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Multiple Solutions for a Class of Quasilinear Equations with Critical Exponential Growth

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Abstract In this paper, we consider a class of quasilinear equations involving a nonlinearity term having critical exponential growth. By using Mountain Pass Theorem, Ekeland's variational principle and inequalities of the type Trudinger-Moser, we obtain the existence of at least two positive weak solutions.

Keywords multiple solutions; quasilinear equations; critical exponential growth

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1. Introduction

This paper concerns with the existence and multiplicity of positive weak solutions for the following class of quasilinear equations

$$\begin{cases} -\Delta_N u + |u|^{N-2}u - \Delta_N(u^2)u = \lambda g(x)|u|^{q-2}u + h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator of u, $N \geq 2$, $1 < q < N$, $\lambda > 0$ is a real parameter, $g(x)$ is a positive function in $L^\theta(\Omega)$ with $\theta = \frac{N}{N-q}$ and function h satisfies the following assumptions:

(h_1) There exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|h(s)|}{e^{\alpha|s|^{\frac{2N}{N-1}}}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

(h_2) $\lim_{s\to 0} \frac{h(s)}{|s|^{N-2}s} = 0$.

(h_3) There exists $\mu > 2N$ such that

$$0 < \mu H(s) \leq sh(s), \text{ for all } |s| > 0,$$

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where $H(s)=\int_0^s h(t)\mathrm{d}t$.

(h_4) There exist constants $l>2N$ and $\gamma>0$ such that

$$h(s)\ge \gamma s^{l-1},\quad \forall s\ge 0,$$

where

$$\begin{aligned}\gamma>\max\Big\{&\frac{1}{C_1^l}(\frac{8^N}{m_0})^{\frac{l-N}{N}}[\frac{\mu(l-N)}{l(\mu-2N)}]^{\frac{l-N}{N}}S_l^l(\frac{\alpha_0}{\alpha_N})^{\frac{(N-1)(l-N)}{N}},\\ &\frac{2}{C_1^l}(\frac{8^N}{m_0})^{\frac{l-2N}{2N}}[\frac{\mu(l-2N)}{l(\mu-2N)}]^{\frac{l-2N}{2N}}S_{\frac{l}{2}}^{\frac{l}{2}}(\frac{\alpha_0}{\alpha_N})^{\frac{(N-1)(l-2N)}{2N}}\Big\}\end{aligned}$$

and C_1, m_0, α_N, S_l, $S_{\frac{l}{2}}$ will be defined later.

In recent years, much attention has been paid to the quasilinear equations of the form

$$-\Delta u+V(x)u-\Delta(u^2)u=g(x,u),\quad x\in\mathbb{R}^N. \tag{1.2}$$

Solutions of (1.2) are related to the existence of standing waves of the following quasilinear Schrödinger equations

$$i\frac{\partial\psi}{\partial t}=-\Delta\psi+W(x)\psi-\kappa\Delta(\rho(|\psi|^2))\rho'(|\psi|^2)\psi-g(x,\psi),\quad x\in\mathbb{R}^N, \tag{1.3}$$

where $W(x)$ is a given potential, κ is a real constant, ρ and g are real functions. We would like to mention that quasilinear equation of the form (1.3) arises in various branches of mathematical physics and has been derived as models of several physical phenomena corresponding to various types of nonlinear term ρ. For instance, the case $\rho(s)=s$ was used for the superfluid film equation in plasma physics by Kurihara [1]. In the case $\rho(s)=(1+s)^{\frac{1}{2}}$, (1.3) models the self-channeling of a high-power ultrashort laser in matter [2, 3]. Eq. (1.3) also appears in fluid mechanics [4], in the theory of Heisenberg ferromagnets and magnons [5], in dissipative quantum mechanics and in condensed matter theory [6].

Compared to the semilinear case ($\kappa=0$), the quasilinear case ($\kappa\ne 0$) becomes much more complicated because of the effects of the quasilinear and non-convex term $\Delta(u^2)u$. One of the main difficulties is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N=1$ (see [7]). Thus, such class of quasilinear problems has been studied extensively in recent years under various hypotheses on the potential and the nonlinearities, see for example [8–17] and references therein.

Motivated by (1.2), there has been considerable attention paid to the following quasilinear Schrödinger equation

$$-\Delta_p u+V(x)|u|^{p-2}u-\Delta_p(u^2)u=g(x,u), \tag{1.4}$$

where $1<p<N$, and see for example [18–25] and references therein. Note that for such case of $p<N$, the majority of problems treated with variational methods, the maximal possible growth for the nonlinearity term is polynomial at infinity and by consequence the corresponding energy functional could be defined in some appropriate Sobolev spaces. But things change when dealing with the case of $p=N$. According to the Trudinger-Moser inequality [26, 27], the maximal growth on the nonlinearities which allows us to treat the problem variationally is the critical

exponential growth. Wang, Yang and Zhang [28] studied problem (1.4) when $p = N$, by using Mountain Pass Theorem, they obtained the existence of a nontrivial solution. More recently, Chen and Song [29] established the existence of a nonnegative and nontrivial solution using the Nehari manifold method and the Schwarz symmetrization with some special techniques.

Motivated by works mentioned above, in this paper, we will study the multiplicity of solutions for problem (1.1) with critical exponential growth. Moreover, as far as we know, there are few works dealing with such problem. Here, we employ variational methods together with Trudinger-Moser inequality for bounded domains.

Our main result is stated as follows.

Theorem 1.1 *Assume that conditions* (h_1)–(h_4) *hold. Then there exists a constant* $\Lambda > 0$ *such that, for* $\lambda \in (0, \Lambda)$, *problem (1.1) has at least two positive weak solutions.*

Remark 1.2 When $N = 2$, the critical exponential growth is defined by $(\widetilde{h}_1)$: There exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|h(s)|}{e^{\alpha|s|^4}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

(see [30]). So we believe that the exponential growth mentioned in (h_1) is the critical growth for problem (1.1).

To finish this introduction, we state a version of the Trudinger-Moser inequality for bounded domains and a technical lemma which is essential to carry out the proof of our result.

Proposition 1.3 ([31, Lemma 2.1]) *Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *be a bounded domain. Given any* $u \in W_0^{1,N}(\Omega)$, *we have*

$$\int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} \mathrm{d}x < \infty, \text{ for every } \alpha > 0.$$

Moreover, there exists a positive constant $C = C(N, |\Omega|)$ *such that*

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} \mathrm{d}x \leq C, \text{ for all } \alpha \leq \alpha_N,$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}} > 0$ *and* ω_{N-1} *is the* $(N-1)$*-dimensional measure of the* $(N-1)$*-sphere.*

Proposition 1.4 ([31, Corollary 2.5]) *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *and* $\{u_n\}$ *be a sequence in* $W_0^{1,N}(\Omega)$ *with*

$$\limsup_{n\to\infty} \|u_n\|^N < (\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Then there exist $\alpha > \alpha_0$, $\sigma > 1$ *and* $C > 0$ *(independent of* n*) such that*

$$\int_\Omega e^{\sigma\alpha|u_n|^{\frac{N}{N-1}}} \mathrm{d}x \leq C, \text{ for all } n \geq n_0,$$

for some n_0 *sufficiently large.*

Throughout the paper, $C, C_1, C_2, \ldots$ denote positive (possibly different) constants.

2. Preliminaries

Since we intend to find positive weak solutions, we suppose that $h(s)=0$ in $(-\infty,0)$.

Consider the Sobolev space $X=W_0^{1,N}(\Omega)$ endowed with the usual norm

$$\|u\|=\Big(\int_\Omega(|\nabla u|^N+|u|^N)\mathrm{d}x\Big)^{\frac{1}{N}},\quad \forall\, u\in X.$$

Obviously, the embedding $X\hookrightarrow L^\tau(\Omega)$, $1\le\tau<\infty$ is compact.

We observe that the natural variational functional for (1.1)

$$I_\lambda(u)=\frac{1}{N}\int_\Omega(1+2^{N-1}|u|^N)|\nabla u|^N\mathrm{d}x+\frac{1}{N}\int_\Omega|u|^N\mathrm{d}x-\frac{\lambda}{q}\int_\Omega g(x)|u|^q\mathrm{d}x-\int_\Omega H(u)\mathrm{d}x$$

is not well defined in X. To overcome this difficulty, we generalize an argument developed in [18, 19]. We make the change of variables $v=f^{-1}(u)$, where f is defined by

$$f'(s)=\frac{1}{(1+2^{N-1}|f(s)|^N)^{\frac{1}{N}}},\quad s\in[0,+\infty),$$
$$f(s)=-f(-s),\quad s\in(-\infty,0].$$

Then, we collect some properties of f.

Lemma 2.1 *The function $f(s)$ enjoys the following properties:*

(f_1) *f is uniquely defined, C^2 function and invertible.*

(f_2) $|f'(s)|\le 1$, $\forall s\in\mathbb{R}$.

(f_3) $|f(s)|\le|s|$, $\forall s\in\mathbb{R}$.

(f_4) $\frac{f(s)}{s}\to 1$, *as* $s\to 0$.

(f_5) $|f(s)|\le 2^{\frac{1}{2N}}|s|^{\frac{1}{2}}$, $\forall s\in\mathbb{R}$.

(f_6) $\frac{1}{2}f(s)\le sf'(s)\le f(s)$, $\forall s\ge 0$.

(f_7) $\frac{f(s)}{\sqrt{s}}\to a>0$, *as* $s\to+\infty$.

(f_8) *There exists a positive constant C_1 such that*

$$|f(s)|\ge\begin{cases} C_1|s|, & \forall|s|\le 1,\\ C_1|s|^{\frac{1}{2}}, & \forall|s|\ge 1.\end{cases}$$

After the change of variables, $I_\lambda(u)$ can be reduced to the following functional

$$J_\lambda(v)=\frac{1}{N}\int_\Omega(|\nabla v|^N+|f(v)|^N)\mathrm{d}x-\frac{\lambda}{q}\int_\Omega g(x)|f(v)|^q\mathrm{d}x-\int_\Omega H(f(v))\mathrm{d}x,$$

which is C^1 on the usual Sobolev space E. Moreover, the critical points of J_λ are the weak solutions of the following problem

$$\begin{cases} -\Delta_N v=-f'(v)[|f(v)|^{N-2}f(v)+\lambda g(x)|f(v)|^{q-2}f(v)+h(f(v))], & x\in\Omega,\\ v=0, & x\in\partial\Omega.\end{cases}\tag{2.1}$$

The following proposition establishes a relation between the solutions of (2.1) and those of (1.1).

Proposition 2.2 *If $v\in X$ is a critical point of J_λ, then $u=f(v)\in X$ is a weak solution of (1.1).*

Proof The proof is analogous to that proved in [19, Proposition 2.2]. □

Denote $\phi(s) = -|f(s)|^{N-2}f(s)f'(s) + h(f(s))f'(s)$ and $\Phi(s) = \int_0^s \phi(t)\mathrm{d}t$. Then, by ($h_2$), ($f_4$) and some direct computation, we can easily deduce there exists $\varrho < 0$ such that

$$\lim_{s\to 0}\frac{\phi(s)}{|s|^{N-2}s} = \varrho. \tag{2.2}$$

In the sequel, we will prove the existence of a weak solution of (1.1) having a positive energy by using the Mountain Pass Theorem in [32] (or see [33]). Firstly, we check that the functional J_λ verifies the Mountain Pass geometry.

Lemma 2.3 *Assume that (h_1)–(h_3) hold. Then there exists $\Lambda_1 > 0$ such that if $\lambda \in (0, \Lambda_1)$, the functional J_λ satisfies:*

(i) There exist $\eta > 0$ and $\rho > 0$ such that $J_\lambda(v) \geq \eta$, for $\|v\| = \rho$.

(ii) There exists $e \in X$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.

Proof First, we show that J_λ satisfies (i). By (2.2), there exist $\epsilon > 0$, $\delta > 0$ such that

$$\Phi(v) \leq \frac{\epsilon+\varrho}{N}|v|^N, \quad |v| \leq \delta.$$

On the other hand, by (h_1), (f_2) and (f_3), for $\nu > N$ and $\alpha > \alpha_0$, there exist $\beta > \alpha$ and $C_\epsilon > 0$ such that $\Phi(v) \leq C_\epsilon |v|^\nu e^{\beta|v|^{\frac{N}{N-1}}}$, $|v| \geq \delta$. These two estimates yield

$$\Phi(v) \leq \frac{\epsilon+\varrho}{N}|v|^N + C_\epsilon|v|^\nu e^{\beta|v|^{\frac{N}{N-1}}}.$$

Fix $M > 0$ verifying $\beta M^{\frac{N}{N-1}} < \alpha_N$ and choose $\sigma_1 > 1$ (sufficiently close to 1) such that $\sigma_1\beta M^{\frac{N}{N-1}} < \alpha_N$ and $\sigma_2 > 1$ such that $\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = 1$. Then it follows from ($f_3$), the Hölder inequality, the Sobolev embedding and Proposition 1.3 that

$$\begin{aligned}
J_\lambda(v) =& \frac{1}{N}\int_\Omega |\nabla v|^N\mathrm{d}x - \frac{\lambda}{q}\int_\Omega g(x)|f(v)|^q\mathrm{d}x - \int_\Omega \Phi(v)\mathrm{d}x\\
\geq& \frac{1}{N}\int_\Omega |\nabla v|^N\mathrm{d}x - \frac{\epsilon+\varrho}{N}\int_\Omega |v|^N\mathrm{d}x - \frac{\lambda}{q}|g|_\theta|v|_N^q - C_\epsilon\Big(\int_\Omega |v|^{\nu\sigma_2}\mathrm{d}x\Big)^{\frac{1}{\sigma_2}}\Big(\int_\Omega e^{\beta\sigma_1|v|^{\frac{N}{N-1}}}\mathrm{d}x\Big)^{\frac{1}{\sigma_1}}\\
\geq& \frac{1}{N}\int_\Omega |\nabla v|^N\mathrm{d}x - \frac{\epsilon+\varrho}{N}\int_\Omega |v|^N\mathrm{d}x - \frac{\lambda}{q}|g|_\theta\|v\|^q - C\|v\|^\nu\Big(\int_\Omega e^{\beta\sigma_1 M^{\frac{N}{N-1}}(\frac{|v|}{\|v\|})^{\frac{N}{N-1}}}\mathrm{d}x\Big)^{\frac{1}{\sigma_1}}\\
\geq& \min\{\frac{1}{N}, -\frac{\epsilon+\varrho}{N}\}\|v\|^N - \frac{\lambda}{q}|g|_\theta\|v\|^q - C\|v\|^\nu.
\end{aligned}$$

Choosing ϵ sufficiently small and $\Lambda_1 = \frac{aq\rho^{N-q}}{2|g|_\theta}$ with $a = \min\{\frac{1}{N}, -\frac{\epsilon+\varrho}{N}\}$, for each $\lambda \in (0, \Lambda_1)$ and $\|v\| = \rho$, we deduce

$$J_\lambda(v) \geq a\rho^N - \frac{\lambda}{q}|g|_\theta\rho^q - C\rho^\nu \geq \frac{a}{2}\rho^N - C\rho^\nu.$$

Thus, if we choose $\rho \in (0, M)$ small enough, there exists $\eta > 0$ such that $J_\lambda(v) \geq \eta$ for $\|v\| = \rho$, which verifies (i).

In order to prove (ii), we fix $w \in X\backslash\{0\}$ and choose $t > 1$ such that $|tw| \geq 1$. Applying (h_3) and (f_8), there exist positive constants C, $D > 0$ such that

$$H(f(tw)) \geq C|f(tw)|^\mu - D \geq Ct^{\frac{\mu}{2}}|w|^{\frac{\mu}{2}} - D.$$

Then

$$\begin{aligned}J_\lambda(tw) =&\frac{t^N}{N}\int_\Omega|\nabla w|^N\mathrm{d}x+\frac{1}{N}\int_\Omega|f(tw)|^N\mathrm{d}x-\frac{\lambda}{q}\int_\Omega g(x)|f(tw)|^q\mathrm{d}x-\int_\Omega H(f(tw))\mathrm{d}x\\ \le&\frac{t^N}{N}\int_\Omega(|\nabla w|^N+|w|^N)\mathrm{d}x-Ct^{\frac{\mu}{2}}\int_\Omega|w|^{\frac{\mu}{2}}\mathrm{d}x+D|\Omega|,\end{aligned}$$

where $|\Omega|$ denotes the measure of Ω. Since $\mu>2N$, $J_\lambda(tw)\to-\infty$ as $t\to+\infty$. Taking $e=\bar{t}w$ with $\bar{t}>1$ sufficiently large, we conclude (ii). $\square$

By a version of the Mountain Pass Theorem found in [32], there exists a Palais Smale sequence $\{v_n\}\subset X$ such that

$$J_\lambda(v_n)\to c \text{ and } J'_\lambda(v_n)\to 0, \text{ as } n\to+\infty, \tag{2.3}$$

where $c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J_\lambda(\gamma(t))$ with

$$\Gamma=\{\gamma\in C([0,1],X)|\gamma(0)=0,\gamma(1)=e\}.$$

Then we have the following results.

Lemma 2.4 *Assume that* (h_1)–(h_4) *hold. Then the mountain level c satisfies*

$$c<\frac{m_0}{8^N}(\frac{1}{N}-\frac{2}{\mu})(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Proof Fix $\omega\in E$ and define

$$S=\frac{(\int_\Omega(|\nabla\omega|^N+|\omega|^N)\mathrm{d}x)^{\frac{1}{N}}}{(\int_\Omega|\omega|^p\mathrm{d}x)^{\frac{1}{p}}},\quad p=l \text{ or } \frac{l}{2}.$$

Case 1. If $|t\omega|\le1$, by the definition of c and (h_4), (f_3) and (f_8), we have

$$\begin{aligned}c\le&\max_{t\ge0}J_\lambda(t\omega)\\ =&\max_{t\ge0}\Big\{\frac{t^N}{N}\int_\Omega|\nabla\omega|^N\mathrm{d}x+\frac{1}{N}\int_\Omega|f(t\omega)|^N\mathrm{d}x-\frac{\lambda}{q}\int_\Omega g(x)|f(t\omega)|^q\mathrm{d}x-\int_\Omega H(f(t\omega))\mathrm{d}x\Big\}\\ \le&\max_{t\ge0}\Big\{\frac{t^N}{N}\int_\Omega(|\nabla\omega|^N+|\omega|^N)\mathrm{d}x-\frac{\gamma C_1^lt^l}{l}\int_\Omega|\omega|^l\mathrm{d}x\Big\}.\end{aligned}$$

Dividing by $(\int_\Omega|\omega|^l\mathrm{d}x)^{\frac{N}{l}}$ and by a direct computation, one has

$$\begin{aligned}\frac{c}{(\int_\Omega|\omega|^l\mathrm{d}x)^{\frac{N}{l}}}\le&\max_{t\ge0}\Big\{\frac{t^N}{N}S_l^N-\frac{\gamma C_1^lt^l}{l}(\int_\Omega|\omega|^l\mathrm{d}x)^{\frac{l-N}{l}}\Big\}\\ =&(\frac{1}{N}-\frac{1}{l})(\gamma C_1^l)^{\frac{-N}{l-N}}S_l^{\frac{lN}{l-N}}(\int_\Omega|\omega|^l\mathrm{d}x)^{\frac{-N}{l}},\end{aligned}$$

which leads to

$$c\le(\frac{1}{N}-\frac{1}{l})(\gamma C_1^l)^{\frac{-N}{l-N}}S_l^{\frac{lN}{l-N}}. \tag{2.4}$$

Case 2. If $|t\omega|>1$, using the same arguments, we can easily get

$$c\le\max_{t\ge0}\Big\{\frac{t^N}{N}\int_\Omega(|\nabla\omega|^N+|\omega|^N)\mathrm{d}x-\frac{\gamma C_1^lt^{\frac{l}{2}}}{l}\int_\Omega|\omega|^{\frac{l}{2}}\mathrm{d}x\Big\}.$$

Dividing by $(\int_\Omega |\omega|^{\frac{l}{2}}\mathrm{d}x)^{\frac{2N}{l}}$, we have

$$\frac{c}{(\int_\Omega |\omega|^{\frac{l}{2}}\mathrm{d}x)^{\frac{2N}{l}}} \le \max_{t\ge 0}\Big\{\frac{t^N}{N}S_{\frac{l}{2}}^N - \frac{\gamma C_1^l t^{\frac{l}{2}}}{l}\Big(\int_\Omega |\omega|^{\frac{l}{2}}\mathrm{d}x\Big)^{\frac{l-2N}{l}}\Big\}$$
$$=2^{\frac{l}{l-2N}}(\frac{1}{2N}-\frac{1}{l})(\gamma C_1^l)^{\frac{-2N}{l-2N}}S_{\frac{l}{2}}^{\frac{lN}{l-2N}}\Big(\int_\Omega |\omega|^{\frac{l}{2}}\mathrm{d}x\Big)^{\frac{-2N}{l}},$$

which implies

$$c \le 2^{\frac{l}{l-2N}}(\frac{1}{2N}-\frac{1}{l})(\gamma C_1^l)^{\frac{-2N}{l-2N}}S_{\frac{l}{2}}^{\frac{lN}{l-2N}}. \tag{2.5}$$

Thus, it follows from (h_4), (2.4) and (2.5) that

$$c < \frac{m_0}{8^N}(\frac{1}{N}-\frac{2}{\mu})(\frac{\alpha_N}{\alpha_0})^{N-1},$$

which completes the proof. □

Lemma 2.5 *Let $\{v_n\}\subset E$ be a $(PS)_c$-sequence associated with J_λ. Then there exists $\Lambda_2>0$ such that if $\lambda\in(0,\Lambda_2)$, there holds*

$$\limsup_{n\to\infty}\|v_n\|^N < \frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Proof Since $\{v_n\}$ satisfies (2.3), for any $\vartheta\in X$, we have

$$J_\lambda(v_n)-\frac{1}{\mu}\langle J_\lambda'(v_n),\vartheta\rangle \le c+o_n(1)+o_n(1)\|v_n\|.$$

On the other hand, choosing $\vartheta=\frac{f(v_n)}{f'(v_n)}$ and then we deduce from the Hölder inequality, (h_3) and (f_3) that

$$\begin{aligned}
J_\lambda(v_n)-\frac{1}{\mu}\langle J_\lambda'(v_n),\vartheta\rangle = &\frac{1}{N}\int_\Omega|\nabla v_n|^N\mathrm{d}x-\frac{1}{\mu}\int_\Omega(1+\frac{2^{N-1}|f(v_n)|^N}{1+2^{N-1}|f(v_n)|^N})|\nabla v_n|^N\mathrm{d}x+\\
&(\frac{1}{N}-\frac{1}{\mu})\int_\Omega|f(v_n)|^N\mathrm{d}x-(\frac{1}{q}-\frac{1}{\mu})\lambda\int_\Omega g(x)|f(v_n)|^q\mathrm{d}x+\\
&\int_\Omega[\frac{1}{\mu}h(f(v_n))f(v_n)-H(f(v_n))]\mathrm{d}x\\
\ge &(\frac{1}{N}-\frac{2}{\mu})\int_\Omega|\nabla v_n|^N\mathrm{d}x+(\frac{1}{N}-\frac{1}{\mu})\int_\Omega|f(v_n)|^N\mathrm{d}x-(\frac{1}{q}-\frac{1}{\mu})\lambda|g|_\theta\|v_n\|^q\\
\ge &(\frac{1}{N}-\frac{2}{\mu})\int_\Omega(|\nabla v_n|^N+|f(v_n)|^N)\mathrm{d}x-(\frac{1}{q}-\frac{1}{\mu})\lambda|g|_\theta\|v_n\|^q.
\end{aligned}$$

Then

$$\begin{aligned}
&(\frac{1}{N}-\frac{2}{\mu})\int_\Omega(|\nabla v_n|^N+|f(v_n)|^N)\mathrm{d}x\\
&\le c+(\frac{1}{q}-\frac{1}{\mu})\lambda|g|_\theta\|v_n\|^q+o_n(1)+o_n(1)\|v_n\|\\
&\le c+\epsilon(\frac{1}{q}-\frac{1}{\mu})\|v_n\|^N+C_\epsilon(\frac{1}{q}-\frac{1}{\mu})\lambda^\theta|g|_\theta^\theta+o_n(1).
\end{aligned} \tag{2.6}$$

Next, we claim that there exists $m_0>0$ such that

$$\int_\Omega(|\nabla v_n|^N+|f(v_n)|^N)\mathrm{d}x \ge m_0\|v_n\|^N. \tag{2.7}$$

In fact, by (h_3), there exists $C_2 > 0$ such that $H(s) \geq C_2 s^{2N}$ for $|s| \geq 1$. Then we deduce from (f_8) that $H(f(s)) \geq C_1^{2N} C_2 |s|^N$, for $|s| \geq 1$, and so

$$\begin{aligned}\int_\Omega |v_n|^N \mathrm{d}x =& \int_{\{|v_n|\leq 1\}} |v_n|^N \mathrm{d}x + \int_{\{|v_n|\geq 1\}} |v_n|^N \mathrm{d}x\\ \leq& \frac{1}{C_1^N}\int_\Omega |f(v_n)|^N \mathrm{d}x + \frac{1}{C_1^{2N}C_2}\int_\Omega H(f(v_n))\mathrm{d}x\\ =& \frac{1}{C_1^N}\int_\Omega |f(v_n)|^N \mathrm{d}x + \frac{1}{C_1^{2N}C_2}\Big[\frac{1}{N}\int_\Omega (|\nabla v_n|^N + |f(v_n)|^N)\mathrm{d}x -\\ & \frac{\lambda}{q}\int_\Omega g(x)|f(v_n)|^q \mathrm{d}x - c + o_n(1)\Big]\\ \leq& C\int_\Omega (|\nabla v_n|^N + |f(v_n)|^N)\mathrm{d}x.\end{aligned}$$

This concludes (2.7).

It follows from (2.6) and (2.7) that

$$(\frac{1}{N} - \frac{2}{\mu})m_0\|v_n\|^N \leq c + \epsilon(\frac{1}{q} - \frac{1}{\mu})\|v_n\|^N + C_\epsilon(\frac{1}{q} - \frac{1}{\mu})\lambda^\theta |g|_\theta^\theta + o_n(1)$$

and then

$$[(\frac{1}{N} - \frac{2}{\mu})m_0 - \epsilon(\frac{1}{q} - \frac{1}{\mu})]\|v_n\|^N \leq c + C_\epsilon(\frac{1}{q} - \frac{1}{\mu})\lambda^\theta |g|_\theta^\theta + o_n(1).$$

Choosing ϵ small enough, we have

$$\frac{m_0}{2}(\frac{1}{N} - \frac{2}{\mu})\limsup_{n\to\infty}\|v_n\|^N \leq c + C_\epsilon(\frac{1}{q} - \frac{1}{\mu})\lambda^\theta |g|_\theta^\theta,$$

which leads to

$$\limsup_{n\to\infty}\|v_n\|^N \leq \frac{c}{\frac{m_0}{2}(\frac{1}{N} - \frac{2}{\mu})} + C_3\lambda^\theta,$$

where

$$C_3 = \frac{C_\epsilon(\frac{1}{q} - \frac{1}{\mu})|g|_\theta^\theta}{\frac{m_0}{2}(\frac{1}{N} - \frac{2}{\mu})}.$$

By Lemma 2.3 and let $\Lambda_2 = (\frac{1}{4C_3})^{\frac{1}{\theta}}(\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{\theta}}$, for any $\lambda \in (0, \Lambda_2)$, we can easily check that $\limsup_{n\to\infty}\|v_n\|^N < \frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{N-1}$. This completes the proof of the lemma. □

Lemma 2.6 *Assume that (h_1)–(h_4) hold. Then the functional $J_\lambda(v)$ satisfies the $(PS)_c$ condition for $c \in (0, \frac{m_0}{8^N}(\frac{1}{N} - \frac{2}{\mu})(\frac{\alpha_N}{\alpha_0})^{N-1})$.*

Proof Since $\{v_n\}$ is bounded in X, going if necessary to a subsequence, we can assume that

$$\begin{aligned}& v_n \rightharpoonup v^1 \text{ in } X,\\ & v_n \to v^1 \text{ in } L^\sigma(\Omega),\ \ 1 \leq \sigma < +\infty,\\ & v_n \to v^1 \text{ a.e. in } \Omega.\end{aligned}$$

First, we claim that there exists $C_4 > 0$ such that

$$\int_\Omega |\nabla(v_n - v^1)|^N \mathrm{d}x + \int_\Omega (|f(v_n)|^{N-2}f(v_n)f'(v_n) - |f(v^1)|^{N-2}f(v^1)f'(v^1))(v_n - v^1)\mathrm{d}x$$

$$\geq C_4\|v_n - v^1\|^N. \tag{2.8}$$

In fact, we may assume that $v_n \neq v^1$ (otherwise, the conclusion is trivial). Similar to the idea of [18, Lemma 2.7], we can easily check that the function $|f(s)|^N$ is convex and then

$$\int_\Omega (|f(v_n)|^{N-2}f(v_n)f'(v_n) - |f(v^1)|^{N-2}f(v^1)f'(v^1))(v_n - v^1)\mathrm{d}x \geq 0.$$

Thus, we deduce from the Poincáre inequality that there exists $C_4 > 0$ such that (2.8) holds.

Now, we claim

$$\lim_{n\to\infty}\int_\Omega g(x)|f(v_n)|^{q-2}f(v_n)f'(v_n)(v_n - v^1)\mathrm{d}x \to 0 \tag{2.9}$$

and

$$\lim_{n\to\infty}\int_\Omega h(f(v_n))f'(v_n)(v_n - v^1)\mathrm{d}x \to 0. \tag{2.10}$$

Applying (f$_3$) and the Hölder inequality, we get

$$\begin{aligned}
&\Big|\int_\Omega g(x)|f(v_n)|^{q-2}f(v_n)f'(v_n)(v_n - v^1)\mathrm{d}x\Big| \\
&\leq \int_\Omega g(x)|v_n|^{q-1}|v_n - v^1|\mathrm{d}x \\
&\leq \Big(\int_\Omega |g(x)|^{\frac{N}{N-q}}\mathrm{d}x\Big)^{\frac{N-q}{N}}\Big(\int_\Omega |v_n|^N\mathrm{d}x\Big)^{\frac{q-1}{N}}\Big(\int_\Omega |v_n - v^1|^N\mathrm{d}x\Big)^{\frac{1}{N}} \\
&\leq |g|_\theta\|v_n\|^{q-1}|v_n - v^1|_N \to 0, \text{ as } n\to\infty,
\end{aligned}$$

which proves (2.9).

It follows from (h_1) and (h_2) that for $\alpha > \alpha_0$,

$$\begin{aligned}
\Big|\int_\Omega h(f(v_n))f'(v_n)(v_n - v^1)\mathrm{d}x\Big| &\leq \int_\Omega |f(v_n)|^{N-1}f'(v_n)|v_n - v^1|\mathrm{d}x + \\
&\int_\Omega e^{\alpha|f(v_n)|^{\frac{2N}{N-1}}}f'(v_n)|v_n - v^1|\mathrm{d}x.
\end{aligned}$$

By (f$_2$), (f$_3$) and the Hölder inequality, we have

$$\begin{aligned}
&\int_\Omega |f(v_n)|^{N-1}f'(v_n)|v_n - v^1|\mathrm{d}x \leq \int_\Omega |v_n|^{N-1}|v_n - v^1|\mathrm{d}x \\
&\leq \Big(\int_\Omega |v_n|^N\mathrm{d}x\Big)^{\frac{N-1}{N}}\Big(\int_\Omega |v_n - v^1|^N\mathrm{d}x\Big)^{\frac{1}{N}} \\
&\leq C\|v_n\|^{N-1}|v_n - v^1|_N \to 0, \text{ as } n\to\infty.
\end{aligned}$$

On the other hand, we deduce from (f$_2$) and (f$_5$) that

$$\int_\Omega e^{\alpha|f(v_n)|^{\frac{2N}{N-1}}}f'(v_n)|v_n - v^1|\mathrm{d}x \leq \int_\Omega e^{\alpha(2^{\frac{1}{N}}|v_n|)^{\frac{N}{N-1}}}|v_n - v^1|\mathrm{d}x.$$

Applying Lemma 2.5 and Proposition 1.4, there exist $\widetilde{\sigma}_1 > 1$ and $C > 0$ such that

$$\begin{aligned}
\int_\Omega e^{\alpha(2^{\frac{1}{N}}|v_n|)^{\frac{N}{N-1}}}|v_n - v^1|\mathrm{d}x \leq &\Big(\int_\Omega e^{\widetilde{\sigma}_1\alpha(2^{\frac{1}{N}}|v_n|)^{\frac{N}{N-1}}}\mathrm{d}x\Big)^{\frac{1}{\widetilde{\sigma}_1}}\Big(\int_\Omega |v_n - v^1|^{\widetilde{\sigma}_2}\mathrm{d}x\Big)^{\frac{1}{\widetilde{\sigma}_2}} \\
\leq &C|v_n - v^1|_{\widetilde{\sigma}_2} \to 0, \text{ as } n\to\infty,
\end{aligned}$$

where $\widetilde{\sigma}_2 = \frac{\widetilde{\sigma}_1}{\widetilde{\sigma}_1 - 1}$. Then (2.10) holds.

Therefore, by (2.9), (2.10) and the fact $v_n \rightharpoonup v$ in X, we have

$$\begin{aligned}
o_n(1) &= \langle J'_\lambda(v_n) - J'_\lambda(v^1), v_n - v^1\rangle \\
&= \int_\Omega (|\nabla v_n|^{N-2}\nabla v_n - |\nabla v^1|^{N-2}\nabla v^1)\nabla(v_n - v^1)\mathrm{d}x + \\
&\quad \int_\Omega (|f(v_n)|^{N-2} f(v_n) f'(v_n) - |f(v^1)|^{N-2} f(v^1) f'(v^1))(v_n - v^1)\mathrm{d}x + o_n(1) \\
&\ge C_N \int_\Omega |\nabla(v_n - v^1)|^N \mathrm{d}x + \\
&\quad \int_\Omega (|f(v_n)|^{N-2} f(v_n) f'(v_n) - |f(v^1)|^{N-2} f(v^1) f'(v^1))(v_n - v^1)\mathrm{d}x \\
&\ge C\|v_n - v^1\|^N + o_n(1),
\end{aligned}$$

where we have used the standard inequality

$$\langle |x|^{N-2}x - |y|^{N-2}y, x - y\rangle \ge C_N|x - y|^N, \quad N \ge 2.$$

This implies that $\|v_n - v^1\| \to 0$ as $n \to \infty$. Thus, the proof is completed. □

3. Proof of Theorem 1.1

In this section, we apply the Mountain Pass Theorem and the Ekeland's variational principle to prove the existence of two positive weak solutions for (1.1).

Lemma 3.1 *Assume that (h_1)–(h_4) hold. Then there exists $\Lambda_3 > 0$ such that if $\lambda \in (0, \Lambda_3)$, problem (2.1) admits a weak solution v^1 satisfying $J_\lambda(v^1) > 0$.*

Proof Let $\Lambda_3 = \min\{\Lambda_1, \Lambda_2\}$. Then the proof follows directly from Lemmas 2.3, 2.6 and the Mountain Pass Theorem in [32] (or see [33]). □

In the following, we prove the existence of the second solution v^2 different from v^1.

Lemma 3.2 *Assume that (h_1)–(h_4) hold. Then there exists $\Lambda_4 > 0$ such that if $\lambda \in (0, \Lambda_4)$, the functional $J_\lambda(v)$ satisfies the $(PS)_{c_0}$ condition with $c_0 \le 0$.*

Proof Fix $c_0 \le 0$ and suppose that $\{v_n\} \subset X$ satisfies

$$J_\lambda(v_n) \to c_0, \quad J'_\lambda(v_n) \to 0, \ \text{as } n \to \infty. \tag{3.1}$$

Proceeding as in (2.6), we derive

$$(\frac{1}{N} - \frac{2}{\mu}) \int_\Omega (|\nabla v_n|^N + |f(v_n)|^N)\mathrm{d}x \le \lambda(\frac{1}{q} - \frac{1}{\mu})|g|_\theta \|v_n\|^q + c_0 + o_n(1)$$

and then

$$(\frac{1}{N} - \frac{2}{\mu}) \int_\Omega |\nabla v_n|^N \mathrm{d}x \le \lambda(\frac{1}{q} - \frac{1}{\mu})|g|_\theta \|v_n\|^q + c_0 + o_n(1).$$

On the other hand, using the Poincáre inequality, there exists $C_5 > 0$ such that

$$\begin{aligned}(\frac{1}{N}-\frac{2}{\mu})\int_\Omega |\nabla v_n|^N \mathrm{d}x \geq & \frac{1}{2}(\frac{1}{N}-\frac{2}{\mu})\int_\Omega |\nabla v_n|^N \mathrm{d}x + \frac{1}{2C_5}(\frac{1}{N}-\frac{2}{\mu})\int_\Omega |v_n|^N \mathrm{d}x \\ \geq & \frac{1}{2}(\frac{1}{N}-\frac{2}{\mu})\min\{1,\frac{1}{C_5}\}\|v_n\|^N.\end{aligned}$$

Then for a subsequence, we have

$$[\frac{1}{2}(\frac{1}{N}-\frac{2}{\mu})\min\{1,\frac{1}{C_5}\}\limsup_{n\to\infty}\|v_n\|^{N-q} - \lambda(\frac{1}{q}-\frac{1}{\mu})|g|_\theta]\limsup_{n\to\infty}\|v_n\|^q \leq 0.$$

Thus,

$$\limsup_{n\to\infty}\|v_n\|^N \leq [\frac{\lambda(\frac{1}{q}-\frac{1}{\mu})|g|_\theta}{\frac{1}{2}(\frac{1}{N}-\frac{2}{\mu})\min\{1,\frac{1}{C_5}\}}]^{\frac{N}{N-q}}.$$

Choosing $\Lambda_4 = \frac{q(\mu-2N)\min\{1,\frac{1}{C_5}\}}{2^{\frac{2N-q}{N}}N(\mu-q)|g|_\theta}(\frac{\alpha_N}{\alpha_0})^{\frac{(N-1)(N-q)}{N}}$, for $\lambda \in (0,\Lambda_4)$, we obtain

$$\limsup_{n\to\infty}\|v_n\|^N < \frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Then, by using the same arguments as Lemam 2.6, we prove that J_λ satisfies the $(PS)_{c_0}$ condition for $c_0 \leq 0$. □

Lemma 3.3 *Assume that* (h$_1$)–(h$_4$) *hold. Then there exists* $\Lambda_5 > 0$ *such that if* $\lambda \in (0,\Lambda_5)$, *problem (2.1) admits a weak solution* v^2 *satisfying* $J_\lambda(v^2) < 0$.

Proof Choosing a function $\varphi \in X\backslash\{0\}$ and for $t > 0$ small enough, we infer from (h$_3$), (f$_3$) and (f$_8$) that

$$\begin{aligned}J_\lambda(t\varphi) = & \frac{t^N}{N}\int_\Omega |\nabla\varphi|^N \mathrm{d}x + \frac{1}{N}\int_\Omega |f(t\varphi)|^N \mathrm{d}x - \frac{\lambda}{q}\int_\Omega g(x)|f(t\varphi)|^q \mathrm{d}x - \int_\Omega H(f(t\varphi))\mathrm{d}x \\ \leq & \frac{t^N}{N}\int_\Omega |\nabla\varphi|^N \mathrm{d}x + \frac{1}{N}\int_\Omega |f(t\varphi)|^N \mathrm{d}x - \frac{\lambda}{q}\int_\Omega g(x)|f(t\varphi)|^q \mathrm{d}x \\ \leq & \frac{t^N}{N}\|\varphi\|^N - \frac{\lambda C_1 t^q}{q}\int_\Omega g(x)|\varphi|^q \mathrm{d}x.\end{aligned}$$

Since $N > q$, $J_\lambda(t\varphi) < 0$ for $t > 0$ sufficiently small. Thus

$$c_0 = \inf_{v\in\overline{B}_\rho} J_\lambda(v) < 0 \text{ and } \inf_{v\in\partial B_\rho} J_\lambda(v) > 0, \tag{3.2}$$

where $\rho > 0$ is given by Lemma 2.3 (i) and B_ρ is an open ball in X centered at the origin with radius ρ. Let $\varepsilon_n \to 0$ be such that

$$0 < \varepsilon_n < \inf_{v\in\partial B_\rho} J_\lambda(v) - \inf_{v\in B_\rho} J_\lambda(v). \tag{3.3}$$

By Ekeland's variational principle, there exists $\{v_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq J_\lambda(v_n) \leq c_0 + \varepsilon_n \tag{3.4}$$

and

$$J_\lambda(v_n) < J_\lambda(v) + \varepsilon_n\|v_n - v\|, \quad \forall v \in \overline{B}_\rho,\ v \neq v_n. \tag{3.5}$$

Then it follows from (3.2)–(3.4) that

$$J_\lambda(v_n) \le c_0 + \varepsilon_n \le \inf_{v\in B_\rho} J_\lambda(v) + \varepsilon_n < \inf_{v\in\partial B_\rho} J_\lambda(v),$$

which leads to $\{v_n\} \subset B_\rho$.

Let $\varpi \in B_1$ and consider the sequence $v_n = u_n + t\varpi$ for $t > 0$ small enough. Then we deduce from (3.5) that

$$\frac{1}{t}[J_\lambda(v_n + t\varpi) - J_\lambda(v_n)] \ge -\varepsilon_n\|\varpi\|. \tag{3.6}$$

Passing to the limit as $t \to 0^+$, (3.6) implies that

$$J'_\lambda(v_n)\varpi \ge -\varepsilon_n\|\varpi\|, \quad \forall\, \varpi \in B_1.$$

Replacing ϖ in (3.6) by $-\varpi$, we have

$$J'_\lambda(v_n)\varpi \le \varepsilon_n\|\varpi\|, \quad \forall\, \varpi \in B_1.$$

Then

$$|J'_\lambda(v_n)\varpi| \le \varepsilon_n, \quad \forall\varpi \in B_1$$

and so

$$\|J'_\lambda(v_n)\| \to 0, \text{ as } n \to \infty.$$

Hence, there exists a sequence $\{v_n\} \subset B_\rho$ such that $J_\lambda(v_n) \to c_0 < 0$ and $J'_\lambda(v_n) \to 0$, as $n \to \infty$. Applying Lemma 3.2, which converges strongly to a function $v^2 \in X$. In this case, $J'_\lambda(v^2) = 0$ and $J_\lambda(v^2) = c_0 < 0$, which completes the proof. □

Proof of Theorem 1.1 It follows from Lemmas 3.1, 3.3 and Proposition 2.2 that problem (1.1) has at least two weak solutions $u^1 = f(v^1)$, $u^2 = f(v^2)$. By using a simple argument as [34], we can assume that $u^1 \ge 0$ and $u^2 \ge 0$ in Ω. Furthermore, as a consequence of Harnack's inequality [35], we have $u^1 > 0$ and $u^2 > 0$ in Ω. Thus, we finish the proof. □

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References

[1] S. KURIHURA. *Large-amplitude quasi-solitons in superfluids films.* J. Phys. Soc. Jpn., 1981, **50**: 3262-3267.

[2] A. BOROVSKII, A. GALKIN. *Dynamical modulation of an ultrashort high-intensity laser pulse in matter.* J. Exp. Theor. Phys., 1983, **77**: 562–573.

[3] H. BRANDI, C. MANUS, G. MAINFRAY, et al. *Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma.* Phys. Fluids B, 1993, **5**: 3539–3550.

[4] L. JEANJEAN, K. TANAKA. *A positive solution for a nonlinear Schrödinger equation on* $\mathbb{R}^N$. Indiana Univ. Math. J., 2005, **54**(2): 443–464.

[5] S. TAKENO, S. HOMMA. *Classical planar Heinsenberg ferromagnet, complex scalar fields and nonlinear excitations.* Prog. Theor. Phys., 1981, **65**: 172–189.

[6] V. G. MAKHANKOV, V. K. FEDYANIN. *Nonlinear effects in quasi-one-dimensional models of condensed matter theory.* Phys. Rep., 1984, **104**: 1–86.

[7] M. POPPENBERG, K. SCHMITT, Zhiqiang WANG. *On the existence of soliton solutions to quasilinear Schrödinger equations.* Calc. Var. Partial Differential Equations, 2002, **14**(3): 329–344.

[8] M. F. FURTADO, E. D. SILVA, M. L. SILVA. *Quasilinear elliptic problems under asymptotically linear conditions at infinity and at the origin.* Z. Angew. Math. Phys., 2015, **66**(2): 277–291.

[9] Minbo YANG, Yanheng DING. *Existence of semiclassical states for a quasilinear Schrödinger equation with critical exponent in* $\mathbb{R}^N$. Ann. Mat. Pura Appl. (4), 2013, **192**(5): 783–804.

[10] Xian WU. *Multiple solutions for quasilinear Schrödinger equations with a parameter*. J. Differential Equations, 2014, **256**(7): 2619–2632.

[11] Jian ZHANG, Xianhua TANG, Wen ZHANG. *Infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential*. J. Math. Anal. Appl., 2014, **420**(2): 1762–1775.

[12] Xianxiong YANG, Wenbo WANG, Fukun ZHAO. *Infinitely many radial and non-radial solutions to a quasilinear Schrödinger equation*. Nonlinear Anal., 2015, **114**: 158–168.

[13] Shaoxiong CHEN, Jiu LIU, Xian WU. *Existence and multiplicity of nontrivial solutions for a class of modified nonlinear fourth-order elliptic equations on* $\mathbb{R}^N$. Appl. Math. Comput., 2014, **248**: 593–601.

[14] O. H. MIYAGAKI, S. I. MOREIRA. *Nonnegative solution for quasilinear Schrödinger equations that include supercritical exponents with nonlinearities that are indefinite in sign*. J. Math. Anal. Appl., 2015, **421**(1): 643–655.

[15] Wen ZHANG, Xiangqing LIU. *Infinitely many sign-changing solutions for a quasilinear elliptic equation in* $\mathbb{R}^N$. J. Math. Anal. Appl., 2015, **427**(2): 722-740.

[16] Ke WU. *Positive solutions of quasilinear Schrödinger equations with critical growth*. Appl. Math. Lett., 2015, **45**: 52–57.

[17] G. SANTOS, G. M. FIGUEIREDO, U. B. SEVERO. *Multiple solutions for a class of singular quasilinear problems*. J. Math. Anal. Appl., 2019, **480**(2): 123405, 14 pp.

[18] C. O. ALVES, G. M. FIGUEIREDO. *Multiple solutions for a quasilinear Schrödinger equation on* $\mathbb{R}^N$. Acta. Appl. Math., 2015, **136**: 91–117.

[19] U. B. SEVERO. *Existence of weak solutions for quasilinear elliptic equations involving the p-Laplacian*. Electron. J. Differential Equations, 2008, **56**: 1–16.

[20] M. J. ALVES, P. C. CARRIâO, O. H. MIYAGAKI. *Non-autonomous perturbations for a class of quasilinear elliptic equations on* $\mathbb{R}$. J. Math. Anal. Appl., 2008, **344**(1): 186–203.

[21] Youjun WANG. *Multiple solutions for quasilinear elliptic equations with critical growth*. J. Korean Math. Soc., 2011, **48**(6): 1269–1283.

[22] Jixiu WANG, Li WANG, Dandan ZHANG. *Solutions of semiclassical states for perturbed p-Laplacian equation with critical exponent*. Bound. Value Probl. 2014, 2014: 243, 22 pp.

[23] C. O. ALVES, G. M. FIGUEIREDO, U. B. SEVERO. *Multiplicity of positive solutions for a class of quasilinear problems*. Adv. Differential Equations, 2009, **14**(9-10): 911–942.

[24] Qin LI, Zuodong YANG. *Existence of solutions for a perturbed p-Laplacian system with critical exponent in* $\mathbb{R}^N$. Appl. Anal., 2017, **96**(11): 1885–1905.

[25] Hongxue SONG, Caisheng CHEN, Wei LIU. *Existence and multiplicity of standing wave solutions for a class of quasilinear Schrödinger systems in* $\mathbb{R}^N$. J. Dyn. Control Syst., 2019, **25**(1): 79–94.

[26] N. S. TRUDINGER. *On imbeddings into Orlicz spaces and some applications*. J. Math. Mech., 1967, **17**: 473–483.

[27] J. MOSER. *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J., 1970/1971, **20**: 1077–1092.

[28] Youjun WANG, Jun YANG, Yimin ZHANG. *Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in* $\mathbb{R}^N$. Nonlinear Anal., 2009, **71**(12): 6157–6169.

[29] Caisheng CHEN, Hongxue SONG. *Soliton solutions for quasilinear Schrödinger equation with critical exponential growth in* $\mathbb{R}^N$. Appl. Math., 2016, **61**(3): 317–337.

[30] J. M. B. DOÓ, O. H. MIYAGAKI, S. H. M. SOARES. *Soliton solutions for quasilinear Schrödinger equations: The critical exponential case*. Nonlinear Anal., 2007, **67**(12): 3357–3372..

[31] C. O. ALVES, L. R. DE FREITAS. *Multiplicity of nonradial solutions for a class of quasilinear equations on annulus with exponential critical growth*. Topol. Methods Nonlinear Anal., 2012, **39**(2): 243–262.

[32] A. AMBROSETTI, P. H. RABINOWITZ. *Dual variational methods in critical point theory and applications*. J. Functional Analysis, 1973, **14**: 349–381.

[33] M. STRUWE. *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer-Verlag, New York, 2000.

[34] C. O. ALVES, G. M. FIGUEIREDO. *On multiplicity and concentration of positive solutions for a class of quasilinear problem with critical exponential growth in* $\mathbb{R}^N$. J. Differential Equations, 2009, **246**(3): 1288–1311.

[35] N. TRUDINGER. *On Harnack type inequalities and their applications to quasilinear elliptic equations*. Comm. Pure Appl. Math., 1967, **20**: 721–747.