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Reducing Subspaces for $T_{z_1^{k_1} z_2^{k_2}+\bar{z}_1^{l_1}\bar{z}_2^{l_2}}$ on Weighted Hardy Space over Bidisk

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Abstract In this paper, we characterize the reducing subspaces for Toeplitz operator $T = M_{z^k} + M_{z^l}^*$, where M_{z^k}, M_{z^l} are the multiplication operators on weighted Hardy space $\mathcal{H}_\omega^2(\mathbb{D}^2)$, $k = (k_1, k_2)$, $l = (l_1, l_2)$, $k \neq l$ and k_i, l_i are positive integers for $i = 1, 2$. It is proved that the reducing subspace for T generated by z^m is minimal under proper assumptions on ω. The Bergman space and weighted Dirichlet spaces $\mathcal{D}_\delta(\mathbb{D}^2)$ $(\delta > 0)$ are weighted Hardy spaces which satisfy these assumptions. As an application, we describe the reducing subspaces for $T_{z^k+\bar{z}^l}$ on $\mathcal{D}_\delta(\mathbb{D}^2)$ $(\delta > 0)$, which generalized the results on Bergman space over bidisk.

Keywords reducing subspaces; weighted Dirichlet space; commutant algebra

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1. Introduction

Let $S \in B(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{M}$ is said to be a reducing subspace for S, if $S\mathcal{M} \subseteq \mathcal{M}$ and $S\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. Or equivalently, $\mathcal{M}$ is a reducing subspace for S if and only if $SP_\mathcal{M} = P_\mathcal{M}S$, where $P_\mathcal{M}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. The space $\mathcal{M}$ is called minimal if there is no nonzero reducing subspace $\mathcal{N}$ for S which is contained in $\mathcal{M}$ properly. In addition, the operator S is irreducible if the only reducing subspaces for S are $\{0\}$ and the whole space $\mathcal{H}$.

Stessin and Zhu [1] completely characterized the reducing subspaces for weighted unilateral shift operators of finite multiplicity. Consequently, multiplication operator M_{z^N} (N is a positive integer) on Bergman space and Dirichlet space over disk has exactly 2^N reducing subspaces. For a finite Blaschke product B, a lot of remarkable progress had been made on reducing subspaces for multiplication operator M_B on the Bergman space over unit disk [1–7]. Some of them are generalized to the Dirichlet space [8–10] and the derivative Hardy space [11].

A naturel theme is to consider the similar question over polydisk. If φ is a polynomial, the reducing subspaces for M_φ on the Bergman space and Dirichlet spaces over bidisk are considered, such as $\varphi = z^N w^M, \alpha z^N + \beta w^M$ with $N, M \geq 0, \alpha, \beta \in \mathbb{C}$ (see [12–18]). Guo and Wang [19] generalized some of above results in view of graded structure for a Hilbert module. Recently,

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Guo and Huang [20] gave a survey on recent developments concerning commutants, reducing subspaces and von Neumann algebras associated with multiplication operators that are defined on both Hardy space and Bergman spaces over bounded domains in $\mathbb{C}^d$.

Since M_{z^N}, M_{w^M} are operator-weighted shifts on weighted Hardy space, Gu [21, 22] characterized the reducing subspaces and common reducing subspaces of operator-weighted shifts, and provided uniform proofs of some results from [12, 13]. In the case that φ is a non-analytic function, the reducing subspaces for $T_{z^k\overline{w}^l}$ and $T_{z^N+\overline{w}^M}$ on Bergman space over bidisk are characterized [23, 24]. Under proper assumptions about the weight coefficients ω, these results can also be generalized to operator-weighted shifts on weighted Hardy space [25, 26]. For $\varphi(z,w) = z^{k_1}w^{k_2} + \overline{z}^{l_1}\overline{w}^{l_2}$, Deng et al. [27] obtained a uniform characterization of the reducing subspaces for T_φ on Bergman space over the bidisk, including the known cases that $\varphi = z^N w^M$ and $\varphi = z^N + \overline{w}^M$. In this paper, we mainly consider the reducing subspaces for T_φ on weighted Hardy space $\mathcal{H}^2_\omega(\mathbb{D}^2)$, where $\mathcal{H}^2_\omega(\mathbb{D}^2)$ is defined by

$$\mathcal{H}^2_\omega(\mathbb{D}^2) = \Big\{f(z) = \sum_{n\in\mathbb{Z}^2_+} f_n z^n : f_n \in \mathbb{C}, \|f\|^2 = \sum_{n\in\mathbb{Z}^2_+} \omega_n |f_n|^2 < \infty\Big\},$$

$\omega_n = \omega_{n_1}\omega_{n_2}$, $\forall n = (n_1, n_2) \in \mathbb{Z}^2_+$, and $\omega = \{\omega_j, j \geq 0\}$ is a sequence of positive numbers such that

$$\liminf_{j\to+\infty} (\sqrt{\omega_j})^{1/j} \geq 1.$$

More details can be seen in [25]. Throughout this paper, let $k = (k_1, k_2)$, $l = (l_1, l_2)$ where $k \neq l$ and k_i, l_i are positive integers for $i = 1, 2$. By computation, we get $\{z^n\}_{n=1}^\infty$ are the eigenvectors of $T^*_\varphi T_\varphi - T_\varphi T^*_\varphi$. Set

$$(T^*_\varphi T_\varphi - T_\varphi T^*_\varphi) z^n = \lambda_n z^n \text{ and } Q_n(p) = \lambda_{n+p(k+l)}, \quad \forall p \in \mathbb{N}.$$

Denote $Q_n(p) \equiv 0$ if $Q_n(p) = 0$, $\forall p \in \mathbb{N}$. Suppose that

(P1) $\lim_{p\to+\infty} \frac{\omega_{m+p(k+l)}}{\omega_{n+p(k+l)}} = 1$.

(P2) If there exists $\{p_j\} \subseteq \mathbb{N}$ such that $\lim_{j\to+\infty} p_j = +\infty$ and $Q_n(p_j) = 0$, then $Q_n(p) \equiv 0$.

(P3) If $Q_n(p) \equiv 0$, then $Q_{n+l}(p) \not\equiv 0, Q_{n+k}(p) \not\equiv 0$.

(P4) If $Q_n(p) \equiv 0$, then

$$\lim_{p\to+\infty} p\Big(\frac{\omega_{n+(p+1)(k+l)}\omega_{n+p(k+l)}}{\omega^2_{n+p(k+l)+l}} - 1\Big) = 0 \text{ or } \lim_{p\to+\infty} p\Big(\frac{\omega_{n+(p+1)(k+l)}\omega_{n+p(k+l)}}{\omega^2_{n+p(k+l)+k}} - 1\Big) = 0.$$

(P5) Let $n \in \Omega_1$, $m \in \Omega_4$. If $Q_n(p) \not\equiv 0$ and $\lambda_n = \lambda_m$, then $Q_m(p) \not\equiv 0$.

(P6) If $n \neq m$ and $Q_n(p) \equiv Q_m(p)$, then the following statements hold:

(i) If $Q_{n+l}(p) \equiv Q_{m+l}(p)$, then $Q_{n+l}(p) \not\equiv 0$, $Q_n(p) \not\equiv 0$;

(ii) If $Q_{n+k}(p) \equiv Q_{m+k}(p)$, then $Q_{n+k}(p) \not\equiv 0$, $Q_n(p) \not\equiv 0$.

(P7) Let $m \in \Delta$ and $n \neq m$. If $\omega_{m+k} = \omega_{n+k}$, $\omega_{m+h(k+l)} = \omega_{n+h(k+l)}$ for $h \in \mathbb{Z}_+$, then $z^n \notin L_m$, where

$$\Delta = \begin{cases} \{(m_1, m_2) \in \mathbb{Z}^2_+ : m_1 \in [0, s_1), m_2 \in [0, \frac{|l_1k_2 - l_2k_1|}{s_1})\}, & k_1l_2 \neq k_2l_1 \\ \{(m_1, m_2) \in \mathbb{Z}^2_+ : m_1 \in [0, s_1) \text{ or } m_2 \in [0, s_2)\}, & k_1l_2 = k_2l_1 \end{cases},$$

$s_i=\gcd\{k_i,l_i\}$, $i=1,2$, and $L_m=\overline{\operatorname{span}}\{z^{m+uk+vl}: m+uk+vl\in\mathbb{Z}_+^2, u,v\in\mathbb{Z}\}$.

Let $[z^m]$ be the reducing subspace for $T_{z^k+\bar z^l}$ on $\mathcal{H}_\omega^2(\mathbb{D}^2)$ generated by z^m. We characterize $[z^m]$ as follows:

Theorem 1.1 *Suppose ω satisfies (P1)–(P7). Let $\varphi=z^{k_1}\omega^{k_2}+\overline{z}^{l_1}\overline{\omega}^{l_2}$, k_i,l_i are positive integers for $i=1,2$ such that $(k_1,k_2)\neq(l_1,l_2)$. For each $m\in\Delta$, $L_m=[z^m]$ is a minimal reducing subspace for T_φ on $\mathcal{H}_\omega^2(\mathbb{D}^2)$.*

In fact, Bergman space over the bidisk is a weighted Hardy space satisfying assumptions (P1)–(P7). So we also get in [27, Theorem 3.3] when k_i,l_i are positive integers. Furthermore, we generalize some results in [27] to the weighted Dirichlet space $\mathcal{D}_\delta(\mathbb{D}^2)$ $(\delta>0)$ over bidisk. For every $\delta>0$, we show that Dirichlet space $\mathcal{D}_\delta(\mathbb{D}^2)$ is a weighted Hardy space which satisfies the assumptions (P1)–(P7), and then we characterize the reducing subspaces for T_φ on $\mathcal{D}_\delta(\mathbb{D}^2)$ and the commutant algebra of $\{T_\varphi,T_\varphi^*\}$ as follows.

Theorem 1.2 *Let $\varphi=z^{k_1}\omega^{k_2}+\overline{z}^{l_1}\overline{\omega}^{l_2}$, where k_i,l_i are positive integers for $i=1,2$ such that $(k_1,k_2)\neq(l_1,l_2)$. If $\mathcal{M}$ is a reducing subspace for T_φ on $\mathcal{D}_\delta(\mathbb{D}^2)$ $(\delta>0)$, then $\mathcal{M}$ is the orthogonal sum of some minimal reducing subspaces. Moreover, $\mathcal{M}$ is a minimal reducing subspace for T_φ if and only if $\mathcal{M}$ has the form as follows:*

(i) If $l_1k_2\neq k_1l_2$, then $\mathcal{M}=L_m$ for some $m\in\Delta$;

(ii) If $l_1k_2=k_1l_2$, then there exist $m\in\Delta$ and $a,b\in\mathbb{C}$ such that $\mathcal{M}=\mathcal{M}_{ab}$ where $\mathcal{M}_{ab}$ is defined by

$$\mathcal{M}_{ab}=\overline{\operatorname{span}}\{(az^m+bz^{m'})z^{uk+vl}: u,v\in\mathbb{Z}, uk+vl+m\succeq 0\},$$

with $m'=(\frac{l_1}{l_2}(m_2+1)-1,\frac{l_2}{l_1}(m_1+1)-1)$. In particular, if $m'\notin\mathbb{Z}_+^2$, then $b=0$.

Theorem 1.3 *Let $\varphi=z^{k_1}\omega^{k_2}+\overline{z}^{l_1}\overline{\omega}^{l_2}$, where k_i,l_i are positive integers for $i=1,2$ such that $(k_1,k_2)\neq(l_1,l_2)$. Then $\mathcal{V}^*(\varphi)$ is a Type I von Neumann algebra. Furthermore, the following statements hold:*

(i) If $k_1l_2\neq k_2l_1$, then $\mathcal{V}^*(\varphi)$ is abelian and is $*$-isomorphic to $\bigoplus_{i=1}^{j}\mathbb{C}$, where $j=|l_1k_2-l_2k_1|$.

(ii) If $k_1l_2=k_2l_1$ and $s=(s_1,s_2)$ with $s_i=\gcd\{k_i,l_i\}$ $(i=1,2)$, then $\mathcal{V}^*(\varphi)=\mathcal{V}^*(z^s)$ and $\mathcal{V}^*(\varphi)$ is never abelian. Moreover, if $s_1=s_2=r$, then $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to

$$\bigoplus_{j=1}^{\infty}M_2(\mathbb{C})\oplus\bigoplus_{i=1}^{r}\mathbb{C};$$

if $s_1\neq s_2$, then $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to the direct sum of countably many $M_2(\mathbb{C})\oplus\mathbb{C}$.

This paper is organized as follows: in Section 2, we give some useful lemmas; in Section 3, we show the proof of Theorem 1.1; in Section 4, we introduce the proof of Theorems 1.2 and 1.3.

2. Preliminaries

Firstly, we follow some notations. More details can be seen in [27] and their references. Denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the set of all positive integers and all nonnegative integers, respectively.

The Toeplitz operator T_φ with non-analytic symbol $\varphi = z^k + \bar{z}^l$ is defined as follows:

$$T_\varphi = T_{z^k+\bar{z}^l} = M_{z^k} + M_{z^l}^*,$$

where $k, l \in \mathbb{N}^2$ and $M_{z^l}^*$ is the adjoint of multiplication operator M_{z^l} on $\mathcal{H}_\omega^2(\mathbb{D}^2)$.

For $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{Z}_+^2$, denote by $a \succeq b$, if $a_1 \geq b_1$ and $a_2 \geq b_2$. Otherwise, denote by $a \not\succeq b$.

By computation,

$$T_\varphi z^n = \begin{cases} z^{n+k}, & n \not\succeq l \\ z^{n+k} + \frac{\omega_n}{\omega_{n-l}} z^{n-l}, & n \succeq l \end{cases}; \ T_\varphi^* z^n = \begin{cases} z^{n+l}, & n \not\succeq k \\ z^{n+l} + \frac{\omega_n}{\omega_{n-k}} z^{n-k}, & n \succeq k \end{cases}.$$

More specifically, let

$$\Omega_1 = \{n \in \mathbb{Z}_+^2 : \ n \not\succeq k, \ n \not\succeq l\}, \quad \Omega_2 = \{n \in \mathbb{Z}_+^2 : \ n \succeq k, \ n \not\succeq l\},$$
$$\Omega_3 = \{n \in \mathbb{Z}_+^2 : \ n \not\succeq k, \ n \succeq l\}, \quad \Omega_4 = \{n \in \mathbb{Z}_+^2 : \ n \succeq k, \ n \succeq l\}.$$

For $n \in \mathbb{Z}_+^2$, $m \in \mathbb{N}^2$, set

$$r(n,m) = \frac{\omega_{n+m}}{\omega_n}, \ \nabla r(n,m) = \frac{\omega_{n+m}}{\omega_n} - \frac{\omega_n}{\omega_{n-m}}, \quad n \succeq m.$$

Denote by $T = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*$, then

$$Tz^n = \lambda_n z^n,$$

where

$$\lambda_n = \begin{cases} r(n,k) - r(n,l), & n \in \Omega_1 \\ \nabla r(n,k) - r(n,l), & n \in \Omega_2 \\ r(n,k) - \nabla r(n,l), & n \in \Omega_3 \\ \nabla r(n,k) - \nabla r(n,l), & n \in \Omega_4 \end{cases}.$$

Let

$$Q_n(p) = \lambda_{n+p(k+l)}, \quad \forall p \in \mathbb{N}.$$

Let $\mathcal{V}^*(\varphi)$ be the commutant algebra of the von Neumann algebra generated by $\{I, T_\varphi, T_\varphi^*\}$. Set $A \in \mathcal{V}^*(\varphi)$. Because $\lambda_\beta \in \mathbb{R}$ and $\lambda_\alpha \langle Az^\alpha, z^\beta \rangle = \langle ATz^\alpha, z^\beta \rangle = \langle TAz^\alpha, z^\beta \rangle = \langle Az^\alpha, Tz^\beta \rangle = \lambda_\beta \langle Az^\alpha, z^\beta \rangle$, we can prove that

$$Az^\alpha = \sum_{\lambda_\beta = \lambda_\alpha} c_\beta z^\beta, \quad \forall \alpha \in \mathbb{Z}_+^2. \tag{2.1}$$

Throughout this paper, let $k = (k_1, k_2)$, $l = (l_1, l_2) \in \mathbb{N}^2$ with $k \neq l$. For $\alpha, \beta \in \mathbb{Z}_+^2$, let

$$\Delta_{\alpha,\beta} = \{p \in \mathbb{Z} : \langle Az^\alpha, z^{\beta+p(k+l)} \rangle \neq 0\},$$
$$H_\beta^0 = \overline{\operatorname{span}}\{z^m : m \neq \beta + p(k+l), \ p \in \mathbb{Z}, m \in \mathbb{Z}_+^2\}.$$

In the following, we provide several lemmas about $\Delta_{\alpha,\beta}$ under the assumptions (P1)–(P6). Given $\alpha \in \Omega_1$, we obtain that if $Q_\alpha(p) \equiv 0$, then $Az^\alpha = cz^\alpha$ for some $c \in \mathbb{C}$ (see Lemma 2.3); if $Q_\alpha(p) \not\equiv 0$, then $Az^\alpha = \sum_{\beta \in \Omega_1} c_\beta z^\beta$ for some $c_\beta \in \mathbb{C}$ (see Lemma 2.5).

Lemma 2.1 *Let $A \in \mathcal{V}^*(\varphi)$. If $\alpha \in \Omega_1$, $\beta \not\succeq k + l$ and $Q_\alpha(p) \equiv 0$, then $\Delta_{\alpha,\beta}$ is a finite set.*

Proof Suppose $\Delta_{\alpha,\beta}$ is infinite. There exist $\{p_j : j \in \mathbb{N}\} \subseteq \Delta_{\alpha,\beta}$ such that $p_j \to +\infty$ as $j \to +\infty$. Thus, $\lambda_\alpha = \lambda_{\beta+p_j(k+l)}$, $\forall j \in \mathbb{N}$. By (P1), we get $\lambda_\alpha = Q_\beta(p_j) \to 0$ as $j \to +\infty$. i.e., $Q_\beta(p_j) = \lambda_\alpha = 0$, $\forall j \in \mathbb{N}$. So (P2) shows that $Q_\beta(p) \equiv 0$. It means $Q_{\beta+l}(p) \not\equiv 0$ by (P3). Replacing α, β by $\alpha + l$, $\beta + l$, respectively, we can prove that $\Delta_{\alpha+l,\beta+l}$ is finite as above. Set

$$Az^\alpha = \sum_{p\in\mathbb{Z}} c_p z^{\beta+p(k+l)} + q(z),$$

where $c_p \in \mathbb{C}$, $q(z) \in H^0_\beta$. By (P4), we will get contradictions in the following two cases.

Case 1. $\lim_{p\to+\infty} p(\frac{r(\beta+p(k+l)+l,k)}{r(\beta+p(k+l),l)} - 1) = 0$. For $\alpha \not\succeq k$, by $AT^*_\varphi = T^*_\varphi A$, we get

$$Az^{\alpha+l} = cz^{\beta-k} + \sum_{p\in\mathbb{Z}} (c_p + c_{p+1} \frac{\omega_{\beta+(p+1)(k+l)}}{\omega_{\beta-k+(p+1)(k+l)}}) z^{\beta+l+p(k+l)} + T^*_\varphi q(z),$$

where $c = 0$ if $\beta \in \Omega_1 \cup \Omega_3$; $c = c_0 \frac{\omega_\beta}{\omega_{\beta-k}}$ if $\beta \in \Omega_2 \cup \Omega_4$, and $T^*_\varphi q(z) \in H^0_{\beta+l}$. Since $\Delta_{\alpha,\beta}$ is infinite and $\Delta_{\alpha+l,\beta+l}$ is finite, equality (2.1) shows that there is $N \in \mathbb{Z}_+$ such that $c_N \neq 0$ and

$$c_p + c_{p+1} \frac{\omega_{\beta+(p+1)(k+l)}}{\omega_{\beta-k+(p+1)(k+l)}} = 0, \quad p \geq N.$$

That is,

$$|c_{p+1}| = |c_p| \frac{\omega_{\beta-k+(p+1)(k+l)}}{\omega_{\beta+(p+1)(k+l)}}, \quad p \geq N.$$

So $c_p \neq 0$ for $p \geq N$ and that

$$\begin{aligned}
\lim_{p\to+\infty} p(\frac{|c_p|^2 \omega_{\beta+p(k+l)}}{|c_{p+1}|^2 \omega_{\beta+(p+1)(k+l)}} - 1) &= \lim_{p\to+\infty} p(\frac{\omega_{\beta+(p+1)(k+l)} \omega_{\beta+p(k+l)}}{\omega^2_{\beta+(p+1)(k+l)-k}} - 1) \\
&= \lim_{p\to+\infty} p(\frac{\omega_{\beta+p(k+l)}}{\omega_{\beta+p(k+l)+l}} \frac{\omega_{\beta+p(k+l)+l+k}}{\omega_{\beta+p(k+l)+l}} - 1) \\
&= \lim_{p\to+\infty} p(\frac{r(\beta+p(k+l)+l,k)}{r(\beta+p(k+l),l)} - 1) = 0.
\end{aligned}$$

By Raabe's convergence test, $\sum_{p\in\mathbb{Z}} |c_p|^2 \omega_{\beta+p(k+l)}$ is divergent, which contradicts $Az^\alpha \in \mathcal{H}^2_\omega(\mathbb{D}^2)$. Hence, $\Delta_{\alpha,\beta}$ is a finite set.

Case 2. $\lim_{p\to+\infty} p(\frac{r(\beta+p(k+l)+k,l)}{r(\beta+p(k+l),k)} - 1) = 0$. For $\alpha \not\succeq l$, by $AT_\varphi = T_\varphi A$ and Raabe's convergence test, we can also get the contradictions. So we complete the proof. □

Lemma 2.2 *Given $\alpha \not\succeq k + l$ and $A \in \mathcal{V}^*(\varphi)$. If $\Delta_{\alpha,\beta}$ is a nonempty and finite set, then* $\max\{p \in \mathbb{Z} : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + h$ *where* $p_0 = \max \Delta_{\alpha,\beta}$ *and* $h \in \mathbb{Z}_+$.

Proof If $h = 0$, it is obviously true by the definition of p_0. For every $N \in \mathbb{Z}_+$, suppose it is true when $h \leq N$. We will prove that it is also true when $h = N + 1$.

By inductive hypothesis, set $Az^{\alpha+N(k+l)} = c_N z^{\beta+(p_0+N)(k+l)} + p_N(z) + h_N(z)$, where $c_N \neq 0$, $p_N \in \overline{\text{span}}\{z^{\beta+p(k+l)} : p < p_0 + N, \beta + p(k + l) \succeq 0\}$ and $h_N \in H^0_\beta$. So $AT^*_\varphi T_\varphi = T^*_\varphi T_\varphi A$ implies that

$$\begin{aligned}
&A(z^{\alpha+(N+1)(k+l)} + \rho z^{\alpha+N(k+l)} + \eta z^{\alpha+(N-1)(k+l)}) \\
&\quad = c_N z^{\beta+(p_0+N+1)(k+l)} + P_N(z) + H_N(z),
\end{aligned} \tag{2.2}$$

where $P_N \in \overline{\mathrm{span}}\{z^{\beta+p(k+l)} : p < p_0+N+1, \beta+p(k+l) \succeq 0\}$, $H_N \in H^0_\beta$, and $\rho, \eta \in \mathbb{R}$. In particular, there is no item $\eta z^{\alpha+(N-1)(k+l)}$ when $N = 0$. Since $\max\{p \in \mathbb{Z} : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + h$ for $h = N$, $N-1$, we get

$$A(\rho z^{\alpha+N(k+l)} + \eta z^{\alpha+(N-1)(k+l)}) \perp z^{\beta+(p_0+N+1)(k+l)}.$$

Thus equality (2.2) shows that $\max\{p \in \mathbb{Z} : \langle Az^{\alpha+(N+1)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + N + 1$. $\square$

Lemma 2.3 *Let $A \in \mathcal{V}^*(\varphi)$. If $\alpha \in \Omega_1$ such that $Q_\alpha(p) \equiv 0$, then $Az^\alpha = cz^\alpha$ for some $c \in \mathbb{C}$.*

Proof If there exists $\beta \not\succeq k+l$ such that $\Delta_{\alpha,\beta}$ is not empty, Lemma 2.1 shows that $\Delta_{\alpha,\beta}$ is a finite set. Let $p_0 = \max \Delta_{\alpha,\beta} \geq 0$. On the one hand, Lemma 2.2 shows that $\lambda_{\alpha+p(k+l)} = \lambda_{\beta+(p_0+p)(k+l)}$ for every $p \in \mathbb{Z}_+$. That is,

$$Q_\alpha(p) \equiv Q_{\beta+p_0(k+l)}(p). \tag{2.3}$$

On the other hand, as in Lemma 2.2, set

$$Az^\alpha = c_{p_0} z^{\beta+p_0(k+l)} + g_{p_0}(z) + h_{p_0}(z),$$

where $c_{p_0} \neq 0$ and $g_{p_0} \in \overline{\mathrm{span}}\{z^{\beta+p(k+l)} : 0 \leq p < p_0\}$ and $h_{p_0} \in H^0_\beta$. By $AT^*_\varphi = T^*_\varphi A$, we get

$$Az^{\alpha+l} = c_{p_0} z^{\beta+l+p_0(k+l)} + cz^{\beta+l+(p_0-1)(k+l)} + G_{p_0}(z) + H_{p_0}(z),$$

where $c = c_{p_0}\frac{\omega_{\beta+p_0(k+l)}}{\omega_{\beta-k+p_0(k+l)}}$, $G_{p_0} \in \overline{\mathrm{span}}\{z^{\beta+p(k+l)} : 0 \leq p < p_0 - 1\}$ and $H_{p_0} \in H^0_\beta$. So

$$\max\{p \in \mathbb{Z} : \langle Az^{\alpha+l}, z^{\beta+l+p(k+l)}\rangle \neq 0\} = p_0.$$

It shows that $\Delta_{\alpha+l,\beta+l}$ is finite. It is easy to see $\alpha + l \not\succeq k + l$ since $\alpha \in \Omega_1$. Using Lemma 2.2 again, we have $\lambda_{\alpha+l+p(k+l)} = \lambda_{\beta+l+(p_0+p)(k+l)}$ for every $p \in \mathbb{Z}_+$. That is,

$$Q_{\alpha+l}(p) \equiv Q_{\beta+l+p_0(k+l)}(p). \tag{2.4}$$

By equalities (2.3), (2.4) and assumption $Q_\alpha(p) \equiv 0$, property (P6) implies that $\alpha = \beta+p_0(k+l) \in \Omega_1$. So $p_0 = 0$ and $\alpha = \beta$, which deduces that $Az^\alpha = cz^\alpha$ for some $c \in \mathbb{C}$. $\square$

Lemma 2.4 *Let $\alpha, \beta \in \mathbb{Z}^2_+$, $\alpha \not\succeq k+l$, and $A \in \mathcal{V}^*(\varphi)$. If $Q_\alpha(p) \not\equiv 0$ and $\Delta_{\alpha,\beta}$ is a nonempty and finite set, then the following two statements hold:*

(i) There is only one element in $\Delta_{\alpha,\beta}$;

(ii) $\min\{p \in \mathbb{Z} : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + h$, where $h \in \mathbb{Z}_+$ and $\{p_0\} = \Delta_{\alpha,\beta}$.

Proof Let $\widetilde{\beta} = \beta + p_1(k+l)$ where $p_1 \in \mathbb{Z}$ such that $\widetilde{\beta} \succeq 0$ and $\widetilde{\beta} \not\succeq k+l$. Then p_0 satisfies the statements for β if and only if $p_0 + p_1$ satisfies the statements for $\widetilde{\beta}$. Therefore, without loss of generality, we assume $\beta \not\succeq k+l$.

Since $Q_\alpha(p) \not\equiv 0$, equality (2.1), properties (P1) and (P2) imply that the set

$$\{h \in \mathbb{Z}_+ : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} \subseteq \{h \in \mathbb{Z}_+ : Q_\alpha(h) = \lambda_{\beta+p(k+l)}\}$$

is a finite set for every $p \in \mathbb{Z}_+$. Let $p_0 = \max \Delta_{\alpha,\beta}$, then

$$E_{p_0} = \bigcup_{0 \leq p \leq p_0} \{h \in \mathbb{Z}_+ : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\}$$

is also finite. Obviously, $0 \in E_{p_0}$. Let $h_0 = \max E_{p_0}$.

Claim. for every $h \in \mathbb{Z}_+$ the following equalities hold:

$$\min\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+(h_0+h+1)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + h + 1, \tag{2.5}$$

$$\langle Az^{\alpha+(h_0+h+q)(k+l)}, z^{\beta+(p_0+h)(k+l)}\rangle = 0, \quad \forall q \in \mathbb{N}. \tag{2.6}$$

If $h = 0$, it is easy to see that (2.6) holds by the definition of h_0. Since $h_0 + 1 \notin E_{p_0}$, set

$$Az^{\alpha+(h_0+1)(k+l)} = d_1 z^{\beta+(p_0+1)(k+l)} + f_1(z) + g_1(z), \tag{2.7}$$

where $d_1 \in \mathbb{C}$, $f_1 \in \overline{\text{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + 2\}$ and $g_1 \in H^0_\beta$. By $AT^*_\varphi T_\varphi = T^*_\varphi T_\varphi A$, we have

$$A(z^{\alpha+(h_0+2)(k+l)} + \eta z^{\alpha+(h_0+1)(k+l)} + \rho z^{\alpha+h_0(k+l)}) = d_1 \frac{\omega_{\beta+(p_0+1)(k+l)}}{\omega_{\beta+p_0(k+l)}} z^{\beta+p_0(k+l)} + F_1(z) + G_1(z),$$

where η, $\rho > 0$, $F_1 \in \overline{\text{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + 1\}$ and $G_1 \in H^0_\beta$. Since $h_0 + 1, h_0 + 2 \notin E_{p_0}$, there is

$$\rho Az^{\alpha+h_0(k+l)} = d_1 \frac{\omega_{\beta+(p_0+1)(k+l)}}{\omega_{\beta+p_0(k+l)}} z^{\beta+p_0(k+l)} + \widetilde{F}_1(z) + \widetilde{G}_1(z), \tag{2.8}$$

where $\widetilde{F}_1 \in \overline{\text{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + 1\}$ and $\widetilde{G}_1 \in H^0_\beta$. By the definition of h_0, there exists some $p \in [0, p_0]$ such that $\langle Az^{\alpha+h_0(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0$. Together with the fact that

$$(\widetilde{F}_1 + \widetilde{G}_1) \perp z^{\beta+p(k+l)}, \quad 0 \leq p \leq p_0,$$

we get $d_1 \neq 0$. So equality (2.7) shows that equality (2.5) holds for $h = 0$. Moreover, (2.8) implies that

$$\min\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+h_0(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0. \tag{2.9}$$

That is, Claim holds when $h = 0$.

Given $N \in \mathbb{Z}_+$. For $h \leq N$, suppose (2.5) and (2.6) hold. Therefore,

$$Az^{\alpha+(h_0+N+1+q)(k+l)} = Az^{\alpha+(h_0+N-j+1+j+q)(k+l)} \perp z^{\beta+(p_0+N-j)(k+l)}, \quad 0 \leq j \leq N.$$

According to $h_0 + 1 + N + q \notin E_{p_0}$, we have $Az^{\alpha+(h_0+1+N+q)(k+l)} \perp z^{\beta+p(k+l)}$ for $0 \leq p \leq p_0$. Thus we can set

$$Az^{\alpha+(h_0+1+N+q)(k+l)} = d_{1+N+q} z^{\beta+(p_0+N+1)(k+l)} + f_{1+N+q}(z) + g_{1+N+q}(z),$$

where $d_{1+N+q} \in \mathbb{C}$, $f_{1+N+q} \in \overline{\text{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + N + 2\}$ and $g_{1+N+q} \in H^0_\beta$. By $AT^*_\varphi T_\varphi = T^*_\varphi T_\varphi A$, it is easy to see that

$$\begin{aligned} &A(z^{\alpha+(h_0+N+2+q)(k+l)} + \eta' z^{\alpha+(h_0+1+N+q)(k+l)} + \rho' z^{\alpha+(h_0+N+q)(k+l)}) \\ &= d_{1+N+q} \frac{\omega_{\beta+(p_0+N+1)(k+l)}}{\omega_{\beta+(p_0+N)(k+l)}} z^{\beta+(p_0+N)(k+l)} + F_{1+N+q}(z) + G_{1+N+q}(z), \end{aligned}$$

where $\eta', \rho' > 0$, $F_{1+N+q}(z) \in \overline{\text{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + N + 1\}$ and $G_{1+N+q}(z) \in H^0_\beta$. Equality (2.6) with $h = N$ shows that $d_{1+N+q} = 0$ for $q \in \mathbb{N}$. It means that (2.6) holds when $h = N + 1$.

By (2.6) with $q = 1$, set

$$Az^{\alpha+(h_0+N+2)(k+l)} = dz^{\beta+(p_0+N+2)(k+l)} + f(z) + g(z), \tag{2.10}$$

where $d \in \mathbb{C}$, $f \in \overline{\operatorname{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + N + 3\}$ and $g \in H_\beta^0$. Then $AT_\varphi^* T_\varphi = T_\varphi^* T_\varphi A$ implies

$$\begin{aligned}&A(z^{\alpha+(h_0+N+3)(k+l)} + \eta'' z^{\alpha+(h_0+N+2)(k+l)} + \rho'' z^{\alpha+(h_0+N+1)(k+l)})\\ &= d\frac{\omega_{\beta+(p_0+N+2)(k+l)}}{\omega_{\beta+(p_0+N+1)(k+l)}} z^{\beta+(p_0+N+1)(k+l)} + F(z) + G(z),\end{aligned}$$

where $F \in \overline{\operatorname{span}}\{z^{\beta+h(k+l)} : h \geq p_0 + N + 2\}$ and $G \in H_\beta^0$. By equality (2.5) with $h = N$, we have $d \neq 0$. Equality (2.10) shows that the equality (2.5) holds for $h = N + 1$. So we finish the proof of Claim.

The equality (2.5) and (2.9) imply $\min\{p \in \mathbb{Z} : \langle Az^{\alpha+(h_0+h)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + h$. i.e., $\lambda_{\alpha+(h_0+h)(k+l)} = \lambda_{\beta+(p_0+h)(k+l)}$. By Lemma 2.2, $p_0 = \max \Delta_{\alpha,\beta}$ shows that $\lambda_{\alpha+h(k+l)} = \lambda_{\beta+(p_0+h)(k+l)}$. Therefore,

$$\lambda_{\alpha+h(k+l)} = \lambda_{\alpha+(h+h_0)(k+l)}, \quad \forall h \in \mathbb{Z}_+.$$

If $h_0 \geq 1$, then $\lambda_{\alpha+h_0(k+l)} = \lambda_{\alpha+nh_0(k+l)} = Q_\alpha(nh_0) = \lim\limits_{n\to+\infty} Q_\alpha(nh_0) = 0$. By (P2) again, we get $Q_\alpha(p) \equiv 0$, which contradicts the assumption. So $h_0 = 0$. The equality (2.9) implies that $p_0 = \min \Delta_{\alpha,\beta}$. So we complete the proof. □

Lemma 2.5 *Let $A \in \mathcal{V}^*(\varphi)$. If $\alpha \in \Omega_1$ such that $Q_\alpha(p) \not\equiv 0$, then $\langle Az^\alpha, z^\beta\rangle = 0$, for every $\beta \in \Omega_2 \cup \Omega_3 \cup \Omega_4$.*

Proof Suppose $\langle Az^\alpha, z^\beta\rangle \neq 0$ for some $\beta \in \Omega_2 \cup \Omega_3 \cup \Omega_4$. Then $0 \in \Delta_{\alpha,\beta}$. Firstly, we show that $\Delta_{\alpha,\beta} = \{0\}$. Otherwise, set $p_0 \in \Delta_{\alpha,\beta}$, then $\lambda_{\beta+p_0(k+l)} = \lambda_\alpha$. If $p_0 \geq 1$, since $Q_\alpha(p) \not\equiv 0$ and $\beta + p_0(k+l) \in \Omega_4$, (P5) shows that $Q_{\beta+p_0(k+l)}(p) \not\equiv 0$. Note that $Q_\beta(p) = Q_{\beta+p_0(k+l)}(p - p_0)$. That is $Q_\beta(p) \not\equiv 0$. By (P1) and (P2), we get $\Delta_{\alpha,\beta} \subseteq \{p \in \mathbb{Z}_+ : Q_\beta(p) = \lambda_\alpha\}$ is finite. Lemma 2.4 implies that there is only one element in $\Delta_{\alpha,\beta}$, which contradicts to $\{0, p_0\} \subseteq \Delta_{\alpha,\beta}$. If $p_0 < 0$, let $\beta_1 = \beta + p_0(k+l) \succeq 0$. As above, we can prove $Q_{\beta_1}(p) \not\equiv 0$ and there is only one element in Δ_{α,β_1}, which contradict to $\{0, -p_0\} \subseteq \Delta_{\alpha,\beta_1}$.

By $\Delta_{\alpha,\beta} = \{0\}$, Lemma 2.2 implies that $Q_\alpha(p) \equiv Q_\beta(p)$. Moreover,

$$Az^\alpha = c_\beta z^\beta + h(z),$$

where $c_\beta \neq 0, h \in H_\beta^0$.

Next, we will get contradictions in two cases respectively.

(i) $\beta \in \Omega_2 \cup \Omega_4$. By $AT_\varphi^* = T_\varphi^* A$, we get

$$Az^{a+l} = c_\beta z^{\beta+l} + c_\beta \frac{\omega_\beta}{\omega_{\beta-k}} z^{\beta-k} + G(z),$$

where $G \in H_\beta^0$. So $\Delta_{\alpha+l,\beta-k} = \{p \in \mathbb{Z} : \langle Az^{\alpha+l}, z^{\beta-k+p(k+l)}\rangle \neq 0\} = \{0, 1\}$ is finite. That is $1 = \max \Delta_{\alpha+l,\beta-k}$. Lemma 2.2 implies that $\lambda_{\alpha+l+h(k+l)} = \lambda_{\beta+l+h(k+l)}$. So $Q_{\alpha+l}(p) \equiv Q_{\beta+l}(p)$. Together with $Q_\alpha(p) \equiv Q_\beta(p)$ and (P6), we get $Q_{\alpha+l}(p) \not\equiv 0$. Then Lemma 2.4 leads to that there is only one element in $\Delta_{\alpha+l,\beta-k}$. This is a contradiction.

(ii) $\beta \in \Omega_3$. Substituting T_φ^* with T_φ, we get

$$Az^{\alpha+k} = c_\beta z^{\beta+k} + c_\beta \frac{\omega_\beta}{\omega_{\beta-l}} z^{\beta-l} + F(z),$$

where $F \in H^0_\beta$. As in (i), we can prove that $\Delta_{\alpha+k,\beta-l} = \{p \in \mathbb{Z} : \langle Az^{\alpha+k}, z^{\beta-l+p(k+l)}\rangle \neq 0\} = \{0, 1\}$, which contradicts to the fact that there is only one element in $\Delta_{\alpha+k,\beta-l}$. □

3. Reducing subspaces for $T_{z^k+\overline{z}^l}$ on weighted Hardy space

In this section, we mainly consider the reducing subspaces for T_φ with symbol $\varphi = z^k + \overline{z}^l$ ($k, l \in \mathbb{N}^2, k \neq l$) on $\mathcal{H}^2_\omega(\mathbb{D}^2)$. It is known that T_φ and T^*_φ share the same reducing subspaces. So k and l are symmetrical. Together with the symmetry of z_1 and z_2, we assume $0 < k_1 < l_1$. For $m \in \mathbb{Z}^2_+$, let

$$L_m = \overline{\mathrm{span}}\{z^{m+uk+vl} : m + uk + vl \in \mathbb{Z}^2_+, u, v \in \mathbb{Z}\}. \tag{3.1}$$

Obviously, L_m are reducing subspaces for T_φ. Let

$$[m] = \{m + uk + vl \in \mathbb{Z}^2_+ : u, v \in \mathbb{Z}\},$$

and

$$\Delta = \begin{cases} \{(m_1, m_2) \in \mathbb{Z}^2_+ : m_1 \in [0, s_1), m_2 \in [0, \frac{|l_1k_2-l_2k_1|}{s_1})\}, & k_1l_2 \neq k_2l_1, \\ \{(m_1, m_2) \in \mathbb{Z}^2_+ : m_1 \in [0, s_1) \text{ or } m_2 \in [0, s_2)\}, & k_1l_2 = k_2l_1, \end{cases}$$

where $s_i = \gcd\{k_i, l_i\}$, $i = 1, 2$. Then $\mathbb{Z}^2_+ = \bigcup_{m\in\Delta}[m]$. The proof can be seen in [27]. Therefore,

$$\mathcal{H}^2_\omega(\mathbb{D}^2) = \bigoplus_{m\in\Delta} L_m.$$

For $m \in \Delta$, let $[z^m]$ be the reducing subspace for $T_{z^k+\overline{z}^l}$ on $\mathcal{H}^2_\omega(\mathbb{D}^2)$ generated by z^m.

If ω satisfies the assumptions (P1)–(P6), we can prove that $[z^m] = L_m$ (see Theorem 3.2). If ω satisfies the assumptions (P1)–(P7), we get that $[z^m]$ is minimal (see Theorem 3.3). By Theorems 3.2 and 3.3, it is easy to obtain Theorem 1.1. To prove Theorem 3.2, we need to show that set Ω is the union of an increasing sequence of sets. So we firstly give the following Lemma.

Lemma 3.1 *Given $m \in \Delta$. Let $c_i = \min\{c \in \mathbb{Z}_+ : m + ck \succeq il\}$, $d_i = \min\{d \in \mathbb{Z}_+ : m + dl \succeq ik\}$, $i \in \mathbb{Z}_+$. Then c_i and d_i are strictly monotonically increasing for $i \in \mathbb{Z}_+$.*

Proof By the definition of c_i, it is easy to see $c_{i+1} \geq c_i \geq 1$. In the following, we will prove that $c_{i+1} > c_i$. For $i \in \mathbb{Z}_+$, since $m+(c_i-1)k \not\succeq il$, we have $m_1+(c_i-1)k_1 < il_1$ or $m_2+(c_i-1)k_2 < il_2$.

Case 1. $m_1 + (c_i - 1)k_1 < il_1$. Then $-m_1 - c_ik_1 + k_1 > -il_1$. By the definition of c_{i+1}, there is $m_1 + c_{i+1}k_1 \geq (i+1)l_1$, which implies that $(c_{i+1} - c_i + 1)k_1 > l_1$. By assumptions $k_1 < l_1$ and $c_i, c_{i+1} \in \mathbb{Z}_+$, we get $c_{i+1} - c_i + 1 \geq 2$. So $c_{i+1} \geq c_i + 1 > c_i$.

Case 2. $m_2 + (c_i - 1)k_2 < il_2$. As in Case 1, it is easy to see $(c_{i+1} - c_i + 1)k_2 > l_2$.

If $k_2 \leq l_2$, then $c_{i+1} \geq c_i + 1 > c_i$.

If $k_2 > l_2$, let $s_i = \gcd\{k_i,\ l_i\}$, then $k_1 = p_1s_1,\ l_1 = q_1s_1,\ k_2 = p_2s_2,\ l_2 = q_2s_2$ for some $p_i, q_i \in \mathbb{N}$ such that $p_1 < q_1$ and $p_2 > q_2$. Assume $c_{i+1} = c_i$. Since $m+c_ik = m+c_{i+1}k \succeq (i+1)l$, we have $m_1 + c_ik_1 \geq (i+1)l_1 \Rightarrow \frac{m_1}{s_1} + c_ip_1 \geq (i+1)q_1$. Since $m \in \Delta$, $\frac{m_1}{s_1} < 1$. Together with the fact that c_ip_1 is an integer, we have $c_ip_1 \geq (i+1)q_1$, i.e.,

$$\frac{c_i}{i+1} \geq \frac{q_1}{p_1} > 1.$$

It follows that $c_i \geq i+2$. Furthermore, we get

$$(i+2)p_2 \leq c_i p_2 < \frac{m_2}{s_2} + c_i p_2 < iq_2 + p_2,$$

where the last inequality comes from the assumption $m_2 + (c_i - 1)k_2 < il_2$. Thus $\frac{p_2}{q_2} < \frac{i}{i+1} < 1$, which contradicts $p_2 > q_2$. Hence, $c_{i+1} > c_i$.

By the same technique, we can prove that $d_{i+1} > d_i$. So we complete the proof. □

Theorem 3.2 *Assume ω satisfies (P1)–(P6). Let $m \in \Delta$, then $[z^m] = L_m$, where L_m is defined by (3.1).*

Proof Clearly, $[z^m] \subseteq L_m$. Denote

$$\Omega \triangleq \{(u,v) \in \mathbb{Z}^2 : m + uk + vl \in \mathbb{Z}_+^2\};\ \widetilde{\Omega} \triangleq \{(u,v) \in \Omega : z^{m+uk+vl} \in [z^m]\}.$$

Clearly, $\widetilde{\Omega} \subseteq \Omega$. It is enough to prove that $\Omega \subseteq \widetilde{\Omega}$. Lemma 3.1 shows that $c_n < c_{n+1}$ and $d_n < d_{n+1}$. Since c_n, d_n are all integers, we have $\lim_{n\to+\infty} c_n = \lim_{n\to+\infty} d_n = +\infty$. Thus

$$\Omega = \bigcup_{n=1}^{\infty} [([-n+1, c_n] \times [-n+1, d_n]) \cap \Omega].$$

By induction, we will prove that the following statements hold for each $n \in \mathbb{N}$:

(T1) $([-n+1, c_n] \times [-n+1, d_n]) \cap \Omega \subseteq \widetilde{\Omega}$;

(T2) $(c_n, -n) \in \widetilde{\Omega}$;

(T3) $(-n, d_n) \in \widetilde{\Omega}$.

Therefore, (T1) implies the desired result.

Step 1. $n = 1$. It is easy to check that

$$T_\varphi^j z^m = z^{m+jk} \in [z^m], \forall j \in [0, c_1];\ T_\varphi^{*j} z^m = z^{m+jl} \in [z^m], \quad \forall j \in [0, d_1].$$

It follows that $([0, c_1] \times \{0\}) \bigcup (\{0\} \times [0, d_1]) \subseteq \widetilde{\Omega}$. If $d_1 = 0$, then (T1) holds for $n = 1$.

For $(u-1, v) \in \Omega$, there is

$$T_\varphi^* z^{m+uk+vl} = z^{m+uk+(v+1)l} + \frac{\omega_{m+uk+vl}}{\omega_{m+(u-1)k+vl}} z^{m+(u-1)k+vl} \in [z^m]. \tag{3.2}$$

By (3.2) and $[0, c_1] \times \{0\} \subseteq \widetilde{\Omega}$, we have $[1, c_1] \times \{1\} \subseteq \widetilde{\Omega}$. If $d_1 = 1$, combining that $\{0\} \times [0, d_1] \subseteq \widetilde{\Omega}$, there is $[0, c_1] \times \{1\} \subseteq \widetilde{\Omega}$. Then (T1) holds when $n = 1$.

If $d_1 \geq 2$, by $[0, c_1] \times \{1\}, \{0\} \times [0, d_1] \subseteq \widetilde{\Omega}$, it can be proved that $[0, c_1] \times \{2\} \subseteq \widetilde{\Omega}$. Therefore, we can prove that (T1) holds when $n = 1$ by repeating the similar process as above a finite number of times.

By the definition of c_1, we have $m + c_1 k - l \succeq 0$. Let $P_{[z^m]}$ be the orthogonal projection from $\mathcal{H}_\omega^2(\mathbb{D}^2)$ onto $[z^m]$. Then (3.2) shows that

$$\begin{aligned} T_\varphi z^{m+c_1 k} &= z^{m+(c_1+1)k} + \frac{\omega_{m+c_1 k}}{\omega_{m+c_1 k - l}} z^{m+c_1 k - l} \in [z^m], \\ T_\varphi z^{m+c_1 k} &= P_{[z^m]} T_\varphi z^{m+c_1 k} = P_{[z^m]} z^{m+(c_1+1)k} + \frac{\omega_{m+c_1 k}}{\omega_{m+c_1 k - l}} P_{[z^m]} z^{m+c_1 k - l}. \end{aligned}$$

It follows that

$$P_{[z^m]}z^{m+(c_1+1)k}-z^{m+(c_1+1)k}=\frac{\omega_{m+c_1k}}{\omega_{m+c_1k-l}}(z^{m+c_1k-l}-P_{[z^m]}z^{m+c_1k-l}). \tag{3.3}$$

By the definition of c_1, we also have $m+c_1k-l\not\succeq l$ and $m+(c_1-1)k\not\succeq l$, i.e., $m+c_1k-l\in\Omega_1$. It is easy to see $m+(c_1+1)k\in\Omega_4$. By Lemmas 2.3 and 2.5, above equality shows that

$$\begin{aligned}\langle P_{[z^m]}z^{m+c_1k-l},z^{m+(c_1+1)k}\rangle=&\langle P_{[z^m]}z^{m+c_1k-l},P_{[z^m]}z^{m+(c_1+1)k}\rangle\\=&\langle z^{m+c_1k-l},P_{[z^m]}z^{m+(c_1+1)k}\rangle=0.\end{aligned}$$

Clearly, $z^{m+c_1k-l}\perp z^{m+(c_1+1)k}$. Therefore, $z^{m+c_1k-l}-P_{[z^m]}z^{m+c_1k-l}\perp P_{[z^m]}z^{m+(c_1+1)k}-z^{m+(c_1+1)k}$ and (3.3) implies that

$$z^{m+c_1k-l}=P_{[z^m]}z^{m+c_1k-l}\in[z^m],$$

that is, (T2) holds when $n=1$. By $P_{[z^m]}T_\varphi^*z^{m+d_1l}=T_\varphi^*z^{m+d_1l}$, similarly, we can get (T3) holds when $n=1$.

Step 2. Assume (T1)–(T3) hold when $n\le p$, we will prove that they also hold when $n=p+1$. Inductive hypothesis (T2) shows that

$$T_\varphi^jz^{m+c_pk-pl}=z^{m+c_pk-pl+jk}\in[z^m],\quad\forall j\in[0,c_{p+1}-c_p].$$

That is $[c_p,c_{p+1}]\times\{-p\}\subseteq\widetilde{\Omega}$. Note that

$$T_\varphi z^{m+uk+vl}=z^{m+(u+1)k+vl}+\frac{\omega_{m+uk+vl}}{\omega_{m+uk+(v-1)l}}z^{m+uk+(v-1)l}\in[z^m],\quad\forall(u,v-1)\in\Omega. \tag{3.4}$$

By (3.4), we can verify the following fact for $j=0,1,\ldots,c_{p+1}-c_p-1$ one by one:

$$\text{since }(c_p+j,-p+1),(c_p+j,-p)\in\widetilde{\Omega},\text{there is }(c_p+j+1,-p+1)\in\widetilde{\Omega}.$$

Furthermore, the following statement holds for $j\in[0,c_{p+1}-c_p-1]$, $h\in[0,d_p+p-1]$:

$$\text{since }(c_p+j,-p+h+1),(c_p+j,-p+h)\in\widetilde{\Omega},\text{there is }(c_p+j+1,-p+1+h)\in\widetilde{\Omega}.$$

Combining inductive hypothesis (T1) with $n\le p$, we have that $([-p,c_{p+1}]\times[-p,d_p])\bigcap\Omega\subseteq\widetilde{\Omega}$.

Similarly, by inductive hypothesis (T3), we have

$$T_\varphi^{*i}z^{m-pk+d_pl}=z^{m-pk+d_pl+il}\in[z^m],\quad\forall i\in[0,d_{p+1}-d_p].$$

Together with $([-p,c_{p+1}]\times\{d_p\})\bigcap\Omega\subseteq\widetilde{\Omega}$, by (3.2) many times, we can prove that

$$([-p,c_{p+1}]\times\{d_p+i\})\bigcap\Omega\subseteq\widetilde{\Omega}\text{ for }i=1,\ldots,d_{p+1}-d_p.$$

So (T1) holds when $n=p+1$.

In particular, statement (T1) shows that $z^{m+c_{p+1}k-pl},\ z^{m+d_{p+1}l-pk}\in[z^m]$. Note that

$$\begin{aligned}T_\varphi z^{m+c_{p+1}k-pl}&=z^{m+(c_{p+1}+1)k-pl}+\frac{\omega_{m+c_{p+1}k-pl}}{\omega_{m+c_{p+1}k-(p+1)l}}z^{m+c_{p+1}k-(p+1)l}\in[z^m],\\T_\varphi^*z^{m+d_{p+1}l-pk}&=z^{m+(d_{p+1}+1)l-pk}+\frac{\omega_{m+d_{p+1}l-pk}}{\omega_{m+d_{p+1}l-(p+1)k}}z^{m+d_{p+1}l-(p+1)k}\in[z^m],\end{aligned}$$

where $m+c_{p+1}k-(p+1)l,m+d_{p+1}l-(p+1)k\in\Omega_1$ and $m+c_{p+1}k-pl,m+d_{p+1}l-pk\in\Omega_4$. By Lemmas 2.3 and 2.5, we can get the desired results as in step 1. $\square$

Theorem 3.3 *Assume ω satisfies (P1)–(P7). Given $m\in\Delta$. Then L_m is a minimal reducing subspace for T_φ.*

Proof Suppose $M\subseteq L_m$ is a reducing subspace. Let P_M be the orthogonal projection from $\mathcal{H}^2_\omega(\mathbb{D}^2)$ onto M. Then $P_MT_\varphi=T_\varphi P_M$ and $P_MT_\varphi^*=T_\varphi^*P_M$. Note that $m\in\Delta\subseteq\Omega_1$. If $Q_m(p)\equiv 0$, Lemma 2.3 shows that $P_Mz^m=cz^m\in M$ for $c\in\mathbb{C}$.

If $Q_m(p)\not\equiv 0$, Lemma 2.5 shows

$$P_Mz^m=\sum_{\beta\in\Omega_1,\lambda_m=\lambda_\beta}a_\beta z^\beta,\tag{3.5}$$

with $a_\beta\in\mathbb{C}$. If $a_\beta\neq 0$, then $\Delta_{m,\beta}=\{0\}$. Lemmas 2.2 and 2.4 induce that

$$\Delta_{m+p(k+l),\beta}=\{p\},\quad\forall p\in\mathbb{Z}_+.\tag{3.6}$$

Thus $P_Mz^{m+p(k+l)}=\sum_{\beta\in\Omega_1,\lambda_m=\lambda_\beta}a_{\beta,p}z^{\beta+p(k+l)}$, $\forall p\in\mathbb{Z}_+$. In the following, we prove that

$$a_{\beta,p}=a_{\beta,q},\ \forall p,q\in\mathbb{Z}_+.$$

Clearly, it holds when $p=0$. For $p\in\mathbb{Z}_+$, suppose $a_{\beta,h}=a_{\beta,q}$, $0\le h,q\le p$. By $T_\varphi^*T_\varphi P_Mz^{m+p(k+l)}=P_MT_\varphi^*T_\varphi z^{m+p(k+l)}$, we get

$$\begin{aligned}&P_M\Big(z^{m+(p+1)(k+l)}+\rho z^{m+p(k+l)}+\frac{\omega_{m+p(k+l)}}{\omega_{m+(p-1)(k+l)}}z^{m+(p-1)(k+l)}\Big)\\&=\sum_{\beta\in\Omega_1,\lambda_m=\lambda_\beta}a_{\beta,p}\Big(z^{\beta+(p+1)(k+l)}+\eta z^{\beta+p(k+l)}+\frac{\omega_{\beta+p(k+l)}}{\omega_{\beta+(p-1)(k+l)}}z^{\beta+(p-1)(k+l)}\Big),\end{aligned}$$

where ρ, $\eta>0$. By (3.6), we have $P_Mz^{m+p(k+l)}\perp z^{\beta+(p+1)k+l}$, $P_Mz^{m+(p-1)(k+l)}\perp z^{\beta+(p+1)k+l}$, $P_Mz^{m+(p+1)(k+l)}\perp z^{\beta+pk+l}$ and $P_Mz^{m+(p+1)(k+l)}\perp z^{\beta+(p-1)k+l}$. Therefore,

$$P_Mz^{m+(p+1)(k+l)}=\sum_{\beta\in\Omega_1,\lambda_m=\lambda_\beta}a_{\beta,p}z^{\beta+(p+1)(k+l)},$$

i.e., $a_{\beta,p}=a_{\beta,p+1}$.

Furthermore, by the expression of $P_Mz^{m+(p-1)(k+l)}$, we have

$$\frac{\omega_{m+p(k+l)}}{\omega_{m+(p-1)(k+l)}}=\frac{\omega_{\beta+p(k+l)}}{\omega_{\beta+(p-1)(k+l)}},\quad\forall p\in\mathbb{N}.$$

So (P1) shows that

$$\frac{\omega_m}{\omega_n}=\frac{\omega_{m+p(k+l)}}{\omega_{n+p(k+l)}}=\lim_{p\to+\infty}\frac{\omega_{m+p(k+l)}}{\omega_{n+p(k+l)}}=1.$$

For $p=0$, $P_MT_\varphi^*T_\varphi z^m=T_\varphi^*T_\varphi P_Mz^m$ implies that

$$P_M\Big(z^{m+k+l}+\frac{\omega_{m+k}}{\omega_m}z^m\Big)=\sum_{\beta\in\Omega_1,\lambda_m=\lambda_\beta}a_\beta\Big(z^{\beta+k+l}+\frac{\omega_{\beta+k}}{\omega_\beta}z^\beta\Big).$$

Thus $\frac{\omega_{m+k}}{\omega_m}=\frac{\omega_{\beta+k}}{\omega_\beta}$ and $\omega_{m+k}=\omega_{n+k}$. By (P7), we have $P_Mz^m=cz^m$ for some $c\in\mathbb{C}$. By Theorem 3.2, we get $M=L_m$ or $M=\{0\}$. $\square$

4. Reducing subspaces for $T_{z^k+\bar{z}^l}$ on Dirichlet space

In this section, we focus on a class of weighted Dirichlet space $\mathcal{D}_\delta(\mathbb{D}^2)$ ($\delta>0$),

$$\mathcal{D}_\delta(\mathbb{D}^2)=\mathcal{H}^2_\omega(\mathbb{D}^2) \text{ with } \omega=\{\omega_n=(n_1+1)^\delta(n_2+1)^\delta, n\in\mathbb{Z}^2_+\}.$$

We also suppose that $0<k_1<l_1$. In this case,

$$\lambda_n=\begin{cases}\prod_{i=1}^2\frac{(n_i+k_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+l_i+1)^\delta}{(n_i+1)^\delta}, & n\in\Omega_1,\\ \prod_{i=1}^2\frac{(n_i+k_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+l_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+1)^\delta}{(n_i-k_i+1)^\delta}, & n\in\Omega_2,\\ \prod_{i=1}^2\frac{(n_i+k_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+l_i+1)^\delta}{(n_i+1)^\delta}+\prod_{i=1}^2\frac{(n_i+1)^\delta}{(n_i-l_i+1)^\delta}, & n\in\Omega_3,\\ \prod_{i=1}^2\frac{(n_i+k_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+l_i+1)^\delta}{(n_i+1)^\delta}-\prod_{i=1}^2\frac{(n_i+1)^\delta}{(n_i-k_i+1)^\delta}+\prod_{i=1}^2\frac{(n_i+1)^\delta}{(n_i-l_i+1)^\delta}, & n\in\Omega_4,\end{cases}$$

and

$$\begin{aligned}Q_n(p)=&\prod_{i=1}^2\frac{(n_i+k_i+p(k_i+l_i)+1)^\delta}{(n_i+p(k_i+l_i)+1)^\delta}-\prod_{i=1}^2\frac{(n_i+l_i+p(k_i+l_i)+1)^\delta}{(n_i+p(k_i+l_i)+1)^\delta}-\\ &\prod_{i=1}^2\frac{(n_i+p(k_i+l_i)+1)^\delta}{(n_i-k_i+p(k_i+l_i)+1)^\delta}+\prod_{i=1}^2\frac{(n_i+p(k_i+l_i)+1)^\delta}{(n_i-l_i+p(k_i+l_i)+1)^\delta}.\end{aligned}$$

Firstly, we will show in this case ω satisfies (P1)–(P7). Clearly, (P1) holds. The next Lemma shows that (P2) holds.

Lemma 4.1 *Let $n\in\mathbb{Z}^2_+$. Then the following statements are equivalent:*

(i) $A_n\triangleq(k_2-l_2)(n_1+1)+(k_1-l_1)(n_2+1)=0$ *and* $k_1k_2=l_1l_2$*;*

(ii) $\frac{k_1}{n_1+1}=\frac{l_2}{n_2+1}$, $\frac{l_1}{n_1+1}=\frac{k_2}{n_2+1}$ *and* $k_1k_2=l_1l_2$*;*

(iii) $Q_n(p)\equiv 0$*;*

(iv) There exist $\{p_j\}\subseteq\mathbb{N}$ *such that* $\lim\limits_{j\to+\infty}p_j=+\infty$ *and* $Q_n(p_j)=0$ *for* $j\in\mathbb{N}$.

Proof Firstly, we prove that (i) holds if and only if (ii) holds. Note that (ii)$\Rightarrow$(i) is obvious. Conversely, if (i) holds,

$$k_1(k_2-l_2)(n_1+1)+k_1(k_1-l_1)(n_2+1)=l_2(l_1-k_1)(n_1+1)+k_1(k_1-l_1)(n_2+1)=0.$$

Since $k_1<l_1$, we get $\frac{k_1}{n_1+1}=\frac{l_2}{n_2+1}$, and then $\frac{l_1}{n_1+1}=\frac{k_2}{n_2+1}$, i.e., (ii) holds.

Secondly, we prove that (ii)$\Rightarrow$(iii). By computation, we have $Q_n(p)=0$ if and only if

$$\begin{aligned}&\prod_{i=1}^2(n_i+p(k_i+l_i)-k_i+1)^\delta(n_i+p(k_i+l_i)-l_i+1)^\delta\times\\ &\quad\Big[\prod_{i=1}^2(n_i+p(k_i+l_i)+k_i+1)^\delta-\prod_{i=1}^2(n_i+p(k_i+l_i)+l_i+1)^\delta\Big]\\ &=\prod_{i=1}^2(n_i+p(k_i+l_i)+1)^{2\delta}\Big[\prod_{i=1}^2(n_i+p(k_i+l_i)-l_i+1)^\delta-\prod_{i=1}^2(n_i+p(k_i+l_i)-k_i+1)^\delta\Big].\end{aligned}$$

If (ii) holds, then

$$\prod_{i=1}^2(n_i+p(k_i+l_i)+k_i+1)^\delta-\prod_{i=1}^2(n_i+p(k_i+l_i)+l_i+1)^\delta$$

$$= \prod_{i=1}^{2}(n_i + p(k_i + l_i) - l_i + 1)^{\delta} - \prod_{i=1}^{2}(n_i + p(k_i + l_i) - k_i + 1)^{\delta} = 0.$$

Therefore, (iii) holds.

Since (iii)⇒(iv) is obvious, we only need to prove that (iv)⇒(i). Let

$$h_1(t) = \prod_{i=1}^{2}(a_i t + 1)^{\delta}(b_i t + 1)^{\delta}\Big(\prod_{i=1}^{2}(c_i t + 1)^{\delta} - \prod_{i=1}^{2}(d_i t + 1)^{\delta}\Big),$$
$$h_2(t) = \prod_{i=1}^{2}(e_i t + 1)^{2\delta}\Big(\prod_{i=1}^{2}(b_i t + 1)^{\delta} - \prod_{i=1}^{2}(a_i t + 1)^{\delta}\Big), \quad t > 0,$$

where

$$e_i = \frac{n_i + 1}{k_i + l_i},\ a_i = e_i - \frac{k_i}{k_i + l_i},\ b_i = e_i - \frac{l_i}{k_i + l_i},\ c_i = e_i + \frac{k_i}{k_i + l_i},\ d_i = e_i + \frac{l_i}{k_i + l_i}, \quad i = 1, 2.$$

Let $x = \frac{k_1k_2 - l_1l_2}{(k_1+l_1)(k_2+l_2)}$. Then

$$\begin{aligned} c_1 + c_2 - d_1 - d_2 &= b_1 + b_2 - a_1 - a_2 = 2x, \\ c_1c_2 - d_1d_2 &= e_1\frac{k_2 - l_2}{k_2 + l_2} + e_2\frac{k_1 - l_1}{k_1 + l_1} + x, \\ b_1b_2 - a_1a_2 &= e_1\frac{k_2 - l_2}{k_2 + l_2} + e_2\frac{k_1 - l_1}{k_1 + l_1} - x. \end{aligned} \tag{4.1}$$

It follows that $\lim_{t\to 0^+}(h_1'(t) - h_2'(t)) = 0$. Since (iv) holds, the definition of $Q_n(p_j)$ shows that

$$h_1(t_j) = h_2(t_j) \text{ for } \ t_j = \frac{1}{p_j}. \tag{4.2}$$

By L'Hospital's Rule, we have

$$\lim_{t\to 0^+}\frac{h_1(t) - h_2(t)}{t^2} = \lim_{t\to 0^+}\frac{h_1'(t) - h_2'(t)}{2t} = \lim_{t\to 0^+}\frac{h_1''(t) - h_2''(t)}{2}.$$

Moreover,

$$\begin{aligned} &\lim_{t\to 0^+}\frac{h_1''(t)}{2} \\ &= (\delta^2(a_1 + a_2 + b_1 + b_2) + \frac{\delta(\delta-1)}{2}(c_1 + c_2 + d_1 + d_2))(c_1 + c_2 - d_1 - d_2) + \delta(c_1c_2 - d_1d_2) \\ &= ((3\delta^2 - \delta)(e_1 + e_2) - \delta^2 - \delta)2x + \delta(e_1\frac{k_2 - l_2}{k_2 + l_2} + e_2\frac{k_1 - l_1}{k_1 + l_1} + x), \\ &\lim_{t\to 0^+}\frac{h_2''(t)}{2} \\ &= (2\delta^2(e_1 + e_2) + \frac{\delta(\delta-1)}{2}(b_1 + b_2 + a_1 + a_2))(b_1 + b_2 - a_1 - a_2) + \delta(b_1b_2 - a_1a_2) \\ &= ((3\delta^2 - \delta)(e_1 + e_2) - \delta^2 + \delta)2x + \delta(e_1\frac{k_2 - l_2}{k_2 + l_2} + e_2\frac{k_1 - l_1}{k_1 + l_1} - x). \end{aligned}$$

By (4.2), we get

$$\lim_{t\to 0^+}\frac{h_1(t)}{t^2} = \lim_{t\to 0^+}\frac{h_2(t)}{t^2}.$$

Since $\delta > 0$, we get $x = 0$, i.e., $k_1k_2 = l_1l_2$.

Furthermore,

$$\begin{aligned} &c_1 + c_2 = d_1 + d_2 = e_1 + e_2 + \frac{k_1}{k_1 + l_1} + \frac{k_2}{k_2 + l_2}, \\ &a_1 + a_2 = b_1 + b_2 = e_1 + e_2 - \frac{k_1}{k_1 + l_1} - \frac{k_2}{k_2 + l_2}, \\ &c_1 c_2 - d_1 d_2 = b_1 b_2 - a_1 a_2. \end{aligned}$$

Case 1. $\delta = 1$. L'Hospital's Rule shows that

$$\lim_{t\to 0^+} \frac{h_1(t) - h_2(t)}{t^3} = \lim_{t\to 0^+} \frac{h_1^{'''}(t) - h_2^{'''}(t)}{6}.$$

On the basis of careful calculation, we get

$$\begin{aligned} &\lim_{t\to 0^+} \frac{h_1^{'''}(t)}{6} = 2\delta^2(e_1 + e_2 - 1)(c_1 c_2 - d_1 d_2), \\ &\lim_{t\to 0^+} \frac{h_2^{'''}(t)}{6} = 2\delta^2(e_1 + e_2)(b_1 b_2 - a_1 a_2). \end{aligned}$$

Therefore, $2(e_1 + e_2 - 1)(c_1c_2 - d_1d_2) = 2(e_1 + e_2)(c_1c_2 - d_1d_2)$, i.e., $c_1c_2 - d_1d_2 = 0$.

Case 2. $\delta \neq 1$. Dividing both sides of (4.2) by $\prod_{i=1}^{2}(e_i t_j + 1)^{2\delta}$, we get

$$f_1(t_j) f_2(t_j) = f_3(t_j),$$

where

$$\begin{aligned} &f_1(t) = \prod_{i=1}^{2} \Big(\frac{(a_i t + 1)(b_i t + 1)}{(e_i t + 1)^2}\Big)^{\delta}, \\ &f_2(t) = \prod_{i=1}^{2} (c_i t + 1)^{\delta} - \prod_{i=1}^{2} (d_i t + 1)^{\delta}, \\ &f_3(t) = \prod_{i=1}^{2} (b_i t + 1)^{\delta} - \prod_{i=1}^{2} (a_i t + 1)^{\delta}, \quad t > 0. \end{aligned}$$

Similarly, by $\lim_{t\to 0^+} f_1(t) = 1$, we get $\lim_{t\to 0^+}(f_2^{'}(t) - f_3^{'}(t)) = \lim_{t\to 0^+}(f_2^{''}(t) - f_3^{''}(t)) = 0$. By L'Hospital's Rule again, we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{f_2(t) - f_3(t)}{t^3} &= \lim_{t\to 0^+} \frac{f_2^{'''}(t) - f_3^{'''}(t)}{6} \\ &= \delta(\delta - 1)(c_1 + c_2)(c_1 c_2 - d_1 d_2) - \delta(\delta - 1)(b_1 + b_2)(b_1 b_2 - a_1 a_2) \\ &= \delta(\delta - 1)(c_1 c_2 - d_1 d_2)(c_1 + c_2 - b_1 - b_2) \\ &= 2\delta(\delta - 1)(c_1 c_2 - d_1 d_2). \end{aligned}$$

So $c_1 c_2 - d_1 d_2 = 0$.

Finally, equality (4.1) implies that $A_n = (n_1 + 1)(k_2 - l_2) + (n_2 + 1)(k_1 - l_1) = 0$. So we complete the proof. □

Lemma 4.2 *The property (P3) holds on $\mathcal{D}_\delta(\mathbb{D}^2)$. That is, if $Q_n(p) \equiv 0$, then $Q_{n+l}(p) \not\equiv 0$ and $Q_{n+k}(p) \not\equiv 0$.*

Proof If $Q_n(p) \equiv 0$, Lemma 4.1 deduces that $A_n = (k_2-l_2)(n_1+1)+(k_1-l_1)(n_2+1) = 0$ and $k_1k_2 = l_1l_2$. By $k_1 < l_1$, we have $k_2 > l_2$. Then $A_{n+l} = A_n + (k_2-l_2)(l_1-k_1) \neq 0$. It follows that $Q_{n+l}(p) \not\equiv 0$. Similarly, we have $Q_{n+k}(p) \not\equiv 0$. □

Lemma 4.3 *The property (P4) holds on $\mathcal{D}_\delta(\mathbb{D}^2)$. That is, if $Q_n(p) \equiv 0$, then*

$$\lim_{p\to+\infty} p(\frac{r(n+p(k+l)+l,k)}{r(n+p(k+l),l)} - 1) = 0.$$

Proof Let

$$e_i = \frac{n_i+1}{k_i+l_i},\ b_i = e_i+1,\ c_i = e_i + \frac{l_i}{k_i+l_i}.$$

By the definition of function $r(n,m)$, we have

$$\frac{r(n+p(k+l)+l,k)}{r(n+p(k+l),l)} - 1 = \frac{\omega_{n+(p+1)(k+l)}}{\omega_{n+p(k+l)+l}}\frac{\omega_{n+p(k+l)}}{\omega_{n+p(k+l)+l}} - 1 = \frac{f_1(\frac{1}{p}) - f_2(\frac{1}{p})}{f_2(\frac{1}{p})},$$

where

$$f_1(t) = \prod_{i=1}^{2}(e_it+1)^\delta(b_it+1)^\delta,\ f_2(t) = \prod_{i=1}^{2}(c_it+1)^{2\delta},\quad \forall\, t>0.$$

By L'Hospital's Rule, we get

$$\begin{aligned}\lim_{t\to0^+}\frac{f_1(t)-f_2(t)}{tf_2(t)} &= \lim_{t\to0^+}\frac{f_1'(t)-f_2'(t)}{(tf_2(t))'}\\ &= \delta(e_1+e_2+b_1+b_2-2c_1-2c_2) = 2\delta\frac{k_1k_2-l_1l_2}{(k_1+l_1)(k_2+l_2)}.\end{aligned}$$

By $Q_n(p)\equiv 0$, Lemma 4.1 shows that $k_1k_2 = l_1l_2$. Hence,

$$\lim_{p\to+\infty} p(\frac{r(n+p(k+l)+l,k)}{r(n+p(k+l),l)} - 1) = \lim_{t\to0^+}\frac{f_1(t)-f_2(t)}{tf_2(t)} = 0. \ \square$$

Lemma 4.4 The property (P5) holds on $\mathcal{D}_\delta(\mathbb{D}^2)$. That is, for $n\in\Omega_1, m\in\Omega_4$, if $Q_n(p)\not\equiv 0$ and $\lambda_n = \lambda_m$, then $Q_m(p)\not\equiv 0$.

Proof Suppose $Q_m(p)\equiv 0$, Lemma 4.1 shows that $l_1l_2 = k_1k_2$. Since $m\in\Omega_4$, we get $\lambda_m = Q_m(0) = 0$. Therefore, $\lambda_n = \lambda_m = 0$. By the definition of λ_n with $n\in\Omega_1$, there is $w_{n+k} = w_{n+l}$, i.e., $(n_1+k_1+1)(n_2+k_2+1) = (n_1+l_1+1)(n_2+l_2+1)$. Together with $l_1l_2 = k_1k_2$, we obtain that

$$A_n = (k_2-l_2)(n_1+1)+(k_1-l_1)(n_2+1) = 0.$$

Lemma 4.1 implies that $Q_n(p)\equiv 0$, which contradicts the assumption. □

Lemma 4.5 *The property (P6) holds on $\mathcal{D}_\delta(\mathbb{D}^2)$. That is, if $Q_n(p)\equiv Q_m(p)$ with $n,m\in\mathbb{Z}_+^2$ and $n\neq m$, then the following statements hold:*

(i) If $Q_{n+l}(p)\equiv Q_{m+l}(p)$, then $Q_{n+l}(p)\not\equiv 0$, $Q_n(p)\not\equiv 0$;

(ii) If $Q_{n+k}(p)\equiv Q_{m+k}(p)$, then $Q_{n+k}(p)\not\equiv 0$, $Q_n(p)\not\equiv 0$.

Proof If $k_1k_2\neq l_1l_2$, Lemma 4.1 implies that (P6) holds.

If $k_1k_2 = l_1l_2$, then $l_2k_1 \neq k_2l_1$. Otherwise, $k_2^2k_1 = k_2l_1l_2 = l_2^2k_1$. It is easy to see $k_2 = l_2$ and $k_1 = l_1$, which contradicts $k \neq l$.

Here, we only prove that if $Q_n(p) \equiv Q_m(p)$ and $Q_{n+l}(p) \equiv Q_{m+l}(p)$, then $Q_{n+l}(p) \not\equiv 0$, since the proof of others is similar.

Suppose $Q_{n+l}(p) \equiv 0$. Then $Q_{m+l}(p) \equiv 0$. Lemma 4.1 implies that

$$\begin{aligned}
&(k_1 - l_1)(n_2 + l_2 + 1) + (k_2 - l_2)(n_1 + l_1 + 1) = 0,\\
&(k_1 - l_1)(m_2 + l_2 + 1) + (k_2 - l_2)(m_1 + l_1 + 1) = 0,\\
&(k_1 - l_1)(n_2 - m_2) + (k_2 - l_2)(n_1 - m_1) = 0.
\end{aligned} \tag{4.3}$$

Let $\nu_n(t) = \prod_{i=1}^2 (\frac{n_i+1}{k_i+l_i}t + 1)^\delta$ for $t > 0$. By $Q_n(p) \equiv Q_m(p)$, there is

$$\nu_m(t)\nu_{m-k}(t)\nu_{m-l}(t)g_n(t) \equiv \nu_n(t)\nu_{n-k}(t)\nu_{n-l}(t)g_m(t), \quad \forall t = \frac{1}{p}, \tag{4.4}$$

where

$$g_n(t) = \nu_{n-k}(t)\nu_{n-l}(t)[\nu_{n+k}(t) - \nu_{n+l}(t)] + \nu_n^2(t)[\nu_{n-k}(t) - \nu_{n-l}(t)].$$

Denote

$$e_i = \frac{n_i + 1}{k_i + l_i},\ \widetilde{e}_i = \frac{m_i + 1}{k_i + l_i},\ x_i = \frac{k_i}{k_i + l_i},\ y_i = \frac{l_i}{k_i + l_i}, \quad i = 1, 2.$$

Set $\xi = e_1(x_2 - y_2) + e_2(x_1 - y_1)$. By (4.3) and $k_1k_1 = l_1l_2$, there is

$$\xi = \widetilde{e}_1(x_2 - y_2) + \widetilde{e}_2(x_1 - y_1) = \frac{(l_1 - k_1)(l_2 - k_2)}{\prod_{i=1}^2(k_i + l_i)} \neq 0.$$

By computation, we have the following equalities:

$$\begin{aligned}
&x_1 + x_2 = y_1 + y_2 = 1,\\
&\lim_{t\to 0^+} \nu_n^{(1)}(t) = \delta(e_1 + e_2),\\
&\lim_{t\to 0^+} \nu_n^{(2)}(t) = \delta(\delta - 1)(e_1 + e_2)^2 + 2\delta e_1e_2,\\
&\lim_{t\to 0^+} (\nu_{n\pm k} - \nu_{n\pm l})^{(1)}(t) = 0,\\
&\lim_{t\to 0^+} (\nu_{n\pm k} - \nu_{n\pm l})^{(2)}(t) = \pm 2\delta\xi,\\
&\lim_{t\to 0^+} [(\nu_{n\pm k} - \nu_{n\pm l})^{(3)}(t) = 6\delta(\delta - 1)(\pm(e_1 + e_2) + 1)\xi,
\end{aligned}$$

Therefore,

$$\lim_{t\to 0^+} g_n(t) = \lim_{t\to 0^+} g_n^{(1)}(t) = \lim_{t\to 0^+} g_n^{(2)}(t) = 0,\ \lim_{t\to 0^+} g_n^{(3)}(t) = -12\delta\xi. \tag{4.5}$$

Note that $\lim_{t\to 0^+} \frac{\nu_m\nu_{m-k}\nu_{m-l}}{\nu_n\nu_{n-k}\nu_{n-l}}(t) = 1$ and $\lim_{t\to 0^+}(g_n^{(3)}(t) - g_m^{(3)}(t)) = 0$. As in Lemma 4.1, equality (4.4) deduces that $\lim_{t\to 0^+} \frac{g_n(t)}{t^4} = \lim_{t\to 0^+} \frac{g_m(t)}{t^4}$. Combining L'Hospital Rule, we get $\lim_{t\to 0^+} \frac{g_n(t)-g_m(t)}{t^4} = \lim_{t\to 0^+} \frac{g_n^{(4)}(t)-g_m^{(4)}(t)}{24} = 0$. Similarly, by

$$\frac{(\nu_{m-k}\nu_{m-l})(t)}{(\nu_{n-k}\nu_{n-l})(t)}\frac{(\nu_m g_n)(t)}{t^4} = \frac{(\nu_n g_m)(t)}{t^4},$$

we get

$$\lim_{t\to 0^+} \frac{(\nu_m g_n)(t) - (\nu_n g_m)(t)}{t^4} = \lim_{t\to 0^+} \frac{(\nu_m g_n)^{(4)}(t) - (\nu_n g_m)^{(4)}(t)}{24} = 0.$$

Since

$$(\nu_m g_n)^{(4)}(t) = \nu_m^{(4)} g_n + 4\nu_m^{(3)} g_n^{(1)} + 6\nu_m^{(2)} g_n^{(2)} + 4\nu_m^{(1)} g_n^{(3)} + \nu_m g_n^{(4)},$$

equality (4.5) shows that

$$\lim_{t\to 0^+} (\nu_m g_n - \nu_n g_m)^{(4)}(t) = 4 \lim_{t\to 0^+} (\nu_m^{(1)} g_n^{(3)} - \nu_n^{(1)} g_m^{(3)})(t) = -48\delta^2(\widetilde{e}_1 + \widetilde{e}_2 - e_1 - e_2)\xi = 0,$$

we obtain $(k_1 + l_1)(n_2 - m_2) + (k_2 + l_2)(n_1 - m_1) = 0$. Together with (4.3), we have

$$\begin{aligned} &k_1(n_2 - m_2) = k_2(m_1 - n_1),\\ &l_2(n_1 - m_1) = l_1(m_2 - n_2),\\ &(k_1 l_2 - k_2 l_1)(n_1 - m_1)(n_2 - m_2) = 0. \end{aligned}$$

Since $k_1 l_2 \neq k_2 l_1$, there must be $n_1 = m_1$, $n_2 = m_2$, which contradicts $n \neq m$. □

Lemma 4.6 *The property (P7) holds on $\mathcal{D}_\delta(\mathbb{D}^2)$. That is, if $n, m \in \Delta$ such that $n \neq m$, $\omega_{m+k} = \omega_{n+k}$ and $\omega_{m+h(k+l)} = \omega_{n+h(k+l)} (\forall h \in \mathbb{Z}_+)$, then $z^n \notin L_m$.*

Proof In fact, we will prove that $l_1 k_2 \neq l_2 k_1$ and $n = (\frac{l_1}{l_2}(m_2 + 1) - 1, \frac{l_2}{l_1}(m_1 + 1) - 1)$. By $\omega_m = \omega_n$, $\omega_{m+k} = \omega_{n+k}$, and $\omega_{m+k+l} = \omega_{n+k+l}$, we get respectively

$$(m_1 + 1)(m_2 + 1) = (n_1 + 1)(n_2 + 1), \tag{4.6}$$

$$(m_1 + k_1 + 1)(m_2 + k_2 + 1) = (n_1 + k_1 + 1)(n_2 + k_2 + 1), \tag{4.7}$$

$$(m_1 + k_1 + l_1 + 1)(m_2 + k_2 + l_2 + 1) = (n_1 + k_1 + l_1 + 1)(n_2 + k_2 + l_2 + 1). \tag{4.8}$$

Putting (4.6) into (4.7), we have

$$k_1(m_2 - n_2) + k_2(m_1 - n_1) = 0. \tag{4.9}$$

Putting (4.7) into (4.8), we have

$$l_1(m_2 - n_2) + l_2(m_1 - n_1) = 0. \tag{4.10}$$

By (4.9) and (4.10), we get $k_1 l_2(m_1 - n_1)(m_2 - n_2) = k_2 l_1(m_1 - n_1)(m_2 - n_2)$.

If $k_1 l_2 \neq k_2 l_1$, then $m_1 = n_1$, $m_2 = n_2$, which contradicts $n \neq m$.

If $k_1 l_2 = k_2 l_1$, equality (4.6) implies

$$m_2 + 1 = \frac{(n_1 + 1)(n_2 + 1)}{m_1 + 1}. \tag{4.11}$$

Now putting (4.11) into (4.10), it means

$$l_1\Big(\frac{(n_1 + 1)(n_2 + 1)}{m_1 + 1} - (n_2 + 1)\Big) + l_2(m_1 - n_1) = 0.$$

Thus,

$$l_1 \frac{n_2 + 1}{m_1 + 1}(n_1 - m_1) = l_2(n_1 - m_1).$$

Therefore,

$$n_2 = \frac{l_2}{l_1}(m_1 + 1) - 1, \quad n_1 = \frac{l_1}{l_2}(m_2 + 1) - 1.$$

Assume $z^n \in L_m$. There are $u, v \in \mathbb{Z}$ such that

$$\frac{l_1}{l_2}(m_2+1)-1 = m_1+uk_1+vl_1 \text{ and } \frac{l_2}{l_1}(m_1+1)-1 = m_2+uk_2+vl_2.$$

That is, $uk_1+vl_1 = -\frac{l_1}{l_2}(uk_2+vl_2)$. Together with $l_1k_2 = k_1l_2$, we get $uk_1+vl_1 = uk_2+vl_2 = 0$ and $m_1 = m_2$, which contradicts $n \neq m$. $\square$

Let $\mathcal{M}$ be a nonzero reducing subspace for T_φ. Let P be the orthogonal projection from $\mathcal{D}_\delta(\mathbb{D}^2)$ onto $\mathcal{M}$. By Lemma 4.6, we have $Pz^m = az^m + bz^{m'}$, where $a, b \in \mathbb{C}$ and $m' = (\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1)$. In particular, if $k_1l_2 \neq k_2l_1$, then $b=0$; if $k_1l_2 = k_2l_1$ and $m' \notin \mathbb{Z}_+^2$, then $b = 0$. And $[az^m+bz^{m'}]\bigoplus[bz^m - az^{m'}] = L_m \bigoplus L_{m'}$ when $a^2+b^2 \neq 0$. Since $\mathcal{D}_\delta(\mathbb{D}^2) = \bigoplus_{m\in\Delta} L_m$ and $\mathcal{M}$ is nonzero, there exists $m_0 \in \Delta$ such that $Pz^{m_0} \neq 0$, and

$$[Pz^{m_0}] = \overline{\text{span}}\{(Pz^{m_0})z^{uk+vl} : u, v \in \mathbb{Z}, m+uk+vl \succeq 0\} \subseteq \mathcal{M}.$$

If $\mathcal{M}$ is minimal, $\mathcal{M} = [Pz^{m_0}]$. As in [27, Theorem 3.8] and [28, Lemma 2.5], we can prove that $\mathcal{M}$ is the orthogonal sum of some minimal reducing subspaces. Therefore, we get Theorem 1.2.

Next, we consider the unitary equivalence of L_m and $L_{m'}$, where $m, m' \in \Delta$. Recall that two reducing subspaces M_1 and M_2 for T_φ are called unitarily equivalent if there exists an operator U on $\mathcal{D}_\delta(\mathbb{D}^2)$ such that $U|_{M_1}$ is unitary from M_1 onto M_2, $U|_{M_1^\perp} = 0$ and U commutes with both T_φ and T_φ^*. On the basis of the results given in section 2 and section 3, we can obtain the following results as in [27].

Lemma 4.7 *Let $k \neq l (k, l \in \mathbb{N}^2)$. Suppose $m,\ m' \in \Delta$, then the following statements hold:*

(i) If $k_1l_2 \neq k_2l_1$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m = m'$.

(ii) If $k_1l_2 = k_2l_1$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m' = m$ or $m' = (\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1)$. In particular, if $m' \notin \Delta$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m' = m$.

Proof Let $U \in \mathcal{V}^*(\varphi)$ and $U|_{L_m}$ be unitary from L_m onto $L_{m'}$. If $Q_n(p) \equiv 0$, Lemma 2.3 shows that $m = m'$ and $Uz^m = cz^m$ for $c \in \mathbb{C}$. By $\|Uz^m\| = \|z^m\|$, we get $c = 1$. If $Q_n(p) \not\equiv 0$, Lemma 4.6 shows that if $k_1l_2 \neq k_2l_1$, then $m = m'$; if $k_1l_2 = k_2l_1$, then $m' \in \{m, (\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1)\}$.

Conversely, the sufficiency of (i) is obvious. Set $U|_{L_m^\perp} = 0$ and

$$U(\frac{z^{m+ik+jl}}{\sqrt{\omega_{m+ik+jl}}}) = (\frac{z^{m'+ik+jl}}{\sqrt{\omega_{m'+ik+jl}}}).$$

It is easy to check that $U|_{L_m}$ is unitary from L_m onto $L_{m'}$. So we get the sufficiency of (ii). $\square$

Finally, by above Lemma and [7, Corollary 8.2.6], we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3 If $k_1l_2 \neq k_2l_1$, then L_m and $L_{m'}$ are not unitarily equivalent when $m \neq m'$. Since the number of elements in Δ is $|l_1k_2 - k_1l_2|$, we have $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to $\bigoplus_{i=1}^{j}\mathbb{C}$, where $j = |l_1k_2 - l_2k_1|$.

If $k_1l_2 = k_2l_1$, let $s_i = \gcd\{k_i, l_i\}$, $k_i = s_ip_i$, $l_i = s_iq_i$, for $i = 1, 2$. Then $p_1q_2 = p_2q_1$. Since $\gcd\{p_1, q_1\} = 1$, $p_2 = sp_1$ for some $s \in \mathbb{Z}_+$. Similarly, $q_1 = tq_2$ for some $t \in \mathbb{Z}_+$. So $p_1q_2 = stp_1q_2$.

It means that $s=t=1$, i.e., $p_2=p_1$ and $q_2=q_1$.

Case 1. $s_1=s_2=r$. Let $m',m\in\Delta$ such that $m'\neq m$. Then L_m and $L_{m'}$ are unitarily equivalent if and only if $m'=(m_2,m_1)$. So

$$\{(m_1,m_2)\in\Delta;\ m_1=m_2=s, s=0,1,2,\ldots,r-1\}=\{m\in\Delta;\ m=m'\},$$

$$\{m\in\Delta;\ m_1\neq m_2\}\subseteq\{m\in\Delta;\ m'\in\Delta, m\neq m'\}.$$

Therefore, $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to $\bigoplus_{j=1}^{\infty}M_2(\mathbb{C})\oplus\bigoplus_{i=1}^{r}\mathbb{C}$.

Case 2. $s_1\neq s_2$. Without loss of generality, we assume $s_2>s_1$.

$$\{(ts_1-1,0):t\in\mathbb{N}\}\subseteq\{m\in\Delta:m'=(\frac{s_1}{s_2}-1,\ ts_2-1)\notin\Delta\},$$

$$\{(s_1-1,ts_2-1):t\in\mathbb{N}\}\subseteq\{m\in\Delta:m'=(ts_1-1,s_2-1)\in\Delta\}.$$

Therefore, $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to the direct sum of countably many $M_2(\mathbb{C})\oplus\mathbb{C}$. $\square$

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References

[1] M. STESSIN, Kehe ZHU. *Reducing subspaces of weighted shift operators*. Proc. Amer. Math. Soc., 2002, **130**(9): 2631–2639.

[2] Shanli SUN, Yuejian WANG. *Reducing subspaces of certain analytic Toeplitz operators on the Bergman space*. Northeast. Math. J., 1998, **14**(2): 147–158.

[3] Kehe ZHU. *Reducing subspaces for a class of multiplication operators*. J. Lond. Math. Soc., 2000, **62**(2): 553–568.

[4] Junyun HU, Shunhua SUN, Xianmin XU, et al. *Reducing subspace of analytic Toeplitz operators on the Bergman space*. Integral Equations Operator Theory, 2004, **49**(3): 387–395.

[5] R. G. DOUGLAS, Shunhua SUN, Dechao ZHENG. *Multiplication operators on the Bergman space via analytic continuation*. Adv. Math., 2011, **226**(1): 541–583.

[6] R. G. DOUGLAS, M. PUTINAR, Kai WANG. *Reducing subspaces for analytic multipliers of the Bergman space*. J. Funct. Anal., 2012, **263**(6): 1744–1765.

[7] Kunyu GUO, Hansong HUANG. *Multiplication Operators on the Bergman Space*. Springer, Heidelberg, 2015.

[8] Liankuo ZHAO. *Reducing subspaces for a class of multiplication operators on the Dirichlet space*. Proc. Amer. Math. Soc., 2009, **137**(9): 3091–3097.

[9] Caixing GU, Shuaibing LUO, Jie XIAO. *Reducing subspaces of multiplication operators on the Dirichlet space via local inverses and Riemann surfaces*. Complex Manifolds, 2017, **4**: 84–119.

[10] Shuaibing LUO. *Reducing subspaces of multiplication operators on the Dirichlet space*. Integral Equations Operator Theory, 2016, **85**(4): 539–554.

[11] Caixing GU, Shuaibing LUO. *Composition and multiplication operators on the derivative Hardy space* $S^2(\mathbb{D})$. Complex Var. Elliptic Equ., 2018, **63**(5): 599–624.

[12] Yufeng LU, Xiaoyang ZHOU. *Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk*. J. Math. Soc. Japan, 2010, **62**(3): 745–765.

[13] Yanyue SHI, Yufeng LU. *Reducing subspaces for Toeplitz operators on the polydisk*. Bull. Korean Math. Soc., 2013, **50**(2): 687–696.

[14] Hongzhao LIN, Yinyin HU, Yufeng LU. *Reducing subspaces of Toeplitz operators on the weighted Dirichlet space of the bidisk*. Chinese Ann. Math. Ser. A , 2016, **37**(3): 311–328. (in Chinese)

[15] Hongzhao LIN. *Reducing subspaces of Toeplitz operators on Dirichlet type spaces of the bidisk*. Turkish J. Math., 2018, **42**(1): 227–242.

[16] Hui DAN, Hansong HUANG. *Multiplication operators defined by a class of polynomials on $L_a^2(\mathbb{D}^2)$*. Integral Equations Operator Theory, 2014, **80**(4): 581–601.

[17] Xudi WANG, Hui DAN, Hansong HUANG. *Reducing subspaces of multiplication operators with the symbol $\alpha z^k + \beta w^l$ on $L_a^2(\mathbb{D}^2)$*. Sci. China Math., 2015, **58**(10): 2167–2180.

[18] Kunyu GUO, Xudi WANG. *Reducing subspaces of tensor products of weighted shifts*. Sci. China Math., 2016, **59**(4): 715–730.

[19] Kunyu GUO, Xudi WANG. *The graded structure induced by operators on a Hilbert space*. J. Math. Soc. Japan., 2018, **70**(2): 853–875.

[20] Kunyu GUO, Hansong HUANG. *Commutants, Reducing Subspaces and Von Neumann Algebras Associated with Multiplication Operators*. CRC Press, Boca Raton, FL, 2019.

[21] Caixing GU. *Reducing subspaces of weighted shifts with operator weights*. Bull. Korean Math. Soc., 2016, **53**(5): 1471–1481.

[22] Caixing GU. *Common reducing subspaces of several weighted shifts with operator weights*. J. Math. Soc. Japan., 2018, **70**(3): 1185–1225.

[23] M. ALBASEER, Yufeng LU, Yanyue SHI. *Reducing subspaces for a class of Toeplitz operators on the Bergman space of the bidisk*. Bull. Korean Math. Soc., 2015, **52**(5): 1649–1660.

[24] Jia DENG, Yufeng LU, Yanyue SHI. *Reducing subspaces for a class of non-analytic Toeplitz operators on the bidisk*. J. Math. Anal. Appl., 2017, **445**(1): 784–796.

[25] Caixing GU. *Reducing subspaces of non-analytic Toeplitz operators on weighted Hardy and Dirichlet spaces of the bidisk*. J. Math. Anal. Appl., 2018, **459**(2): 980–996.

[26] Xu TANG, Caixing GU, Yufeng LU, et al. *Reducing subspaces for the product of a forward and a backward operator-weighted shifts*. J. Math. Anal. Appl., 2021, **501**(2): 125206.

[27] Jia DENG, Yufeng LU, Yanyue SHI, et al. *Reducing subspaces for a class of non-analytic Toeplitz operators*. Banach J. Math. Anal. Appl., 2018, **12**(2): 456-480.

[28] Rebo KOU, Yanyue SHI, Xiaoping XU. *Reducing subspaces of weighted shift operators on weighted Hardy space*. Sci Sin Math, doi: 10.1360/SSM-2020-0076. (in Chinese)