\nabla\cdot(\frac{u}{v}\nabla v)\|_{L^\infty(\Omega)}\mathrm{d}s\\ \le& c_5\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\nabla\cdot(\frac{u}{v}\nabla v)\|_{L^p(\Omega)}\mathrm{d}s\\ =&c_5\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\frac{\nabla u\cdot\nabla v}{v}-\frac{u|\nabla v|^2}{v^2}+\frac{u}{v}\Delta v\|_{L^p(\Omega)}\mathrm{d}s\\ \le& c_5\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\frac{\nabla u\cdot\nabla v}{v}\|_{L^p(\Omega)}\mathrm{d}s+\\ &c_5\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\frac{u|\nabla v|^2}{v^2}\|_{L^p(\Omega)}\mathrm{d}s+\\ &c_5\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\frac{u}{v}\Delta v\|_{L^p(\Omega)}\mathrm{d}s, \end{aligned}$$

where c_i $(i=2,3,\dots,5)$ and λ_1 are positive constants. According to Lemma 2.1, we know $\|v(\cdot,t)\|_{W^{1,\infty}}\le c_6$ with $c_6>0$. Hence, through the second equation in (1.3), we can gain

$$\|\Delta v(\cdot,t)\|_{L^\infty(\Omega)}\le\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|u(\cdot,t)\|_{L^\infty(\Omega)}\le c_7\ \text{ for all }\ t>1. \tag{2.6}$$

By choosing $p>n$, we ensure the finiteness of the integral

$$\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\mathrm{d}s.$$

Then we know from Lemmas 2.1, 2.2 and (2.6) that

$$I_2\le c_8\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\nabla u\|_{L^p(\Omega)}\mathrm{d}s+c_9. \tag{2.7}$$

It follows from (2.3)–(2.5), (2.7) that

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}\le c_8\int_{t-1}^{t}[1+(t-s)^{-\frac12-\frac n2\cdot\frac1p}]e^{-\lambda_1(t-s)}\|\nabla u\|_{L^p(\Omega)}\mathrm{d}s+c_{10}, \tag{2.8}$$

where $c_{10}=c_2+c_4+c_9$. Applying the Gaglirado-Nirenberg inequality

$$\|D^l\phi\|_{L^p(\Omega)}\le C_{GN}\|D^k\phi\|^{\theta}_{L^q(\Omega)}\|\phi\|^{1-\theta}_{L^r(\Omega)}+\|\phi\|^{\theta}_{L^r(\Omega)}$$

with $0 < p < \infty, 1 \le q, r \le +\infty$ and

$$\frac{1}{p} - \frac{l}{n} = \theta(\frac{1}{q} - \frac{k}{n}) + (1-\theta)\frac{1}{r},$$

there are some positive constants c_{11}, c_{12} and c_{13} satisfying

$$\begin{aligned}\|\nabla u(\cdot,s)\|_{L^p(\Omega)} &\le c_{11}\|\nabla u(\cdot,s)\|^{\theta}_{L^\infty(\Omega)}\|u(\cdot,s)\|^{1-\theta}_{L^\infty(\Omega)} + c_{12}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\ &\le c_{13}(M^\theta(T)+1) \quad \text{for all} \quad s \in (1,T),\end{aligned} \tag{2.9}$$

where $\theta = \frac{p-n}{p} \in (0,1)$ due to $p > n$. Finally, since $\frac{1}{2} + \frac{n}{2p} < 1$, substituting (2.9) into (2.8), we can get some positive constants c_{14} and c_{15} such that

$$M(T) \le c_{14}M^\theta(T) + c_{15} \quad \text{for all} \quad T > 2,$$

which implies

$$M(T) \le \max\big\{(\frac{c_{15}}{c_{14}})^{\frac{1}{\theta}}, (2c_{14})^{\frac{1}{1-\theta}}\big\} \quad \text{for all} \quad T > 2.$$

Hence, we get an estimate of (2.2). And the proof of Lemma 2.4 is completed. □

Finally, we give the following lemma.

Lemma 2.5 *Assume that $f : (1,\infty) \to \mathbb{R}$ is a uniformly continuous nonnegative function such that $\int_1^\infty f(t)\mathrm{d}t < \infty$. Then we have $f(t) \to 0$ as $t \to \infty$.*

Proof A detailed proof can be found in [21]. Thus we omit the details. □

3. Proof of Theorem 1.1

This section is devoted to the proof of main results. In order to simplify notation, let $\kappa := (\frac{r}{\mu})^{\frac{1}{k-1}}$ in this section. As a preparation, we first have the following simple estimate.

Lemma 3.1 *Let (u, v) be a global bounded classical solution of (1.3). Then there exists $\delta > 0$ such that*

$$\int_\Omega \frac{|\nabla v|^2}{v^2}\mathrm{d}x \le \frac{1}{4\kappa\delta}\int_\Omega (u(x,t)-\kappa)^2\mathrm{d}x \quad \textit{for all} \quad t > 0. \tag{3.1}$$

Proof Multiply the second equation in (1.3) by $\frac{v-\kappa}{v}$ and integrate by parts to obtain

$$\kappa\int_\Omega \frac{|\nabla v|^2}{v^2}\mathrm{d}x = -\int_\Omega \frac{1}{v}(v(x,t)-\kappa)^2\mathrm{d}x + \int_\Omega \frac{1}{v}(u(x,t)-\kappa)(v(x,t)-\kappa)\mathrm{d}x.$$

By Lemma 2.2, there exists $\delta > 0$ such that $v(x,t) \ge \delta$ for all $(x,t) \in \Omega \times (0,\infty)$. Then it is obtained that by Cauchy inequality

$$\begin{aligned}\kappa\int_\Omega \frac{|\nabla v|^2}{v^2}\mathrm{d}x &\le -\int_\Omega \frac{1}{v}(v(x,t)-\kappa)^2\mathrm{d}x + \int_\Omega \frac{1}{v}(v(x,t)-\kappa)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \frac{1}{v}(u(x,t)-\kappa)^2\mathrm{d}x \\ &\le \frac{1}{4\delta}\int_\Omega (u(x,t)-\kappa)^2\mathrm{d}x.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.1 Define

$$E(t) := \int_\Omega F(u(x,t))\mathrm{d}x, \quad t > 0$$

with

$$F(s) := s - \kappa - \kappa \ln(\frac{s}{\kappa}), \quad s > 0.$$

Then we shall prove the nonnegativity of $E(t)$. Fixing (x, t) and using two-order Taylor's formula, we can derive that there exists $\tau \in (0,1)$ such that

$$F(u(x,t)) - F(\kappa) = F'(\kappa)(u(x,t) - \kappa) + \frac{1}{2}F''(\tau u(x,t) + (1-\tau)\kappa) \cdot (u(x,t) - \kappa)^2$$
$$= \frac{\kappa}{2(\tau u(x,t) + (1-\tau)\kappa)^2}(u(x,t) - \kappa)^2 \geq 0$$

for all $x \in \Omega$ and $t > 0$. Thus $E(t) \geq 0$ for $t > 0$. Differentiate $E(t)$ to arrive at

$$\begin{aligned} E'(t) &= \int_\Omega (1 - \frac{\kappa}{u})u_t \mathrm{d}x \\ &= -\kappa \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + \chi\kappa \int_\Omega \frac{\nabla u \cdot \nabla v}{uv}\mathrm{d}x - \mu \int_\Omega (u-\kappa)(u^{k-1} - \frac{r}{\mu})\mathrm{d}x \\ &\leq \frac{\kappa\chi^2}{4} \int_\Omega \frac{|\nabla v|^2}{v^2}\mathrm{d}x - (\mu r^{k-2})^{\frac{1}{k-1}} \int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x \ \text{ for all } \ t > 0, \end{aligned}$$

where the Cauchy inequality is used. According to (3.1), we obtain that

$$E'(t) \leq -D \int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x \ \text{ for all } \ t > 0, \tag{3.2}$$

where $D = (\mu r^{k-2})^{\frac{1}{k-1}} - \frac{\chi^2}{16\delta} > 0$ if we choose r large enough to satisfy

$$r^{k-2} > (\frac{\chi^2}{16\rho})^{k-1},$$

which implies

$$\mu > r^{2-k}(\frac{\chi^2}{16\delta})^{k-1}$$

due to Lemma 2.2. Integrating (3.2) over $(1, t)$, we get

$$E(t) - E(1) \leq -D \int_1^t \int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x\mathrm{d}s$$

and thus

$$\int_1^t \int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x\mathrm{d}s \leq \frac{E(1)}{D} < \infty.$$

It follows from Lemma 2.3 that (u, v) is Hölder continuous uniformly in regard to $t > 1$ in $\bar{\Omega} \times [t, t+1]$, so we derive that $\int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x$ is uniformly continuous in $(1, \infty)$. Then from Lemma 2.5, we have

$$\int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x \to 0 \text{ as } t \to \infty. \tag{3.3}$$

Next, we further indicate that the L^2-normal of $u - \kappa$ decays exponentially. More precisely, there exists $t_0 > 0$ such that

$$\int_\Omega (u(x,t) - \kappa)^2 \mathrm{d}x \leq c_0 e^{-Lt}, \quad t > t_0. \tag{3.4}$$

Application of the Gagliardo-Nirenberg inequality together with Lemma 2.4 provides us with some positive constants C_1 and C_2 such that

$$\begin{aligned}\|u(\cdot,t)-\kappa\|_{L^\infty(\Omega)} &\le C_1\|u(\cdot,t)-\kappa\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|u(\cdot,t)-\kappa\|^{\frac{2}{n+2}}_{L^2(\Omega)}\\ &\le C_2\|u(\cdot,t)-\kappa\|^{\frac{2}{n+2}}_{L^2(\Omega)}. \end{aligned} \tag{3.5}$$

In conjunction with (3.3), this entails that

$$\|u(\cdot,t)-\kappa\|_{L^\infty(\Omega)} \to 0 \text{ as } t\to\infty.$$

Using L'Hôpital's rule, one can derive

$$\lim_{s\to\kappa}\frac{F(s)-F(\kappa)}{(s-\kappa)^2}=\frac{1}{2\kappa}.$$

Because $\|u(\cdot,t)-\kappa\|_{L^\infty(\Omega)} \to 0$ as $t\to\infty$, we can obtain that there exists $t_0>0$ such that

$$\begin{aligned}&\int_\Omega \big(u(x,t)-\kappa-\kappa\ln(\frac{u(x,t)}{\kappa})\big)\mathrm{d}x=\int_\Omega(F(u(x,t))-F(\kappa))\mathrm{d}x\\ &\le\frac{1}{\kappa}\int_\Omega(u(x,t)-\kappa)^2\mathrm{d}x\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\int_\Omega \big(u(x,t)-\kappa-\kappa\ln(\frac{u(x,t)}{\kappa})\big)\mathrm{d}x=\int_\Omega(F(u(x,t))-F(\kappa))\mathrm{d}x\\ &\ge\frac{1}{4\kappa}\int_\Omega(u(x,t)-\kappa)^2\mathrm{d}x\end{aligned} \tag{3.7}$$

for all $t>t_0$. We can show that (3.6) implies $c_1E(t)\le\int_\Omega(u(x,t)-\kappa)^2\mathrm{d}x$ with $c_1>0$ for all $t>t_0$. According to (3.2), we can see

$$E'(t)\le -D\int_\Omega(u(x,t)-\kappa)^2\mathrm{d}x\le -c_1DE(t)\quad\text{for all}\quad t>t_0,$$

which on integration shows that there exist $c_2>0$ and $L>0$ such that

$$E(t)\le c_2e^{-Lt}\quad\text{for all}\quad t>t_0. \tag{3.8}$$

Hence the combination of (3.7) and (3.8) gives

$$\int_\Omega(u(x,t)-\kappa)^2\mathrm{d}x\le c_3E(t)\le c_2c_3e^{-Lt}$$

with $c_3>0$ for all $t>t_0$. Now according to (3.4) and (3.5), we obtain

$$\| u(.,t)-\kappa \|_{L^\infty(\Omega)}\le C_2c_0^{\frac{1}{n+2}}e^{-\frac{L}{n+2}t}\quad\text{for all}\quad t>t_0. \tag{3.9}$$

Set $w=v-\kappa$. Then we have

$$\begin{cases} -\Delta w+w=u-\kappa, & x\in\Omega,\, t>0,\\ \frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega,\, t>0.\end{cases}$$

It follows by the maximum principle that

$$\|v(\cdot,t)-\kappa\|_{L^\infty(\Omega)}=\|w(\cdot,t)\|_{L^\infty(\Omega)}\le\|u(\cdot,t)-\kappa\|_{L^\infty(\Omega)}\le C_2c_0^{\frac{1}{n+2}}e^{-\frac{L}{n+2}t}\quad\text{for all}\quad t>t_0. \tag{3.10}$$

Noting $\kappa = (\frac{r}{\mu})^{\frac{1}{k-1}}$, by (3.9) and (3.10), we complete the proof of Theorem 1.1 by taking $\lambda = -\frac{L}{n+2}$ and $C = C_2 c_0^{\frac{1}{n+2}}$. □

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