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M^k-Type Sharp Estimates and Boundedness on Morrey Space for Toeplitz Type Operators Associated to Fractional Integral and Singular Integral Operator with Non-Smooth Kernel

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Abstract In this paper, we prove the M^k-type sharp maximal function estimates for the Toeplitz type operators associated to the fractional integral and singular integral operator with non-smooth kernel. As an application, we obtain the boundedness of the operators on the Morrey space.

Keywords Toeplitz type operator; singular integral operator; fractional integral operator; sharp maximal function; BMO; Morrey space

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1. Introduction

As the development of singular integral operators [1, 2], their commutators have been well studied. In [3,4], the authors proved that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo [4] proved a similar result when singular integral operators are replaced by the fractional integral operators. In [5,6], some singular integral operators with non-smooth kernel were introduced, and the boundedness for the operators and their commutators was obtained [7–21]. In [22–24], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators were introduced, and the boundedness for the operators generated by BMO and Lipschitz functions was obtained. In this paper, we will study the Toeplitz type operators generated by the fractional integral and singular integral operators with non-smooth kernel and the BMO functions.

2. Preliminaries

In this paper, we will study some singular integral operators as follows.

Definition 2.1 *A family of operators D_t, $t > 0$ is said to be an "approximation to the identity"*

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if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y) f(y) \mathrm{d}y$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2} \rho(|x - y|^2/t),$$

where ρ is a positive, bounded and decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\epsilon} \rho(r^2) = 0$$

for some $\epsilon > 0$.

Definition 2.2 *A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x, y)$ such that*

$$T(f)(x) = \int_{R^n} K(x, y) f(y) \mathrm{d}y$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an "approximation to the identity" $\{B_t, t > 0\}$ such that TB_t has the associated kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} |K(x, y) - k_t(x, y)| \mathrm{d}x \leq c_2 \text{ for all } y \in R^n.$$

(2) There exists an "approximation to the identity" $\{A_t, t > 0\}$ such that $A_t T$ has the associated kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \leq c_4 t^{-n/2} \text{ if } |x - y| \leq c_3 t^{1/2},$$

and

$$|K(x, y) - K_t(x, y)| \leq c_4 t^{\delta/2} |x - y|^{-n-\delta} \text{ if } |x - y| \geq c_3 t^{1/2},$$

for some $\delta > 0$, c_3, $c_4 > 0$.

Let b be a locally integrable function on R^n and T be the singular integral operator with non-smooth kernel. The Toeplitz type operators associated to T are defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2}$$

and

$$S_b = \sum_{k=1}^{m} (T^{k,3} M_b I_\alpha T^{k,4} + T^{k,5} I_\alpha M_b T^{k,6}),$$

where $T^{k,1}$ and $T^{k,3}$ are the singular integral operator with non-smooth kernel T or $\pm I$ (the identity operator), $T^{k,2}$, $T^{k,4}$ and $T^{k,6}$ are the linear operators, $T^{k,5} = \pm I$, $k = 1, \ldots, m$, $M_b(f) = bf$ and I_α is the fractional integral operator ($0 < \alpha < n$).

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operators T_b and S_b. The Toeplitz type operators are the non-trivial generalizations of the commutator. It is well-known that commutators are of great interest in harmonic analysis and have been widely studied by many authors. The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operators T_b and S_b. As the application, we obtain the boundedness on the Morrey space for the Toeplitz type operators T_b and S_b.

Now, let us introduce some notation. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|\mathrm{d}y,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)\mathrm{d}x$. It is well-known that [14]

$$f^{\#}(x)\approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q |f(y)-c|\mathrm{d}y.$$

We say that f belongs to $\mathrm{BMO}(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and define $\|f\|_{\mathrm{BMO}}=\|f^{\#}\|_{L^\infty}$. It has been known that [14]

$$\|f-f_{2^kQ}\|_{\mathrm{BMO}}\le Ck\|f\|_{\mathrm{BMO}}.$$

For $0<r<\infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x)=[(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y.$$

For $\eta>0$, let $M_\eta(f)=M(|f|^\eta)^{1/\eta}$. For $k\in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)=M(f)$ and

$$M^k(f)=M(M^{k-1}(f)) \text{ when } k\ge 2.$$

For $0<\eta<n$ and $1\le r<\infty$, set

$$M_{\eta,r}(f)(x)=\sup_{Q\ni x}\Big(\frac{1}{|Q|^{1-r\eta/n}}\int_Q |f(y)|^r\mathrm{d}y\Big)^{1/r}.$$

The sharp maximal function $M_A(f)$ associated with the "approximation to the identity" $\{A_t, t>0\}$ is defined by

$$M_A^{\#}(f)(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)-A_{t_Q}(f)(y)|\mathrm{d}y,$$

where $t_Q=l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $\eta>0$, let

$$M_{A,\eta}^{\#}(f)=M_A^{\#}(|f|^\eta)^{1/\eta}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. We denote the Φ-average by, for a function f,

$$\|f\|_{\Phi,Q}=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_Q \Phi(\frac{|f(y)|}{\lambda})\mathrm{d}y\le 1\Big\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1+\log t)$ and $\tilde{\Phi}(t) = \exp(t)$, and the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L),Q}$, $M_{L(\log L)}$ and $\|\cdot\|_{\exp L,Q}$, $M_{\exp L}$. Following [14], we know the generalized Hölder's inequality and the following inequalities hold:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\mathrm{d}y \le \|f\|_{\Phi,Q}\|g\|_{\tilde{\Phi},Q},$$

$$\|f\|_{L(\log L),Q} \le M_{L(\log L)}(f) \le CM^2(f),$$

$$\|f - f_Q\|_{\exp L,Q} \le C\|f\|_{\mathrm{BMO}}.$$

The A_1 weight is defined by [14]

$$A_1 = \{0 < w \in L^1_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x),\ \text{a.e.}\}.$$

The $A_{p,q}$ weight is defined by [14], for $1 < p \le q < \infty$,

$$A_{p,q} = \Big\{0 < w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)^q \mathrm{d}x\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q w(x)^{-p/(p-1)}\mathrm{d}x\Big)^{(p-1)/p} < \infty\Big\}.$$

Given a non-negative weight function w and $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \Big(\int_{R^n} |f(x)|^p w(x)\mathrm{d}x\Big)^{1/p} < \infty.$$

Throughout this paper, φ will denote a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \text{ for } t \ge 0.$$

Let f be a locally integrable function on R^n. Set, for $0 \le \eta < n$ and $1 \le p < n/\eta$,

$$\|f\|_{L^{p,\eta,\varphi}} = \sup_{x\in R^n, d>0} \Big(\frac{1}{\varphi(d)^{1-p\eta/n}}\int_{Q(x,d)} |f(y)|^p \mathrm{d}y\Big)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized fractional Morrey space is defined by

$$L^{p,\eta,\varphi}(R^n) = \{f \in L^1_{\mathrm{loc}}(R^n) : \|f\|_{L^{p,\eta,\varphi}} < \infty\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey space. If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue space [14].

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces [14].

3. Theorems and Lemmas

We shall prove the following theorems.

Theorem 3.1 *Let T be the singular integral operator as Definition 2.2, $0 < r < 1$, $1 < s < \infty$ and $b \in \mathrm{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}_{A,r}(T_b(f))(\tilde{x}) \le C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{m} (M^2(T^{k,2}(f))(\tilde{x}) + M_s(T^{k,2}(f))(\tilde{x})).$$

Theorem 3.2 *Let T be the singular integral operator as Definition 2.2, $0 < r < 1$, $1 < s < \infty$ and $b \in \mathrm{BMO}(R^n)$. If $S_1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}_{A,r}(S_b(f))(\tilde{x}) \le C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{m} (M^2(I_\alpha T^{k,4}(f))(\tilde{x}) + M_s(I_\alpha T^{k,4}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,6}(f))(\tilde{x})).$$

Theorem 3.3 *Let T be the singular integral operator as Definition 2.2, $1 < p < \infty$, $0 < D < 2^n$ and $b \in \mathrm{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^{p,\varphi}(R^n)$, $k = 1, \ldots, m$, then T_b is bounded on $L^{p,\varphi}(R^n)$.*

Theorem 3.4 *Let T be the singular integral operator as Definition 2.2, $0 < D < 2^n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in \mathrm{BMO}(R^n)$. If $S_1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,4}$ and $T^{k,6}$ are the bounded operators on $L^{p,\alpha,\varphi}(R^n)$, $k = 1, \ldots, m$, then S_b is bounded from $L^{p,\alpha,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.*

Corollary 3.5 *Let $[b, T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator T with non-smooth kernel and b. Then Theorems 3.1–3.4 hold for $[b, T]$.*

Remark 3.6 In Theorems 3.3 and 3.4, the condition $0 < D < 2^n$ is a natural requirement because of Morrey space, which is a doubling condition about φ.

To prove the theorems, we need the following lemmas.

Lemma 3.7 ([14]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define that, for $1/r = 1/p - 1/q$*

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \quad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

Lemma 3.8 ([14]) *We have*

$$\frac{1}{|Q|} \int_Q |f(x)g(x)| \mathrm{d}x \le \|f\|_{\exp L, Q} \|g\|_{L(\log L), Q}.$$

Lemma 3.9 ([14]) *Let T be the singular integral operator with non-smooth kernel as Definition 2.2. Then T is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1) bounded.*

Lemma 3.10 ([14,15]) *Let $\{A_t, t > 0\}$ be an "approximation to the identity". For any $\gamma > 0$,* *there exists a constant $C>0$ independent of γ such that*

$$|\{x\in R^n: M(f)(x)>D\lambda, M_A^{\#}(f)(x)\le\gamma\lambda\}|\le C\gamma|\{x\in R^n: M(f)(x)>\lambda\}|$$

for $\lambda>0$, where D is a fixed constant which only depends on n. Thus, for $f\in L^p(R^n)$, $1<p<\infty$, $0<\eta<\infty$ and $w\in A_1$,

$$\|M_\eta(f)\|_{L^p(w)}\le C\|M_{A,\eta}^{\#}(f)\|_{L^p(w)}.$$

Lemma 3.11 ([14]) *Let $\{A_t, t>0\}$ be an "approximation to the identity" and $\tilde{K}_{\alpha,t}(x,y)$ be the kernel of difference operator $I_\alpha-A_tI_\alpha$. Then*

$$|\tilde{K}_{\alpha,t}(x,y)|\le C\frac{t}{|x-y|^{n+2-\alpha}}.$$

Lemma 3.12 ([15]) *Assume the following conditions are satisfied:*

(i) The holomorphic semigroup e^{-zL}, $0\le|\arg(z)|<\pi/2-\theta$ is represented by the kernels $a_z(x,y)$ which satisfy, for all $\nu>\theta$, an upper bound

$$|a_z(x,y)|\le c_\nu h_{|z|}(x,y)$$

for $x,y\in R^n$, and $0\le|\arg(z)|<\pi/2-\theta$, where $h_t(x,y)=Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty}r^{n+\epsilon}s(r^2)=0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu>\theta$ and $g\in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies

$$\|g(L)(f)\|_{L^2}\le c_\nu\|g\|_{L^\infty}\|f\|_{L^2}.$$

Then the operator L has a bounded functional calculus in $L^p(R^n)$ for $1<p<\infty$.

Lemma 3.13 *Let w be a non-negative weight function. Then*

(I) $\|M_s(f)\|_{L^p(w)}\le C\|f\|_{L^p(w)}$ for $w\in A_1$ and $1\le s<p<\infty$;

(II) $\|M_{\alpha,s}(f)\|_{L^q(w^q)}\le C\|f\|_{L^p(w^p)}$ for $0<\alpha<n$, $1\le s<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $w\in A_{p,q}$;

(III) $\|I_\alpha(f)\|_{L^q(w^q)}\le C\|f\|_{L^p(w^p)}$ for $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $w\in A_{p,q}$.

Proof (I) By $w\in A_1$ and [15], we have

$$M_s(f)(x)\le M_{s,w}(f)(x),$$

where

$$M_{s,w}(f)(x)=\sup_{Q\ni x}\Big(\frac{1}{w(Q)}\int_Q|f(y)|^s w(y)\mathrm{d}y\Big)^{1/s},$$

thus, (I) follows from the $L^p(w)$-boundedness of $M_{s,w}$ and interpolation theorem [14]. (II) and (III) see [15]. □

Lemma 3.14 *Let $\{A_t, t>0\}$ be an "approximation to the identity" and $0<D<2^n$. Then* *(a)* $\|M_\eta(f)\|_{L^{p,\varphi}} \le C\|M^{\#}_{A,\eta}(f)\|_{L^{p,\varphi}}$ *for* $1 < p < \infty$ *and* $0 < \eta < \infty$*;*

(b) $\|I_\alpha(f)\|_{L^{q,\varphi}} \le C\|f\|_{L^{p,\alpha,\varphi}}$ *for* $0 < \alpha < n$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$*;*

(c) $\|M_{\alpha,s}(f)\|_{L^{q,\varphi}} \le C\|f\|_{L^{p,\alpha,\varphi}}$ *for* $0 \le \alpha < n$, $1 \le s < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$.

Proof (a) For any cube $Q = Q(x_0, d)$ in R^n, we know $(M(\chi_Q))^\delta \in A_1$ for any cube $Q = Q(x, d)$ and $0 < \delta < 1$. Write $w = M(\chi_Q)$, $w^\delta = (M(\chi_Q))^\delta \in A_1$, w^δ satisfies the reverse of Hölder's inequality:

$$\Big(\frac{1}{|B|}\int_B (w(x)^\delta)^r \mathrm{d}x\Big)^{1/r} \le \frac{C}{|B|}\int_B w(x)^\delta \mathrm{d}x$$

for all cube B and some $1 < r < \infty$ (see [14]), thus, taking $\delta = 1/r$, we get

$$\frac{1}{|B|}\int_B w(x)\mathrm{d}x \le C\Big(\frac{1}{|B|}\int_B w^\delta(x)\mathrm{d}x\Big)^{1/\delta}$$

and

$$M(w) \le C(M(w^\delta))^{1/\delta} \le C(w^\delta)^{1/\delta} = Cw,$$

that is $w = M(\chi_Q) \in A_1$. Now, noticing that

$$M(\chi_Q) \le 1 \ \text{ and } \ M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$$

if $x \in Q^c$, we have

$$\begin{aligned}
&\int_Q M_\eta(f)(y)^p \mathrm{d}y = \int_{R^n} M_\eta(f)(y)^p \chi_Q(y)\mathrm{d}y \\
&\le \int_{R^n} M_\eta(f)(y)^p M(\chi_Q)(y)\mathrm{d}y \le C\int_{R^n} M^{\#}_{A,\eta}(f)(y)^p M(\chi_Q)(y)\mathrm{d}y \\
&= C\Big(\int_Q M^{\#}_{A,\eta}(f)(y)^p M(\chi_Q)(y)\mathrm{d}y + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M^{\#}_{A,\eta}(f)(y)^p M(\chi_Q)(y)\mathrm{d}y\Big) \\
&\le C\Big(\int_Q M^{\#}_{A,\eta}(f)(y)^p \mathrm{d}y + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M^{\#}_{A,\eta}(f)(y)^p \frac{|Q|}{|2^{k+1}Q|}\mathrm{d}y\Big) \\
&\le C\|M^{\#}_{A,\eta}(f)\|^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} 2^{-nk}\varphi(2^{k+1}d) \\
&\le C\|M^{\#}_{A,\eta}(f)\|^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} (2^{-n}D)^k\varphi(d) \\
&\le C\|M^{\#}_{A,\eta}(f)\|^p_{L^{p,\varphi}}\varphi(d),
\end{aligned}$$

thus

$$\|M_\eta(f)\|_{L^{p,\varphi}} \le C\|M^{\#}_{A,\eta}(f)\|_{L^{p,\varphi}}.$$

This completes the proof of (a).

For (b) and (c), let $T_\alpha = I_\alpha$ or $M_{\alpha,s}$. We know $c_1 \le M(\chi_Q) \le 1$ for some $0 < c_1 \le 1$, thus

$$\Big(\frac{1}{|Q|}\int_Q M(\chi_Q)(x)^q \mathrm{d}x\Big)^{1/q} \le 1$$

and

$$\Big(\frac{1}{|Q|}\int_Q M(\chi_Q)(x)^{-p/(p-1)}\mathrm{d}x\Big)^{(p-1)/p} \le C,$$

that is $M(\chi_Q) \in A_{p,q}$ for any cube $Q = Q(x,d)$. Now, similar to the proofs of (a), we get

$$\begin{aligned}
&\Big(\int_Q |T_\alpha(f)(y)|^q \mathrm{d}y\Big)^{1/q} \le \Big(\int_{R^n} |T_\alpha(f)(y)M(\chi_Q)(y)|^q \mathrm{d}y\Big)^{1/q}\\
&\le C\Big(\int_{R^n} |f(y)M(\chi_Q)(y)|^p \mathrm{d}y\Big)^{1/p}\\
&\le C\Big(\int_Q |f(y)M(\chi_Q)(y)|^p \mathrm{d}y + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} |f(y)\frac{|Q|}{|2^{k+1}Q|}|^p \mathrm{d}y\Big)^{1/p}\\
&\le C\Big(\|f\|^p_{L^{p,\alpha,\varphi}} \sum_{k=0}^{\infty} 2^{-npk}\varphi(2^{k+1}d)^{1-p\alpha/n}\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\alpha,\varphi}} \sum_{k=0}^{\infty} 2^{-nk}(D^k\varphi(d))^{1/p-\alpha/n}\\
&\le C\|f\|_{L^{p,\alpha,\varphi}} \sum_{k=0}^{\infty} (2^{-n}D^{1/q})^k\varphi(d)^{1/q}\\
&\le C\|f\|_{L^{p,\alpha,\varphi}}\varphi(d)^{1/q},
\end{aligned}$$

where the last inequality follows from the fact that $D^{1/q} \le D < 2^n$ if $D \ge 1$ or $2^{-n}D^{1/q} < 1$ if $D < 1$,

$$\sum_{k=0}^{\infty}(2^{-n}D^{1/q})^k < \infty,$$

thus

$$\|T_\alpha(f)\|_{L^{q,\varphi}} \le C\|f\|_{L^{p,\alpha,\varphi}}.$$

This completes the proof of (b) and (c). $\square$

4. Proofs of Theorems

Now we prove the theorems in the paper.

Proof of Theorem 3.1 It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\Big(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^r \mathrm{d}x\Big)^{1/r} \le C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{m}(M^2(T^{k,2}(f))(\tilde{x}) + M_s(T^{k,2}(f))(\tilde{x})),$$

where $t_Q = (l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are T $(k = 1,\dots,m)$. Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$. By $T_1(g) = 0$, we have

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = U_1(x) + U_2(x)$$

and

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^r \mathrm{d}x\Big)^{1/r} \le \Big(\frac{C}{|Q|}\int_Q |U_1(x)|^r \mathrm{d}x\Big)^{1/r} +\\
&\quad \Big(\frac{C}{|Q|}\int_Q |A_{t_Q}(U_1)(x)|^r \mathrm{d}x\Big)^{1/r} + \Big(\frac{C}{|Q|}\int_Q |U_2(x) - A_{t_Q}(U_2)(x)|^r \mathrm{d}x\Big)^{1/r}\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For I, by Lemmas 3.7–3.9, we obtain

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^r \mathrm{d}x\Big)^{1/r}\\
&\le |Q|^{-1}\frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\le C|Q|^{-1}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}\\
&\le C|Q|^{-1}\|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{L^1}\\
&\le C|Q|^{-1}\int_{2Q}|b(x)-b_{2Q}||T^{k,2}(f)(x)|\mathrm{d}x\\
&\le C\|b-b_{2Q}\|_{\exp L,2Q}\|T^{k,2}(f)\|_{L(\log L),2Q}\\
&\le C\|b\|_{\mathrm{BMO}}M^2(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{I} \le& C\sum_{k=1}^m\Big(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r \mathrm{d}x\Big)^{1/r}\\
\le& C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For II, by the condition on h_{t_Q} and notice for $x\in Q$, $y\in 2^{j+1}Q\backslash 2^jQ$, then

$$h_{t_Q}(x,y)\le Ct_Q^{-n/2}\rho(2^{2(j-1)}).$$

By Lemma 3.9, we obtain, for $1<p<s$,

$$\begin{aligned}
\mathrm{II} \le& \sum_{k=1}^m\frac{C}{|Q|}\int_Q |A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f))(x)|\mathrm{d}x\\
\le& \sum_{k=1}^m\frac{C}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|\mathrm{d}y\mathrm{d}x+\\
&\sum_{k=1}^m\frac{C}{|Q|}\int_Q\int_{(2Q)^c} h_{t_Q}(x,y)|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|\mathrm{d}y\mathrm{d}x\\
\le& \sum_{k=1}^m\frac{C}{|Q|}\int_Q\int_{2Q} t_Q^{-n/2}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|\mathrm{d}y\mathrm{d}x+\\
&C\sum_{k=1}^m\sum_{j=1}^\infty t_Q^{-n/2}\rho(2^{2(j-1)})(2^jl(Q))^n\frac{1}{|2^{j+1}Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|\mathrm{d}y\\
\le& C\sum_{k=1}^m\Big(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^p\mathrm{d}y\Big)^{1/p}+\\
&C\sum_{k=1}^m\sum_{j=1}^\infty t_Q^{-n/2}\rho(2^{2(j-1)})(2^jl(Q))^n\Big(\frac{1}{|2^{j+1}Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^p\mathrm{d}y\Big)^{1/p}\\
\le& C\sum_{k=1}^m\Big(\frac{1}{|Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^p\mathrm{d}x\Big)^{1/p}+
\end{aligned}$$

$$
\begin{aligned}
&C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\Big(\frac{1}{|2^{j+1}Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^p\mathrm{d}y\Big)^{1/p}\\
\le&C\sum_{k=1}^{m}\Big(\frac{1}{|2Q|}\int_{2Q}|T^{k,2}(f)(x)|^s\mathrm{d}x\Big)^{1/s}\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{sp/(s-p)}\mathrm{d}y\Big)^{(s-p)/sp}+\\
&C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/p)}\Big(\frac{1}{|2Q|}\int_{2Q}|T^{k,2}(f)(y)|^s\mathrm{d}y\Big)^{1/s}\times\\
&\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{sp/(s-p)}\mathrm{d}y\Big)^{(s-p)/sp}\\
\le&C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

For III, by Lemma 3.11, we get, for $1/s+1/s'=1$,

$$
\begin{aligned}
\mathrm{III}\le&\frac{C}{|Q|}\int_Q\sum_{k=1}^{m}\int_{(2Q)^c}|b(y)-b_{2Q}||K(x-y)-K_{t_Q}(x-y)||T^{k,2}(f)(y)|\mathrm{d}y\\
\le&C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{l(Q)^\delta}{|x_0-y|^{n+\delta}}|T^{k,2}(f)(y)|\mathrm{d}y\\
\le&C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-j\delta}\Big(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}|^{s'}\mathrm{d}y\Big)^{1/s'}\Big(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s\mathrm{d}y\Big)^{1/s}\\
\le&C\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j\delta}\|b\|_{\mathrm{BMO}}M_s(T^{k,2}(f))(\tilde{x})\\
\le&C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2 It suffices to prove for $f\in C_0^\infty(R^n)$, the following inequality holds:

$$
\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q|S_b(f)(x)-A_{t_Q}(S_b(f))(x)|^r\mathrm{d}x\Big)^{1/r}\\
&\quad\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}(M^2(I_\alpha T^{k,4}(f))(\tilde{x})+M_s(I_\alpha T^{k,4}(f))(\tilde{x})+M_{\alpha,s}(T^{k,6}(f))(\tilde{x})).
\end{aligned}
$$

Without loss of generality, we may assume $T^{k,3}$ are T $(k=1,\dots,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Write, by $T_1(g)=0$,

$$
\begin{aligned}
S_b(f)(x)=&\sum_{k=1}^{m}T^{k,3}M_bI_\alpha T^{k,4}(f)(x)+\sum_{k=1}^{m}T^{k,5}I_\alpha M_bT^{k,6}(f)(x)\\
=&V_b(x)+V_b(x)=V_{b-b_{2Q}}(x)+W_{b-b_{2Q}}(x),
\end{aligned}
$$

where

$$
V_{b-b_{2Q}}(x)=\sum_{k=1}^{m}T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(x)+\sum_{k=1}^{m}T^{k,3}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,4}(f)(x)
$$

$$=V_1(x)+V_2(x)$$

and

$$\begin{aligned}W_{b-b_{2Q}}(x)=&\sum_{k=1}^m T^{k,5}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,6}(f)(x)+\sum_{k=1}^m T^{k,5}I_\alpha M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,6}(f)(x)\\=&W_1(x)+W_2(x).\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q|S_b(f)(x)-A_{t_Q}(S_b(f))(x)|^r\mathrm{d}x\Big)^{1/r}\le\Big(\frac{C}{|Q|}\int_Q|V_1(x)|^r\mathrm{d}x\Big)^{1/r}+\\&\quad\Big(\frac{C}{|Q|}\int_Q|W_1(x)|^r\mathrm{d}x\Big)^{1/r}+\Big(\frac{C}{|Q|}\int_Q|A_{t_Q}(V_1)(x)|^r\mathrm{d}x\Big)^{1/r}+\Big(\frac{C}{|Q|}\int_Q|A_{t_Q}(W_1)(x)|^r\mathrm{d}x\Big)^{1/r}+\\&\quad\Big(\frac{C}{|Q|}\int_Q|V_2(x)-A_{t_Q}(V_2)(x)|^r\mathrm{d}x\Big)^{1/r}+\Big(\frac{C}{|Q|}\int_Q|W_2(x)-A_{t_Q}(W_2)(x)|^r\mathrm{d}x\Big)^{1/r}\\&=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6.\end{aligned}$$

By using a similar argument as in the proof of Theorem 3.1, we get, for $1<v<s$ with $1/v=1/u-\alpha/n$ and $1<p<s$,

$$\begin{aligned}\mathrm{I}_1\le&C\sum_{k=1}^m\Big(\frac{1}{|Q|}\int_{R^n}|T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(x)|^r\mathrm{d}x\Big)^{1/r}\\\le&C\sum_{k=1}^m|Q|^{-1}\frac{\|T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\\le&C\sum_{k=1}^m|Q|^{-1}\|T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)\|_{WL^1}\\\le&C\sum_{k=1}^m|Q|^{-1}\|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)\|_{L^1}\\\le&C\sum_{k=1}^m|Q|^{-1}\int_{2Q}|b(x)-b_{2Q}||I_\alpha T^{k,4}(f)(x)|\mathrm{d}x\\\le&C\sum_{k=1}^m\|b-b_{2Q}\|_{\exp L,2Q}\|I_\alpha T^{k,4}(f)\|_{L(\log L),2Q}\\\le&C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m M^2(I_\alpha T^{k,4}(f))(\tilde{x}),\\\mathrm{I}_2\le&C\sum_{k=1}^m\Big(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,6}(f)(x)|^v\mathrm{d}x\Big)^{1/v}\\\le&C\sum_{k=1}^m|Q|^{-1/v}\Big(\int_{2Q}(|b(x)-b_{2Q}||T^{k,6}(f)(x)|)^u\mathrm{d}x\Big)^{1/u}\\\le&C\sum_{k=1}^m\Big(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,6}(f)(x)|^s\mathrm{d}x\Big)^{1/s}\times\end{aligned}$$

$$
\begin{aligned}
&\Big(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{su/(s-u)}\mathrm{d}x\Big)^{(s-u)/su}\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,6}(f))(\tilde{x}),\\
\mathrm{I}_3 &\le C\sum_{k=1}^{m}\Big(\frac{1}{|Q|}\int_{R^n}|T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(y)|^p\mathrm{d}y\Big)^{1/p}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}t_Q^{-n/2}\rho(2^{2(j-1)})(2^jl(Q))^n\Big(\frac{1}{|2^{j+1}Q|}\int_{R^n}|T^{k,3}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(y)|^p\mathrm{d}y\Big)^{1/p}\\
&\le C\sum_{k=1}^{m}\Big(\frac{1}{|Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(x)|^p\mathrm{d}x\Big)^{1/p}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\Big(\frac{1}{|2^{j+1}Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,4}(f)(y)|^p\mathrm{d}y\Big)^{1/p}\\
&\le C\sum_{k=1}^{m}\Big(\frac{1}{|2Q|}\int_{2Q}|I_\alpha T^{k,4}(f)(x)|^s\mathrm{d}x\Big)^{1/s}\times\\
&\quad\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{sp/(s-p)}\mathrm{d}y\Big)^{(s-p)/sp}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/p)}\Big(\frac{1}{|2Q|}\int_{2Q}|I_\alpha T^{k,4}(f)(y)|^s\mathrm{d}y\Big)^{1/s}\times\\
&\quad\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{sp/(s-p)}\mathrm{d}y\Big)^{(s-p)/sp}\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_s(I_\alpha T^{k,4}(f))(\tilde{x}),\\
\mathrm{I}_4 &\le C\sum_{k=1}^{m}t_Q^{-n/2}|Q|^{1-1/v}\Big(\int_{R^n}|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,6}(f)(y)|^v\mathrm{d}y\Big)^{1/v}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}t_Q^{-n/2}\rho(2^{2(j-1)})|Q|^{1-1/v}\Big(\int_{R^n}|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,6}(f)(y)|^v\mathrm{d}y\Big)^{1/v}\\
&\le C\sum_{k=1}^{m}|Q|^{-1/v}\Big(\int_{2Q}|(b(y)-b_{2Q})T^{k,6}(f)(y)|^u\mathrm{d}y\Big)^{1/u}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\rho(2^{2(j-1)})|Q|^{-1/v}\Big(\int_{2Q}|(b(y)-b_{2Q})T^{k,6}(f)(y)|^u\mathrm{d}y\Big)^{1/u}\\
&\le C\sum_{k=1}^{m}\Big(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,6}(f)(y)|^s\mathrm{d}y\Big)^{1/s}\times\\
&\quad\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{su/(s-u)}\mathrm{d}y\Big)^{(s-u)/su}+\\
&\quad C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(n+\epsilon)}\Big(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,6}(f)(y)|^s\mathrm{d}y\Big)^{1/s}\times
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\frac{1}{|2Q|}\int_{2Q}|b(y)-b_{2Q}|^{su/(s-u)}\mathrm{d}y\Big)^{(s-u)/su}\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,6}(f))(\tilde{x}),\\
\mathrm{I}_5 &\le\frac{C}{|Q|}\int_Q\sum_{k=1}^{m}\int_{(2Q)^c}|b(y)-b_{2Q}||K(x-y)-K_{t_Q}(x-y)||I_\alpha T^{k,4}(f)(y)|\mathrm{d}y\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{l(Q)^\delta}{|x_0-y|^{n+\delta}}|I_\alpha T^{k,4}(f)(y)|\mathrm{d}y\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-j\delta}\Big(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}|^{s'}\mathrm{d}y\Big)^{1/s'}\times\\
&\Big(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,4}(f)(y)|^{s}\mathrm{d}y\Big)^{1/s}\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j\delta}\|b\|_{\mathrm{BMO}}M_s(I_\alpha T^{k,4}(f))(\tilde{x})\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_s(I_\alpha T^{k,4}(f))(\tilde{x}),\\
\mathrm{I}_6 &\le\sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}||\tilde{K}_{t_Q}(x-y)||T^{k,6}(f)(y)|\mathrm{d}y\mathrm{d}x\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{t_Q}{|x-y|^{n+2-\alpha}}|T^{k,6}(f)(y)|\mathrm{d}y\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-2j}\Big(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}|^{s'}\mathrm{d}y\Big)^{1/s'}\times\\
&\Big(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,6}(f)(y)|^{s}\mathrm{d}y\Big)^{1/s}\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-2j}\|b\|_{\mathrm{BMO}}M_{\alpha,s}(T^{k,6}(f))(\tilde{x})\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,6}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3 Choose $1<s<p$ in Theorem 3.1, we get, by Lemma 3.15,

$$
\begin{aligned}
\|T_b(f)\|_{L^{p,\varphi}} &\le\|M_r(T_b(f))\|_{L^{p,\varphi}}\le C\|M^{\#}_{A,r}(T_b(f))\|_{L^{p,\varphi}}\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}(\|M^2(T^{k,2}(f))\|_{L^{p,\varphi}}+\|M_s(T^{k,2}(f))\|_{L^{p,\varphi}})\\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^{p,\varphi}}
\end{aligned}
$$

$$\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p,\varphi}}.$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4 Choose $1 < s < p$ in Theorem 3.2, we get, by Lemma 3.15,

$$\begin{aligned}
\|S_b(f)\|_{L^{q,\varphi}} &\leq \|M_r(S_b(f))\|_{L^{q,\varphi}(w)} \leq C\|M^{\#}_{A,r}(S_b(f))\|_{L^{q,\varphi}} \\
&\leq C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}(\|M^2(I_\alpha T^{k,4}(f))\|_{L^{q,\varphi}} + \|M_s(I_\alpha T^{k,4}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{k,6}(f))\|_{L^{q,\varphi}}) \\
&\leq C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}(\|I_\alpha T^{k,4}(f)\|_{L^{q,\varphi}} + \|I_\alpha T^{k,4}(f)\|_{L^{q,\varphi}} + \|T^{k,6}(f)\|_{L^{p,\alpha,\varphi}}) \\
&\leq C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{m}(\|T^{k,4}(f)\|_{L^{p,\alpha,\varphi}} + \|T^{k,6}(f)\|_{L^{p,\alpha,\varphi}}) \\
&\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p,\alpha,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 3.4. □

5. Applications

In this section we shall apply Theorems 3.1–3.4 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus [5]. Given $0 \leq \theta < \pi$. Define

$$S_\theta = \{z \in C : |\arg(z)| \leq \theta\}\bigcup\{0\}$$

and its interior by S^0_θ. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|\|(\eta I - L)^{-1}\| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S^0_\mu) = \{f : S^0_\theta \to C : f \ \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. Set

$$\Psi(S^0_\mu) = \{g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c\frac{|z|^s}{1+|z|^{2s}}\}.$$

If L is of type θ and $g \in H_\infty(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1}\int_\Gamma(\eta I - L)^{-1}g(\eta)\mathrm{d}\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S^0_\mu)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if $\|g(L)\| \le N\|g\|_{L^\infty}$ for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$. Applying Lemma 3.12 and Theorems 3.1–3.4, we get

Corollary 5.1 *Assume the following conditions are satisfied:*

(i) The holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ is represented by the kernels $a_z(x, y)$ which satisfy, for all $\nu > \theta$, an upper bound

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for $x, y \in R^n$, and $0 \le |\arg(z)| < \pi/2 - \theta$, where $h_t(x, y) = Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies

$$\|g(L)(f)\|_{L^2} \le c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$

Let $g(L)_b$ be the Toeplitz type operator associated to $g(L)$. Then Theorems 3.1–3.4 hold for $g(L)_b$.

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