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Characterizations of Additive Jordan Left $*$-Derivations on C^*-Algebras

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Abstract An additive mapping δ from a $*$-algebra $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$ is called an additive Jordan left $*$-derivation if $\delta(A^2) = A\delta(A) + A^*\delta(A)$ for every A in $\mathcal{A}$. In this paper, we prove that every additive Jordan left $*$-derivation from a complex unital C^*-algebra into its unital Banach left module is equal to zero. An additive mapping δ from a $*$-algebra $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$ is called left $*$-derivable at G in $\mathcal{A}$ if $\delta(AB) = A\delta(B) + B^*\delta(A)$ for each A, B in $\mathcal{A}$ with $AB = G$. We prove that every continuous additive left $*$-derivable mapping at the unit element I from a complex unital C^*-algebra into its unital Banach left module is equal to zero.

Keywords additive mapping; Jordan left $*$-derivation; left $*$-derivable mapping; C^*-algebra

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1. Introduction

Let $\mathcal{R}$ be an associative ring. By an involution on $\mathcal{R}$, we mean a mapping $*$ from $\mathcal{R}$ into itself, such that $(AB)^* = B^*A^*$ and $(A^*)^* = A$ for each A, B in $\mathcal{R}$. A ring equipped with an involution is called a $*$-ring. In [1], Brešar and Vukman gave the concept of additive Jordan $*$-derivations. An additive mapping δ from a $*$-ring $\mathcal{R}$ into its bimodule $\mathcal{M}$ is called an additive Jordan $*$-derivation if

$$\delta(A^2) = \delta(A)A^* + A\delta(A)$$

for every A in $\mathcal{R}$. It is easy to show that an additive mapping δ from a $*$-algebra $\mathcal{A}$ into its bimodule $\mathcal{M}$ is an additive Jordan $*$-derivation if and only if

$$\delta(AB) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)$$

for each A, B in $\mathcal{A}$.

The study of additive Jordan $*$-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones. It turns out that the question of

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whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of additive Jordan $*$-derivations. For the results concerning this problem we refer to [2–6].

In [1], the authors studied some algebraic properties of additive Jordan $*$-derivations. As a special case of [1, Theorem 1], we know that every additive Jordan $*$-derivation δ from a complex unital $*$-algebra $\mathcal{A}$ into itself is of the form $\delta(A) = TA^* - AT$ for some T in $\mathcal{A}$. For non-unital $*$-algebras, Brešar and Zalar [7] proved that every additive Jordan $*$-derivation δ from an algebra of all compact linear operators on a complex Hilbert space $\mathcal{H}$ into itself is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$. But it is also an open question whether above result in [7] remains true in the real case.

Roughly speaking, it is much more difficult to study additive Jordan $*$-derivations on real algebras than on complex algebras.

Nevertheless, Šemrl [8] proved that every additive Jordan $*$-derivation on $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where $\mathcal{H}$ is a real Hilbert space with $\dim\mathcal{H} > 1$, and in [7], the authors gave a new proof of this result. Šemrl [9] showed that every additive Jordan $*$-derivation from a standard operator algebra $\mathcal{A}$ on $\mathcal{H}$ into $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where $\mathcal{H}$ is a real or complex Hilbert space with $\dim\mathcal{H} > 1$.

Inspired by the definition of additive Jordan $*$-derivations, Ali et al. [10] gave the definitions of additive left $*$-derivations and additive Jordan left $*$-derivations. Suppose that $\mathcal{R}$ is a $*$-ring and $\mathcal{M}$ is a left $\mathcal{R}$-module. An additive mapping δ from $\mathcal{R}$ into $\mathcal{M}$ is called an additive left $*$-derivation if

$$\delta(AB) = A\delta(B) + B^*\delta(A)$$

for each A, B in $\mathcal{R}$; δ is called an additive Jordan left $*$-derivation if

$$\delta(A^2) = A\delta(A) + A^*\delta(A)$$

for every A in $\mathcal{R}$. Obviously, every additive left $*$-derivation is an additive Jordan left $*$-derivation. The converse is, in general, not true. Ali et al. [10] proved that every additive left $*$-derivation from a noncommutative prime $*$-ring into itself is equal to zero.

This paper is organized as follows. In Section 2, we prove that every additive Jordan left $*$-derivation from a complex unital C^*-algebra into its Banach left module is equal to zero.

An additive mapping δ from a $*$-algebra $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$ is called left $*$-derivable at G in $\mathcal{A}$ if

$$\delta(AB) = A\delta(B) + B^*\delta(A)$$

for each A, B in $\mathcal{A}$ with $AB = G$.

In Section 3, we suppose that $\mathcal{A}$ is a complex unital C^*-algebra and $\mathcal{M}$ is a unital left $\mathcal{A}$-module, and we show that every continuous additive left $*$-derivable mapping at the unit element I in $\mathcal{A}$ is equal to zero.

2. Jordan left $*$-derivations

The following lemma will be used repeatedly in this section.

Lemma 2.1 *Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a left $\mathcal{A}$-module. If δ is an additive Jordan left $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$, then for each A, B in $\mathcal{A}$, we have that*

$$\delta(AB + BA) = (A + A^*)\delta(B) + (B + B^*)\delta(A).$$

Proof By the definition of additive Jordan left $*$-derivations, we can obtain that

$$\delta((A + B)^2) = (A + B)\delta(A + B) + (A + B)^*\delta(A + B)$$

for each A, B in $\mathcal{A}$. By a simple calculation, it follows that

$$\delta(AB + BA) = (A + A^*)\delta(B) + (B + B^*)\delta(A)$$

for each A, B in $\mathcal{A}$. □

Proposition 2.2 *Let $\mathcal{A}$ be a complex unital $*$-algebra and $\mathcal{M}$ be a unital left $\mathcal{A}$-module. If δ is an additive Jordan left $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\delta(A) = \frac{i}{2}(A^* - A)\delta(i)$ for every A in $\mathcal{A}$.*

Proof By Lemma 2.1, we have that

$$2\delta(A) = \delta(2A) = \delta(i(-iA) + (-iA)i) = (-iA + iA^*)\delta(i) = i(A^* - A)\delta(i)$$

for every A in $\mathcal{A}$. □

By Proposition 2.2, we can obtain the following two results immediately.

Corollary 2.3 *Let $\mathcal{A}$ be a complex unital $*$-algebra and $\mathcal{M}$ be a unital left $\mathcal{A}$-module. Then every additive Jordan left $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$ is real linear.*

Corollary 2.4 *Let $\mathcal{A}$ be a complex unital C^*-algebra and $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module. Then every additive Jordan left $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$ is automatically continuous.*

The following theorem is the main result in this section.

Theorem 2.5 *Let $\mathcal{A}$ be a complex unital C^*-algebra and $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module. Then every additive Jordan left $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$ is equal to zero.*

Proof Denote by $\mathcal{A}^{\sharp\sharp}$ and $\mathcal{M}^{\sharp\sharp}$ the second dual space of $\mathcal{A}$ and $\mathcal{M}$, respectively.

It is well known that $\mathcal{M}^{\sharp\sharp}$ turns into a Banach left $(\mathcal{A}^{\sharp\sharp}, \diamond)$-module with the operation defined by

$$A^{\sharp\sharp} \cdot M^{\sharp\sharp} = \lim_{\lambda} \lim_{\mu} A_\lambda M_\mu$$

for every $A^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$ and every $M^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, where (A_λ) is a net in $\mathcal{A}$ with $\|A_\lambda\| \leqslant \|A^{\sharp\sharp}\|$ and $(A_\lambda) \to A^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$, (M_μ) is a net in $\mathcal{M}$ with $\|M_\mu\| \leqslant \|M^{\sharp\sharp}\|$ and $(M_\mu) \to M^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

By [11, p. 26], we can define a product $\diamond$ in $\mathcal{A}^{\sharp\sharp}$ by $A^{\sharp\sharp} \diamond B^{\sharp\sharp} = \lim_\lambda \lim_\mu A_\lambda B_\mu$ for each $A^{\sharp\sharp}$, $B^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$, where (A_λ) and (B_μ) are two nets in $\mathcal{A}$ with $\|A_\lambda\| \leqslant \|A^{\sharp\sharp}\|$ and $\|B_\mu\| \leqslant \|B^{\sharp\sharp}\|$, such that $A_\lambda \to A^{\sharp\sharp}$ and $B_\mu \to B^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. Moreover, we can define an involution $*$ in $\mathcal{A}^{\sharp\sharp}$ by

$$(A^{\sharp\sharp})^*(\rho) = \overline{A^{\sharp\sharp}(\rho^*)}, \quad \rho^*(A) = \overline{\rho(A^*)},$$

where $A^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$, ρ in $A^{\sharp}$ and A in $\mathcal{A}$. By [12, p. 726], we know that $\mathcal{A}^{\sharp\sharp}$ is $*$-isomorphic to a von Neumann algebra under the product $\diamond$ and the involution $*$, and so we may assume that $(\mathcal{A}^{\sharp\sharp},\diamond)$ is a complex von Neumann algebra.

By Corollary 2.4, we know that $\delta^{\sharp\sharp} : (\mathcal{A}^{\sharp\sharp}, \diamond) \to \mathcal{M}^{\sharp\sharp}$ is the complex linear and weak*-continuous extension of δ to the double duals of $\mathcal{A}$ and $\mathcal{M}$.

Let $A^{\sharp\sharp}$ be in $\mathcal{A}^{\sharp\sharp}$, and let (A_λ) be a net in $\mathcal{A}$ with $\|A_\lambda\| \leqslant \|A^{\sharp\sharp}\|$ and $A^{\sharp\sharp} = \lim_\lambda A_\lambda$ in $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. We have that

$$\begin{aligned}\delta^{\sharp\sharp}(A^{\sharp\sharp} \diamond A^{\sharp\sharp}) &= \delta^{\sharp\sharp}(\lim_\lambda \lim_\lambda A_\lambda A_\lambda) = \lim_\lambda \lim_\lambda \delta(A_\lambda A_\lambda)\\ &= \lim_\lambda \lim_\lambda A_\lambda \delta(A_\lambda) + \lim_\lambda \lim_\lambda A_\lambda^* \delta(A_\lambda)\\ &= A^{\sharp\sharp}\delta^{\sharp\sharp}(A^{\sharp\sharp}) + (A^{\sharp\sharp})^*\delta^{\sharp\sharp}(A^{\sharp\sharp}).\end{aligned}$$

It means that $\delta^{\sharp\sharp}$ is a Jordan left $*$-derivation from $\mathcal{A}^{\sharp\sharp}$ into $\mathcal{M}^{\sharp\sharp}$.

It is well known that every element $A^{\sharp\sharp}$ in complex von Neumann algebra $\mathcal{A}^{\sharp\sharp}$ can be expressed in the form $H^{\sharp\sharp} + iK^{\sharp\sharp}$, where $H^{\sharp\sharp}, K^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$ with $H^{\sharp\sharp} = (H^{\sharp\sharp})^*$ and $K^{\sharp\sharp} = (K^{\sharp\sharp})^*$. Since $\delta^{\sharp\sharp}$ is a complex linear mapping, and by Proposition 2.2, it is easy to show that

$$\delta^{\sharp\sharp}(A^{\sharp\sharp}) = 0$$

for every $A^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$; hence $\delta(A) = 0$ for every A in $\mathcal{A}$. $\square$

3. Left $*$-derivable mappings

Recall an additive mapping δ from a $*$-algebra $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$ is an additive left $*$-derivable mapping at G in $\mathcal{A}$ if $\delta(AB) = A\delta(B) + B^*\delta(A)$ for each A, B in $\mathcal{A}$ with $AB = G$.

For a unital algebra $\mathcal{A}$ and a unital left $\mathcal{A}$-module $\mathcal{M}$, we call an element W in $\mathcal{A}$ a left separating point of $\mathcal{M}$ if $WM = 0$ implies $M = 0$ for every M in $\mathcal{M}$. It is easy to see that every left invertible element in $\mathcal{A}$ is a left separating point of $\mathcal{M}$.

Theorem 3.1 *Suppose that $\mathcal{A}$ is a complex unital C^*-algebra, $\mathcal{M}$ is a unital Banach left $\mathcal{A}$-module and G is a left separating point of $\mathcal{M}$. If $GA = AG$ for every A in $\mathcal{A}$ and δ is a continuous additive left $*$-derivable mapping at G from $\mathcal{A}$ into $\mathcal{M}$, then δ is equal to zero.*

Proof Since $GI = G$, it follows that $G\delta(I)+\delta(G) = \delta(G)$. By the definition of the left separating point, we know that $\delta(I) = 0$.

Let A be a non-zero element in $\mathcal{A}$. It is well known that $I - tA$ is invertible in $\mathcal{A}$ for every t in $\mathbb{R}$ with $|t| < \|A\|^{-1}$, and we have that

$$(I - tA)^{-1} = \sum_{n=0}^{\infty}(tA)^n = \sum_{n=0}^{\infty} t^n A^n. \tag{3.1}$$

Since δ is a continuous additive mapping, it is easy to prove that δ is real linear. Thus $\delta(t^n B) = t^n \delta(B)$ for every B in $\mathcal{A}$ and every positive integer n.

By $G(I - tA)(I - tA)^{-1} = G$, we can obtain that

$$(G - tGA)\delta((I - tA)^{-1}) + (I - tA^*)^{-1}\delta(G - tGA) = \delta(G).$$

By (2.1) we have that

$$(G - tGA)\delta(\sum_{n=0}^{\infty} t^n A^n) + \sum_{n=0}^{\infty} t^n (A^*)^n \delta(G - tGA) = \delta(G).$$

Since δ is a continuous additive mapping, it follows that

$$\begin{aligned}\delta(G) = &\sum_{n=0}^{\infty} t^n G\delta(A^n) - \sum_{n=0}^{\infty} t^{n+1} GA\delta(A^n) + \sum_{n=0}^{\infty} t^n (A^*)^n \delta(G) - \\ &\sum_{n=0}^{\infty} t^{n+1} (A^*)^n \delta(GA) \\ = &\sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + (A^*)^n \delta(G) - (A^*)^{n-1}\delta(GA)] + \\ &G\delta(I) + \delta(G).\end{aligned}$$

By $\delta(I) = 0$, it implies that

$$\sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + (A^*)^n \delta(G) - (A^*)^{n-1}\delta(GA)] = 0$$

for every t in $\mathbb{R}$ with $|t| < \|A\|^{-1}$. Consequently,

$$G\delta(A^n) - GA\delta(A^{n-1}) + (A^*)^n \delta(G) - (A^*)^{n-1}\delta(GA) = 0 \tag{3.2}$$

for all $n = 1, 2, \ldots$. In particular, choose $n = 1$ and $n = 2$ in (3.2), respectively, we have the following two identities:

$$G\delta(A) + A^*\delta(G) - \delta(GA) = 0 \tag{3.3}$$

and

$$G\delta(A^2) - GA\delta(A) + (A^*)^2\delta(G) - A^*\delta(GA) = 0. \tag{3.4}$$

Comparing (3.3) and (3.4), we can obtain that

$$G\delta(A^2) - GA\delta(A) - A^*G\delta(A) = 0.$$

Since $GA = AG$ and by the definition of the left separating point, we have that

$$\delta(A^2) = A\delta(A) + A^*\delta(A).$$

Thus δ is a Jordan left $*$-derivation. By Theorem 2.5, we know that $\delta \equiv 0$. □

Corollary 3.2 *Suppose that $\mathcal{A}$ is a complex unital C^*-algebra and $\mathcal{M}$ is a unital Banach left $\mathcal{A}$-module. If δ is a continuous additive left $*$-derivable mapping at I from $\mathcal{A}$ into $\mathcal{M}$, then $\delta \equiv 0$.*

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