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Some Remarks on Higgs Bundles over Compact Generalized Kähler Manifolds

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Abstract In this paper, we first solve the Dirichlet problem for α-Hermite-Yang-Mills-Higgs equations on $I_{\pm}$-Higgs bundles over a compact generalized Kähler manifold. Then we prove that the α-semi-stability on $I_{\pm}$-Higgs bundles over closed generalized Kähler manifolds implies the existence of approximate α-Hermite-Yang-Mills-Higgs structure.

Keywords $I_{\pm}$-Higgs bundle; generalized Kähler manifold; Gauduchon metric; α-Hermite-Yang-Mills-Higgs metric; approximate α-Hermite-Yang-Mills-Higgs structure

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1. Introduction

A manifold M is called a generalized Kähler manifold [1] if it carries the data (g, I_+, I_-, b), where g is a Riemann metric on M, $I_{\pm}$ are two complex structures on M, b is a 2-form on M. Moreover, $I_{\pm}$ are parallel with respect to the connections $\nabla_{\pm} = \nabla + \frac{1}{2} g^{-1}\mathbb{H}$, respectively, where ∇ is the Levi-Civita connection of g and $\mathbb{H} = \mathrm{d}b$. The generalized Calabi-Yau manifold is an important kind of this generalized Kähler manifold. Let (M, g, I_+, I_-, b) be an n-dimensional generalized Kähler manifold. Let E be a holomorphic vector bundle on M endowed with two holomorphic structures $\bar{\partial}_+$ and $\bar{\partial}_-$ with respect to the complex structures I_+ and I_-, respectively. An $I_{\pm}$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ over (M, g, I_+, I_-, b) is an $I_{\pm}$-holomorphic vector bundle $(E, \bar{\partial}_+, \bar{\partial}_-)$ together with a Higgs field $\phi \in \Omega_M^{1,0}(\mathrm{End}(E))$ satisfying $\bar{\partial}_{\pm}\phi = 0$.

Suppose H is a Hermitian metric on $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$. Let $F_{\pm}^H$ be the curvatures of the Chern connections $\nabla_{\pm}^H$ on $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ associated to the Hermitian metric H and the holomorphic structures $\bar{\partial}_{\pm}$. We consider the Hitchin-Simpson connection [2]

$$\nabla_{\pm,\phi}^H = \nabla_{\pm}^H + \phi + \phi^{*_H},$$

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where ϕ^{*_H} is the adjoint of ϕ with respect to the metric H. The curvature of this connection is

$$F^H_{\pm,\phi} = F^H_{\pm} + [\phi, \phi^{*_H}] + \partial^H_{\pm}\phi + \bar{\partial}_{\pm}\phi^{*_H}.$$

A Hermitian metric H on $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ is said to be α-Hermite-Yang-Mills-Higgs if

$$\sqrt{-1}(\alpha F^H_{+,\phi} \wedge \omega_+^{n-1} + (1-\alpha) F^H_{-,\phi} \wedge \omega_-^{n-1}) = (n-1)! \lambda \cdot \mathrm{Id}_E \cdot \mathrm{dvol}_g, \tag{1.1}$$

where $\alpha \in (0,1)$, $\lambda \in \mathbb{R}$, and $\omega_\pm(\cdot,\cdot) = g(I_\pm \cdot, \cdot)$ are the fundamental 2-forms of g. Once $I_+ = I_-$, (1.1) reduces to the Hermite-Yang-Mills-Higgs equation, also known as Hermite-Einstein equation. When $\phi = 0$, (1.1) is nothing but α-Hermite-Einstein equation for $I_\pm$-holomorphic vector bundle introduced by Hu, Moraru and Seyyedali [3].

During the past three decades, the existence of Hermite-Yang-Mills metrics on holomorphic bundles has attracted a lot of attention. The classical Donaldson-Uhlenbeck-Yau theorem states that the stability of holomorphic vector bundle over closed Kähler manifold implies the existence of Hermite-Yang-Mills metric [4, 5]. The inverse of this theorem is also true due to the work of Kobayashi [6] and Lübke [12]. So we have a correspondence, also called Hitchin-Kobayashi correspondence, which exhibits a deep relation between the stability in the sense of algebraic geometry and the existence of special metrics. And there are many interesting generalized Hitchin-Kobayashi correspondence (or Donaldson-Uhlenbeck-Yau theorem) along different directions [2, 3, 8–12, 12–24]. As for manifolds with boundary, Donaldson first solved the Dirichlet problem for Hermite-Yang-Mills equations over compact Kähler manifolds with non-empty boundary [25]. This result was generalized to the general Hermitian manifolds in [26]. Just very recently, the author [23] considered the Dirichlet boundary value problem for α-Hermite-Yang-Mills equations on $I_\pm$-holomorphic vector bundles. In this paper, we will consider a more general setting and prove the following theorem.

Theorem 1.1 *Let (M, g, I_+, I_-, b) be a compact generalized Kähler manifold with non-empty boundary ∂M such that $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. Suppose $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ is an $I_\pm$-Higgs bundle on M. Then for any Hermite metric $\widetilde{H}$ on the restriction of $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ to ∂M, there is a unique α-Hermite-Yang-Mills-Higgs metric H on $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ such that $H = \widetilde{H}$ on ∂M.*

An $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ over a generalized Kähler manifold M is said to be admitting an approximate α-Hermite-Yang-Mills-Higgs structure, if for every $\varepsilon > 0$, there exists a Hermitian metric H_ε on $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ such that

$$\sup_M |\sqrt{-1}(\alpha F^{H_\varepsilon}_{+,\phi} \wedge \omega_+^{n-1} + (1-\alpha) F^{H_\varepsilon}_{-,\phi} \wedge \omega_-^{n-1}) - (n-1)! \lambda \cdot \mathrm{Id}_E \cdot \mathrm{dvol}_g|_{H_\varepsilon} < \varepsilon. \tag{1.2}$$

Kobayashi [27] first introduced this notion for holomorphic vector bundles (i.e., $I_+ = I_-$ and $\phi = 0$). He proved that a holomorphic vector bundle over a compact Kähler manifold admitting such a structure must be semi-stable. Later, Bruzzo and Graña Otero [28] generalized this result to numerically effective Higgs bundles. When X is projective, Kobayashi [27] also solved the inverse that a semi-stable holomorphic vector bundle must admit an approximate Hermite-Einstein structure. Moreover, he conjectured that this should be true for general Kähler manifolds. This was confirmed by [15, 29, 30]. In 2016, Nie and Zhang [31] generalized this correspondence to the non-Kähler case. In 2018, Zhang et al. [22] showed this is also true for a class of non-compact Gauduchon manifolds.

In this paper, we are interested in the existence of approximate α-Hermite-Yang-Mills-Higgs structures on $I_\pm$-Higgs bundles over closed generalized Kähler manifolds. And we will prove the following theorem.

Theorem 1.2 *Let (M, g, I_+, I_-, b) be a closed generalized Kähler manifold such that g is Gauduchon with respect to both I_+ and I_-, and $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. Suppose $(E, \bar\partial_+, \bar\partial_-, \phi)$ is an $I_\pm$-Higgs bundle on M. Then $(E, \bar\partial_+, \bar\partial_-, \phi)$ is α-semi-stable if and only if it admits an approximate α-Hermite-Yang-Mills-Higgs structure, i.e., satisfying the inequality (1.2).*

Remark 1.3 The proof of the necessity is the same as in [31], so we omit it. The proof of the existence of approximate α-Hermite-Yang-Mills-Higgs structure is based on Uhlenbeck-Yau's continuity method. Since the stability condition is not a strict inequality, we cannot apply the methods in [3,5,16] directly. To fix this, we will adopt Li-Zhang's arguments [15] and Nie-Zhang's arguments [31] to our settings.

This paper is organized as follows. In Section 2, we will introduce the α-Hermite-Yang-Mills-Higgs flow on generalized Kähler manifolds and prove the long-time existence of the flow over a compact generalized Kähler manifold. In Section 3, we deal with convergence of the α-Hermite-Yang-Mills-Higgs flow over a compact generalized Kähler manifold with boundary, in which we complete the proof of Theorem 1.1. At last, we will prove Theorem 1.2 in detail.

2. α-Hermite-Yang-Mills-Higgs flow on generalized Kähler manifold

Suppose $(E, \bar\partial_+, \bar\partial_-, \phi)$ is an $I_\pm$-Higgs bundle on a generalized Kähler manifold (M, g, I_+, I_-, b) whose associated bi-Hermitian structure (g, I_+, I_-) is such that $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. Let us fix the $I_\pm$-holomorphic structures $\bar\partial_\pm$ and a Hermitian metric H_0 on $(E, \bar\partial_+, \bar\partial_-, \phi)$. For any positive-definite Hermitian endomorphism $h \in \mathrm{Herm}^+(E, H_0)$, define the Hermitian metric $H := H_0 h$ by $\langle s, t\rangle_H := \langle hs, t\rangle_{H_0}$, where $s, t \in C^\infty(E)$. Let $\nabla_\pm^H = \bar\partial_\pm + \partial_\pm^H$ be the corresponding Chern connections. Denote by $\nabla_{\pm,\phi}^H = \nabla_\pm^H + \phi + \phi^{*H}$ the Hitchin-Simpson connections. The relation between $\nabla_{\pm,\phi}^H$ and $\nabla_{\pm,\phi}^{H_0}$ is given by

$$\nabla_{\pm,\phi}^H = \nabla_{\pm,\phi}^{H_0} + h^{-1}(\partial_\pm^{H_0} + \phi^{*H_0})h. \tag{2.1}$$

Then the curvatures with respect to $\nabla_\pm^H$ and $\nabla_\pm^{H_0}$ satisfy

$$F_{\pm,\phi}^H = F_{\pm,\phi}^{H_0} + (\bar\partial_\pm + \phi)(h^{-1}(\partial_\pm^{H_0} + \phi^{*H_0})h). \tag{2.2}$$

Let us turn to a family of Hermitian metrics $H(t)$ on $(E, \bar\partial_+, \bar\partial_-, \phi)$ with an initial metric $H(0) = H_0$. We will follow the classical heat flow method to derive the existence of α-Hermite-Yang-Mills-Higgs metric. We consider the following α-Hermite-Yang-Mills-Higgs flow:

$$H^{-1}\frac{\partial}{\partial t}H = -(\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi, \phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi, \phi^{*H}]) - \lambda \cdot \mathrm{Id}_E), \tag{2.3}$$

where $\Lambda_\pm$ are the contraction operators associated to $\omega_\pm$, respectively.

If $\phi = 0$, (2.3) is the Hermite-Yang-Mills flow considered in [23]. After taking a local holomorphic basis $\{e_a\}_{a=1}^r$ on E and local complex coordinates $\{z^i\}_{i=1}^n$ on M, the flow (2.3) can be written as follows:

$$\begin{aligned}\frac{\partial H}{\partial t} =& -\alpha\sqrt{-1}\Lambda_+\bar{\partial}_+\partial_+ H + \alpha\sqrt{-1}\Lambda_+\bar{\partial}_+ H H^{-1}\partial_+ H - \alpha\sqrt{-1}\Lambda_+ H[\phi,\phi^{*_H}] - \\ &(1-\alpha)\sqrt{-1}\Lambda_-\bar{\partial}_-\partial_- H + (1-\alpha)\sqrt{-1}\Lambda_-\bar{\partial}_- H H^{-1}\partial_- H - \\ &(1-\alpha)\sqrt{-1}\Lambda_- H[\phi,\phi^{*_H}] + \lambda\cdot H,\end{aligned} \tag{2.4}$$

where $\partial_\pm$ denote the $(1,0)$-parts of the exterior differential d with respect to the complex structures $I_\pm$, respectively. Hence the above evolution equation is non-linear strictly parabolic.

For later use, we define

$$\Delta_{\bar{\partial},\alpha} := \alpha\Delta_{\bar{\partial}_+} + (1-\alpha)\Delta_{\bar{\partial}_-},$$

where

$$\Delta_{\bar{\partial}_\pm} := -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm\partial_\pm.$$

We first prove the following proposition.

Proposition 2.1 *Let $H(t)$ be a solution of the flow (2.3). Then*

$$\Big(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t}\Big)|\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E|_H^2 \geq 0.$$

Proof For simplicity, set

$$\xi = \alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E.$$

Then from (2.1) and (2.2), we have

$$\begin{aligned}\Delta_{\bar{\partial}_\pm}|\xi|_H^2 =& -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm\partial_\pm \mathrm{tr}\{\xi H^{-1}\bar{\xi}^T H\} \\ =& -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm \mathrm{tr}\{\partial_\pm\xi H^{-1}\bar{\xi}^T H - \xi H^{-1}\partial_\pm H H^{-1}\bar{\xi}^T H + \\ &\xi H^{-1}\overline{\bar{\partial}_\pm\xi}^T H + \xi H^{-1}\bar{\xi}^T H H^{-1}\partial_\pm H\} \\ =& 2\mathrm{Re}\langle -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm\partial_\pm^H\xi,\xi\rangle_H + |\partial_\pm^H\xi|_H^2 + |\bar{\partial}_\pm\xi|_H^2,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(\sqrt{-1}\Lambda_\pm F_{\pm,\phi}^H) =& \sqrt{-1}\Lambda_\pm\Big\{\bar{\partial}_\pm\Big(-h^{-1}\frac{\partial h}{\partial t}h^{-1}\partial_\pm^{H_0}h + h^{-1}\partial_\pm^{H_0}\Big(\frac{\partial h}{\partial t}\Big)\Big) - \\ &\Big[\phi, h^{-1}\frac{\partial h}{\partial t}h^{-1}\phi^{*_{H_0}}h\Big] + \Big[\phi, h^{-1}\phi^{*_{H_0}}\frac{\partial h}{\partial t}\Big]\Big\} \\ =& -\sqrt{-1}\Lambda_\pm\{\bar{\partial}_\pm\partial_\pm^H\xi + [\phi,[\phi^{*_H},\xi]]\}.\end{aligned}$$

Hence

$$\begin{aligned}\Big(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t}\Big)|\xi|_H^2 =& \Delta_{\bar{\partial},\alpha}|\xi|_H^2 - 2\mathrm{Re}\Big\langle\frac{\partial}{\partial t}\xi,\xi\Big\rangle_H \\ =& \alpha(|\partial_+^H\xi|_H^2 + |\bar{\partial}_+\xi|_H^2) + (1-\alpha)(|\partial_-^H\xi|_H^2 + |\bar{\partial}_-\xi|_H^2) - \\ &2\alpha\mathrm{Re}\sqrt{-1}\Lambda_+\langle[\phi^{*_H},\xi],[\phi^{*_H},\xi]\rangle -\end{aligned}$$

$$2(1-\alpha)\mathrm{Re}\sqrt{-1}\Lambda_-\langle[\phi^{*_H},\xi],[\phi^{*_H},\xi]\rangle \geq 0. \quad \square$$

Let us recall the Donaldson's distance on the space of Hermitian metrics.

Definition 2.2 *Let H and $\widetilde{H}$ be two Hermitian metrics on $(E,\bar\partial_+,\bar\partial_-,\phi)$. We define*

$$\sigma(H,\widetilde{H}) = tr(H^{-1}\widetilde{H}) + tr(\widetilde{H}^{-1}H) - 2r,$$

where $r = \mathrm{rk(E)}$.

It is well-known that a sequence of metrics H_i converges to some H in the usual C^0-topology iff $\sup_M \sigma(H_i,H)\to 0$.

Proposition 2.3 *Let H, $\widetilde{H}$ be two Hermitian metrics satisfying (1.1). Then*

$$\Delta_{\bar\partial,\alpha}\sigma(H,\widetilde{H}) \geq 0.$$

Proof Let $h = \widetilde{H}^{-1}H$. From (2.2) we have

$$\begin{aligned}\mathrm{tr}\{\sqrt{-1}h(\Lambda_\pm F^H_{\pm,\phi} - \Lambda_\pm F^{\widetilde{H}}_{\pm,\phi})\} = &-\Delta_{\bar\partial_\pm}\mathrm{tr}(h) + \mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h h^{-1}\partial^{\widetilde{H}}_\pm h) + \\ &\mathrm{tr}(\sqrt{-1}h\Lambda_\pm[\phi,\phi^{*_H}-\phi^{*_{H_0}}]),\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\{\sqrt{-1}h^{-1}(\Lambda_\pm F^{\widetilde{H}}_{\pm,\phi} - \Lambda_\pm F^H_{\pm,\phi})\} = &-\Delta_{\bar\partial_\pm}\mathrm{tr}(h^{-1}) + \mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h^{-1} h\partial^{H}_\pm h^{-1}) + \\ &\mathrm{tr}(\sqrt{-1}h^{-1}\Lambda_\pm[\phi,\phi^{*_{H_0}}-\phi^{*_H}]).\end{aligned}$$

On the other hand [4],

$$\mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h h^{-1}\partial^{\widetilde{H}}_\pm h) \geq 0, \quad \mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h^{-1}h\partial^H_\pm h^{-1}) \geq 0.$$

Hence we complete the proof by the following identities:

$$\mathrm{tr}(\sqrt{-1}h\Lambda_\pm[\phi,\phi^{*_H}-\phi^{*_{H_0}}]) = |[\phi,h]h^{-\frac{1}{2}}|^2_{H_0}, \mathrm{tr}(\sqrt{-1}h^{-1}\Lambda_\pm[\phi,\phi^{*_{H_0}}-\phi^{*_H}]) = |[\phi,h^{-1}]h^{\frac{1}{2}}|^2_{H_0}. \quad \square$$

Next, given two solutions $H = H(t)$, $\widetilde{H} = \widetilde{H}(t)$ of the flow (2.3) with the same initial data H_0, it is easy to check the following proposition.

Proposition 2.4 *Assume $H = H(t)$, $\widetilde{H} = \widetilde{H}(t)$ are two solutions of the α-Hermite-Yang-Mills-Higgs flow (2.3) with the same initial data H_0, then we have*

$$(\Delta_{\bar\partial,\alpha} - \frac{\partial}{\partial t})\sigma(H(t),\widetilde{H}(t)) \geq 0.$$

Proof Set $h(t) = \widetilde{H}(t)^{-1}H(t)$. Notice that

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{tr}(h) &= \mathrm{tr}(\widetilde{H}^{-1}HH^{-1}\frac{\partial}{\partial t}H - \widetilde{H}^{-1}\frac{\partial}{\partial t}\widetilde{H}\widetilde{H}^{-1}H),\\ \frac{\partial}{\partial t}\mathrm{tr}(h^{-1}) &= \mathrm{tr}(-H^{-1}\frac{\partial}{\partial t}HH^{-1}\widetilde{H} + H^{-1}\widetilde{H}\widetilde{H}^{-1}\frac{\partial}{\partial t}\widetilde{H}).\end{aligned}$$

These two identities together with Proposition 2.3 show that

$$(\Delta_{\bar\partial,\alpha} - \frac{\partial}{\partial t})(\mathrm{tr}(h) + \mathrm{tr}(h^{-1})) \geq 0. \quad \square$$

We are going to prove the long-time existence of the α-Hermite-Yang-Mills-Higgs flow (2.3) over a compact generalized Kähler manifold. If the base manifold M is closed, we consider the following problem:

$$\begin{cases} H^{-1}\frac{\partial}{\partial t}H = -(\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E), \\ H(0) = H_0. \end{cases} \tag{2.5}$$

And when M is a compact manifold with a non-empty smooth boundary ∂M, for any given initial metric $\widetilde{H}$ over ∂M we will consider the following boundary value problem:

$$\begin{cases} H^{-1}\frac{\partial}{\partial t}H = -(\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E), \\ H(0) = H_0, \\ H|_{\partial M} = \widetilde{H}. \end{cases} \tag{2.6}$$

From the standard parabolic PDE theory [32], we first give the short-time existence.

Theorem 2.5 *For sufficiently small $\epsilon > 0$, (2.5) and (2.6) have a smooth solution $H(t)$ defined for $0 \le t < \epsilon$.*

Next, we can show the long-time existence by a standard argument in [4].

Lemma 2.6 *Suppose that a smooth solution H_t to (2.5) or (2.6) is defined for $0 \le t < T$. Then H_t converges in C^0 to some continuous non-degenerate metric H_T as $t \to T$.*

Proof We first prove the convergence. It suffices to prove that, given any $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\sup_M \sigma(H_t, H_{t'}) < \varepsilon, \text{ for all } t, t' > T - \delta.$$

This can be easily derived from the continuity at $t = 0$ combining with the maximum principle and Proposition 2.4.

Now we are left to prove H_T is non-degenerate. By Proposition 2.1 we know that

$$\sup_{M\times[0,T)} |\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E|_H^2 < C,$$

where $C = C(H_0)$ is a uniform constant. A direct calculation yields

$$|\frac{\partial}{\partial t}(\log \mathrm{tr} h)| \le |\alpha\sqrt{-1}\Lambda_+ F_{+,\phi}^H + (1-\alpha)\sqrt{-1}\Lambda_- F_{-,\phi}^H - \lambda\cdot \mathrm{Id}_E|_H.$$

Similarly,

$$|\frac{\partial}{\partial t}(\log \mathrm{tr} h^{-1})| \le |\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi,\phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi,\phi^{*H}]) - \lambda\cdot \mathrm{Id}_E|_H.$$

Hence we conclude that $\sigma(H, H_0)$ is uniformly bounded on $M \times [0,T)$, which means that H_T is non-degenerate. $\square$

For further consideration, one can prove the following lemma in the very same way as [4, Lemma 19] and [2, Lemma 6.4]. So the proof is omitted here.

Lemma 2.7 *Let (M, g, I_+, I_-, b) be a compact generalized Kähler manifold without boundary* *(or compact with non-empty boundary). Let $H(t)$, for $0 \leq t < T$, be a 1-parameter family of Hermitian metrics on $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ over (M, g, I_+, I_-) (satisfying the Dirichlet boundary condition), satisfying*

(1) $H(t)$ converges in C^0 to some continuous metric H_T as $t \to T$;

(2) $\sup_M |\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi, \phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi, \phi^{*H}]) - \lambda \cdot \mathrm{Id}_E|_{H_0}$ is uniformly bounded for $t < T$.

Then $H(t)$ is bounded in C^1, and also bounded in L_2^p $(1 < p < +\infty)$ uniformly in t.

Now we are ready to prove the long-time existence.

Theorem 2.8 *Eqs. (2.5) and (2.6) have a unique solution $H(t)$ which exists for $0 \leq t < \infty$.*

Proof From Theorem 2.5, there is a solution $H(t)$ to (2.5) or (2.6) existing for $0 \leq t < T$. And from Lemma 2.6, $H(t)$ converges in C^0 to a continuous non-degenerate metric H_T. This together with $\sup_M |\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi, \phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi, \phi^{*H}]) - \lambda \cdot \mathrm{Id}_E|_{H_0}$ is bounded uniformly in t implies that $H(t)$ are bounded in C^1, and also bounded in L_2^p $(1 < p < +\infty)$ uniformly in t. Since (2.5) and (2.6) are quadratic in the first derivative of H, one can apply Hamilton's techniques [32] to deduce that $H(t) \to H_T$ in C^∞, and the solution can be extended past T. Hence we have proved the long-time existence of problem (2.5) and (2.6). The uniqueness comes from Proposition 2.4 and the maximum principle. □

3. Dirichlet problem for α-Hermite-Yang-Mills-Higgs equations

In the previous section, we have proved the long-time existence of (2.6). So it remains to show that the solution $H(t)$ converges to a metric H_∞ as the time $t \to +\infty$, and that the limit H_∞ is α-Hermite-Yang-Mills-Higgs.

Suppose $H(t)$ is a solution to (2.6) for $0 \leq t < \infty$. We still set

$$\xi = \alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi, \phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi, \phi^{*H}]) - \lambda \cdot \mathrm{Id}_E.$$

From Proposition 2.1 we have

$$(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})|\xi|_H \geq 0, \tag{3.1}$$

here we also used the fact that $|\nabla|\zeta|_H|^2 \leq |\nabla_\pm^H \zeta|_H^2$ holds for any section ζ of $\mathrm{End}(E)$.

Next, using [33, Proposition 1.8 of Chapter 5], we can solve the following Dirichlet problem:

$$\begin{cases} \Delta_{\bar{\partial},\alpha} v = -|\alpha\sqrt{-1}\Lambda_+(F_+^H + [\phi, \phi^{*H}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^H + [\phi, \phi^{*H}]) - \lambda \cdot \mathrm{Id}_E|_{H_0}, \\ v|_{\partial M} = 0. \end{cases} \tag{3.2}$$

For convenience, we set $w(x,t) = \int_0^t |\xi|_H(x,\varsigma)\mathrm{d}\varsigma - v(x)$, where $v(x)$ is a solution to the problem above. Using (3.1), (3.2) and the boundary condition satisfied by H, we can conclude that for $t > 0$, $|\xi|_H$ vanishes over the boundary ∂M. Then we have

$$\begin{cases} (\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})w(x,t) \geq 0, \\ w(x,0) = -v(x), \\ w(x,t)|_{\partial M} = 0. \end{cases} \tag{3.3}$$

Hence the maximum principle implies that

$$\int_0^t |\xi|_H(x,\varsigma)\mathrm{d}\varsigma \le \sup_{\hat{x}\in M} v(\hat{x}), \tag{3.4}$$

for any $\hat{x} \in M$ and $0 \le t < +\infty$.

Let $0 \le t_1 \le t$, $\bar{h} = H^{-1}(x,t_1)H(x,t)$. Obviously, $\bar{h}$ satisfies

$$\bar{h}^{-1}\frac{\partial}{\partial t}\bar{h} = -\xi,$$

which means

$$\frac{\partial}{\partial t}\log(\mathrm{tr}\bar{h}) \le |\xi|_H.$$

Integrating it over $[t_1, t]$ gives

$$\mathrm{tr}(\bar{h}) = \mathrm{tr}(H^{-1}(x,t_1)H(x,t)) \le r\exp\Big(\int_{t_1}^t |\xi|_H \mathrm{d}\varsigma\Big).$$

One can also get a similar estimate for $\bar{h}^{-1}$. Combining them together, we have

$$\sigma(H(x,t), H(x,t_1)) \le 2r\Big(\exp(\int_{t_1}^t |\xi|_H\mathrm{d}\varsigma) - 1\Big). \tag{3.5}$$

By using (3.4) and (3.5), we conclude that $H(t)$ converges in C^0 to some continuous metric H_∞ as $t \longrightarrow +\infty$. Then using Lemma 2.7 again, we know that $H(t)$ has uniform C^1 and L_2^p bounds. This together with the fact that $|H^{-1}\frac{\partial}{\partial t}H|$ is uniformly bounded and the standard elliptic regularity arguments shows that, by passing to a subsequence if necessary, $H(t) \to H_\infty$ in C^∞ topology. And from (3.4) we have

$$\alpha\sqrt{-1}\Lambda_+(F_+^{H_\infty} + [\phi, \phi^{*H_\infty}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^{H_\infty} + [\phi, \phi^{*H_\infty}]) - \lambda\cdot \mathrm{Id}_E,$$

i.e., H_∞ is the desired α-Hermite-Yang-Mills-Higgs metric satisfying the Dirichlet boundary condition. The uniqueness comes from Proposition 2.3 and the maximum principle. Therefore, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Suppose $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ is an $I_\pm$-Higgs bundle on the generalized Kähler manifold (M, g, I_+, I_-, b) with $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. In this section, we assume that the Riemann metric g is Gauduchon with respect to both I_+ and I_-, i.e., $\mathrm{dd}_\pm^c\omega_\pm^{n-1} = 0$, where $\mathrm{d}_\pm^c = I_\pm \circ \mathrm{d} \circ I_\pm$. Then we can associate to $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ two degrees $\deg_\pm(E)$ and two slopes $\mu_\pm(E)$ in the standard way [16, Definition 1.4.1]:

$$\deg_\pm(E) = \frac{\sqrt{-1}}{2\pi}\int_M \mathrm{tr}(F_\pm^H) \wedge \frac{\omega_\pm^{n-1}}{(n-1)!}$$

and

$$\mu_\pm(E) = \frac{\deg_\pm(E)}{\mathrm{rank}(E)}.$$

Note that $\deg_\pm(E)$ are independent of the choice of H on $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$. Given these degrees and slopes, we now define the α-degree $\deg_\alpha(E)$ and α-slope $\mu_\alpha(E)$ as [3, Definition 3.3]:

$$\deg_\alpha(E) = \alpha\deg_+(E) + (1-\alpha)\deg_-(E)$$

and

$$\mu_\alpha(E) = \alpha\mu_+(E) + (1-\alpha)\mu_-(E),$$

respectively.

An $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$ is called α-stable resp., α-semistable), if for any proper coherent ϕ-invariant subsheaf $\mathcal{F}$ of $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$, one has

$$\mu_\alpha(\mathcal{F}) < \mu_\alpha(E) \text{ (resp., } \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(E)).$$

By Uhlenbeck-Yau's continuity method [5], we are going to show that the α-semi-stability implies approximate α-Hermite-Yang-Mills-Higgs structure.

Set

$$\mathrm{Herm}^+(E, H) = \{\xi \in \mathrm{End}(E) | \xi^{*H} = \xi, \ \xi > 0\}.$$

Fixing a proper background Hermitian metric H_0 on $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$, we consider the following perturbed equation

$$\Phi(H_\varepsilon) + \varepsilon \log h_\varepsilon = 0, \quad \varepsilon \in (0,1], \tag{4.1}$$

where

$$\Phi(H_\varepsilon) = \alpha\sqrt{-1}\Lambda_+(F_+^{H_\varepsilon} + [\phi, \phi^{*H_\varepsilon}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^{H_\varepsilon} + [\phi, \phi^{*H_\varepsilon}]) - \lambda \cdot \mathrm{Id}_E$$

and $h_\varepsilon = H_0^{-1}H_\varepsilon \in \mathrm{Herm}^+(E, H_0)$. By the results in [14], (4.1) is solvable for all $\varepsilon \in (0,1]$. Using the assumption of α-semi-stability of $I_\pm$-Higgs bundle $(E, \bar{\partial}_+, \bar{\partial}_-, \phi)$, we can show that

$$\lim_{\varepsilon \to 0} \varepsilon \max_M |\log h_\varepsilon|_{H_0} = 0. \tag{4.2}$$

This implies that $\max_M |\Phi(H_\varepsilon)|_{H_\varepsilon}$ converges to zero as $\varepsilon \to 0$.

By an appropriate conformal change [16, 31], we can assume that H_0 satisfies

$$\mathrm{tr}(\Phi(H_0)) = 0.$$

Fix a background Hermitian metric H_0 satisfying $\mathrm{tr}(\Phi(H_0)) = 0$. From (4.1), we have

$$0 = -\Delta_{\bar{\partial},\alpha}(\mathrm{tr}\log h_\varepsilon) + \varepsilon \mathrm{tr}(\log h_\varepsilon),$$

which together with the maximum principle gives

$$\det h_\varepsilon = 1.$$

Using the arguments in [3, 31], the following lemma can be easily proved.

Lemma 4.1 *If $h_\varepsilon \in \mathrm{Herm}^+(E, H_0)$ satisfies $L_\varepsilon(h_\varepsilon) = 0$ for some $\varepsilon > 0$, then*

(1) $-\frac{1}{2}\Delta_{\bar{\partial},\alpha}(|\log h_\varepsilon|_{H_0}^2) + \varepsilon|\log h_\varepsilon|_{H_0}^2 \leq |\log h_\varepsilon|_{H_0}|\Phi(H_0)|_{H_0}$;

(2) $\max_M |\log h_\varepsilon|_{H_0} \leq \frac{1}{\varepsilon} \cdot \max_M |\Phi(H_0)|_{H_0}$;

(3) $\max_M |\log h_\varepsilon|_{H_0} \leq C \cdot (\max_M |\Phi(H_0)|_{H_0} + \|\log h_\varepsilon\|_{L^2})$, *where C depends only on g and* H_0.

Let us recall some basic facts. Fixing $\xi \in \mathrm{Herm}(E, H)$, from [16, p. 237], one can choose an open dense subset $W \subseteq M$ satisfying at each $y \in W$ there exist an open neighbourhood U of y, a local unitary basis $\{e_i\}_{i=1}^r$ w.r.t. H and smooth real functions $\{\lambda_i\}_{i=1}^r$ such that

$$\xi(x)=\sum_{i=1}^r \lambda_i(x)\cdot e_i(x)\otimes e^i(x)$$

for all $x\in U$, where $\{e^i\}_{i=1}^r$ denotes the dual basis. Let $\Omega\in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $A=\sum_{i,j=1}^r A_j^i e_i\otimes e^j\in \mathrm{End}(E)$, where we assume $\mathrm{rk}(E)=r$. We denote $\Omega(\xi)(A)$ by

$$\Omega(\xi)(A)=\Omega(\lambda_i,\lambda_j)A_j^i e_i\otimes e^j.$$

By [31, Proposition 3.1], we have:

Proposition 4.2 *For some $\varepsilon>0$, if $h_\varepsilon\in \mathrm{Herm}^+(E,H_0)$ solves (4.1), then*

$$\begin{aligned}\int_M \mathrm{tr}(\Phi(H_0)s_\varepsilon)\frac{\omega_\pm^n}{n!}+\alpha\int_M \langle\Omega(s_\varepsilon)((\bar\partial_++\phi)s_\varepsilon),(\bar\partial_++\phi)s_\varepsilon\rangle_{H_0}\frac{\omega_\pm^n}{n!}+\\ (1-\alpha)\int_M\langle\Omega(s)((\bar\partial_-+\phi)s_\varepsilon),(\bar\partial_-+\phi)s_\varepsilon\rangle_{H_0}\frac{\omega_\pm^n}{n!}=-\varepsilon\|s_\varepsilon\|_{L^2}^2,\end{aligned}\tag{4.3}$$

where $s_\varepsilon=\log h_\varepsilon$ and

$$\Omega(x,y)=\begin{cases}\frac{e^{y-x}-1}{y-x}, & x\neq y;\\ 1, & x=y.\end{cases}$$

We are ready to prove the following theorem.

Theorem 4.3 *If the $I_\pm$-Higgs bundle $(E,\bar\partial_+,\bar\partial_-,\phi)$ is α-semi-stable, then it admits an approximate α-Hermite-Yang-Mills-Higgs structure.*

Proof Let $\{h_\varepsilon\}_{0<\varepsilon\le 1}$ be the solutions of equation (4.1) with the background metric H_0. Then

$$\int_X|\log h_\varepsilon|^2\frac{\omega_\pm^n}{n!}=-\frac{1}{\varepsilon}\int_M\langle\Phi(H_\varepsilon),\log h_\varepsilon\rangle_{H_\varepsilon}\frac{\omega_\pm^n}{n!}.$$

Case 1. There exists a constant $C_1>0$ such that $\|\log h_\varepsilon\|_{L^2}<C_1<+\infty$. By Lemma 4.1,

$$\max_M|\Phi(H_\varepsilon)|_{H_\varepsilon}=\varepsilon\cdot\max_M|\log h_\varepsilon|_{H_\varepsilon}<\varepsilon C\cdot(C_1+\max_M|\Phi(H_0)|_{H_0}).$$

Hence $\max_M|\Phi(H_\varepsilon)|_{H_\varepsilon}\to 0$ as $\varepsilon\to 0$.

Case 2. $\overline{\lim}_{\varepsilon\to 0}\int_X|\log h_\varepsilon|^2\frac{\omega_\pm^n}{n!}\to\infty$.

Claim. If the $I_\pm$-Higgs bundle $(E,\bar\partial_+,\bar\partial_-,\phi)$ is α-semi-stable, then

$$\lim_{\varepsilon\to 0}\max_M|\Phi(H_\varepsilon)|_{H_\varepsilon}=\lim_{\varepsilon\to 0}\varepsilon\max_M|\log h_\varepsilon|_{H_\varepsilon}=0.\tag{4.4}$$

We will show that if the above claim does not hold, there exists an $I_\pm$-Higgs subsheaf contradicting the α-semi-stability.

Let us assume that the claim does not hold, then there exist $\delta>0$ and a subsequence $\varepsilon_i\to 0,\ i\to+\infty$ so that

$$\int_X|\log h_\varepsilon|^2\frac{\omega_\pm^n}{n!}\to+\infty$$

and

$$\max_M|\Phi(H_{\varepsilon_i})|_{H_{\varepsilon_i}}=\varepsilon_i\max_M|\log h_{\varepsilon_i}|_{H_{\varepsilon_i}}\ge\delta.\tag{4.5}$$

Let $u_{\varepsilon_i} = s_{\varepsilon_i}/l_i$, where $s_{\varepsilon_i} = \log h_{\varepsilon_i}$ and $l_i = \|s_{\varepsilon_i}\|_{L^2}$. Then we have $\mathrm{tr}(u_{\varepsilon_i}) = 0$ and $\|u_{\varepsilon_i}\|_{L^2} = 1$. Then using (4.5) and Lemma 4.1, we have

$$l_i \geq -\max_M |\Phi(H_0)|_{H_0} + \frac{\delta}{C\varepsilon_i} \tag{4.6}$$

and

$$\max_M |u_{\varepsilon_i}| \leq \frac{C}{l_i}(\max_M |\Phi(H_0)|_{H_0} + l_i) < C_2 < +\infty. \tag{4.7}$$

Step 1. We first show that $\|u_{\varepsilon_i}\|_{L_1^2}$ are uniformly bounded. Since $\|u_{\varepsilon_i}\|_{L^2} = 1$, it suffices to prove $\|\nabla u_{\varepsilon_i}\|_{L^2}$ are uniformly bounded.

By Proposition 4.2, for each h_{ε_i}, it holds

$$\begin{aligned}\int_M \mathrm{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!} + \alpha l_i \int_M \langle \Omega(l_i u_{\varepsilon_i})((\bar\partial_+ + \phi)u_{\varepsilon_i}), (\bar\partial_+ + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} + \\ (1-\alpha)l_i \int_M \langle \Omega(l_i u_{\varepsilon_i})((\bar\partial_- + \phi)u_{\varepsilon_i}), (\bar\partial_- + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} = -\varepsilon_i l_i.\end{aligned} \tag{4.8}$$

Substituting (4.6) into (4.8) gives

$$\begin{aligned}\frac{\delta}{C} + \int_M \mathrm{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!} + \alpha l_i \int_M \langle \Omega(l_i u_{\varepsilon_i})((\bar\partial_+ + \phi)u_{\varepsilon_i}), (\bar\partial_+ + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} + \\ (1-\alpha)l_i \int_M \langle \Omega(l_i u_{\varepsilon_i})((\bar\partial_- + \phi)u_{\varepsilon_i}), (\bar\partial_- + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} \leq \varepsilon_i \max_M |\Phi(H_0)|_{H_0},\end{aligned} \tag{4.9}$$

Consider the function

$$l\Omega(lx, ly) = \begin{cases} \frac{e^{l(y-x)}-1}{y-x}, & x \neq y; \\ l, & x = y. \end{cases}$$

We have

$$l\Omega(lx, ly) \to \begin{cases} +\infty, & x \leq y; \\ (x-y)^{-1}, & x > y, \end{cases} \tag{4.10}$$

increases monotonically as $l \to +\infty$. Let $\xi \in C^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^+)$ satisfy $\xi(x,y) < (x-y)^{-1}$ whenever $x > y$. From (4.9), (4.10) and the arguments in [2, Lemma 5.4], it follows

$$\begin{aligned}\frac{\delta}{C} + \int_M \mathrm{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!} + \alpha \int_M \langle \xi(u_{\varepsilon_i})((\bar\partial_+ + \phi)u_{\varepsilon_i}), (\bar\partial_+ + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} + \\ (1-\alpha) \int_M \langle \xi(u_{\varepsilon_i})((\bar\partial_- + \phi)u_{\varepsilon_i}), (\bar\partial_- + \phi)u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!} \leq \varepsilon_i \max_M |\Phi(H_0)|_{H_0}\end{aligned} \tag{4.11}$$

for $i \gg 1$. From (4.7), we can assume that $(x,y) \in [-\hat{C}, \hat{C}] \times [-\hat{C}, \hat{C}]$. In particular, we can safely take $\xi(x,y) = \frac{1}{3\hat{C}}$. When $(x,y) \in [-\hat{C}, \hat{C}] \times [-\hat{C}, \hat{C}]$ and $x > y$, we have $\frac{1}{3\hat{C}} < \frac{1}{x-y}$. Therefore,

$$\begin{aligned}\frac{\delta}{C} + \int_M \mathrm{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!} + \int_M \frac{1}{3\hat{C}}(\alpha|(\bar\partial_+ + \phi)u_{\varepsilon_i}|_{H_0}^2 + (1-\alpha)|(\bar\partial_- + \phi)u_{\varepsilon_i}|_{H_0}^2)\frac{\omega_\pm^n}{n!} \\ \leq \varepsilon_i \max_M |\Phi(H_0)|_{H_0}\end{aligned} \tag{4.12}$$

for $i \gg 1$. Hence

$$\int_M (\alpha|(\bar\partial_+ + \phi)u_{\varepsilon_i}|_{H_0}^2 + (1-\alpha)|(\bar\partial_- + \phi)u_{\varepsilon_i}|_{H_0}^2)\frac{\omega_\pm^n}{n!} \leq 3\hat{C}^2 \max_M |\Phi(H_0)|_{H_0}\mathrm{Vol}(M, g),$$

which means u_{ε_i} are bounded in L_1^2. So we can choose a subsequence $\{u_{\varepsilon_{i_j}}\}$ such that $u_{\varepsilon_{i_j}} \rightharpoonup u_\infty$ in L_1^2, denoted by $\{u_{\varepsilon_i}\}$ for simplicity. Noting that $L_1^2 \hookrightarrow L^2$,

$$1 = \int_M |u_{\varepsilon_i}|_{H_0}^2 \to \int_M |u_\infty|_{H_0}^2.$$

This implies that $\|u_\infty\|_{L^2} = 1$. Therefore, u_∞ is non-trivial.

Using (4.11) and following a similar discussion as in [2, Lemma 5.4], we have

$$\begin{aligned}\frac{\delta}{C} + \int_M \mathrm{tr}\{\Phi(H_0)u_\infty\}\frac{\omega_\pm^n}{n!} + \alpha\int_M \langle \xi(u_\infty)((\bar\partial_+ + \phi)u_\infty), (\bar\partial_+ + \phi)u_\infty\rangle_{H_0}\frac{\omega_\pm^n}{n!} + \\ (1-\alpha)\int_M \langle \xi(u_\infty)((\bar\partial_- + \phi)u_\infty), (\bar\partial_- + \phi)u_\infty\rangle_{H_0}\frac{\omega_\pm^n}{n!} \le 0.\end{aligned} \tag{4.13}$$

Step 2. We will construct an $I_\pm$-Higgs subsheaf which contradicts the α-semi-stability of $(E, \bar\partial_+, \bar\partial_-, \phi)$ by Uhlenbeck and Yau's trick in [5].

From (4.13) and [2, Lemma 5.5], the eigenvalues of u_∞ are constant almost everywhere. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_l$ be the distinct eigenvalues of u_∞. In the meanwhile, $\mathrm{tr}(u_\infty) = \mathrm{tr}(u_{\varepsilon_i}) = 0$ and $\|u_\infty\|_{L^2} = 1$ force $2 \le l \le r$. For each λ_j $(1 \le j \le l-1)$, construct a function $\Pi_j : \mathbb{R} \to \mathbb{R}$ such that $\Pi_j = 0$ when $x \ge \lambda_{j+1}$ and $\Pi_j = 1$ when $x \le \lambda_j$.

Set $\pi_j = \Pi_j(u_\infty)$. As in [2, 3], π_j is an L_1^2-subsystem with the following properties: $\pi_j = \pi_j^2 = \pi_j^{*H_0}$; $(\mathrm{Id}_E - \pi_j)\bar\partial_\pm \pi_j = 0$; $(\mathrm{Id}_E - \pi_j)[\phi, \pi_j] = 0$.

From the regularity statement of L_1^2-subbundle in [5], $\{\pi_j\}_{j=1}^{l-1}$ determine $l-1$ $I_\pm$-Higgs subsheaves of $(E, \bar\partial_+, \bar\partial_-, \phi)$. Since $\mathrm{tr}(u_\infty) = 0$ and noting that

$$u_\infty = \lambda_l \cdot \mathrm{Id}_E - \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)\pi_j,$$

we have

$$\lambda_l \cdot \mathrm{rk}(E) = \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)\mathrm{rk}(E_j), \tag{4.14}$$

where $E_j = \pi_j(E)$.

Construct

$$\nu = \lambda_l \cdot \deg_\alpha(E) - \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)\deg_\alpha(E_j).$$

On the one hand, substituting (4.14) into ν yields

$$\nu = \sum_{\alpha=1}^{l-1}(\lambda_{j+1} - \lambda_j)\mathrm{rk}(E_j)\Big(\frac{\deg_\alpha(E)}{\mathrm{rk}(E)} - \frac{\deg_\alpha(E_j)}{\mathrm{rk}(E_j)}\Big). \tag{4.15}$$

On the other hand, we have the following Chern-Weil formula [3]

$$\deg_\alpha(E_j) = \frac{1}{2\pi}\int_M \Big(\mathrm{tr}(\pi_j \Upsilon_{H_0}) - \alpha|(\bar\partial_+ + \phi)\pi_j|_{H_0}^2 - (1-\alpha)|(\bar\partial_- + \phi)\pi_j|_{H_0}^2\Big)\frac{\omega_\pm^n}{n!}, \tag{4.16}$$

where

$$\Upsilon_{H_0} = \alpha\sqrt{-1}\Lambda_+(F_+^{H_0} + [\phi, \phi^{*H_0}]) + (1-\alpha)\sqrt{-1}\Lambda_-(F_-^{H_0} + [\phi, \phi^{*H_0}]).$$

Substituting (4.16) into ν, we have

$$
\begin{aligned}
2\pi\nu =&\lambda_l \int_M \mathrm{tr}(\Upsilon_{H_0})\frac{\omega_\pm^n}{n!} - \sum_{j=1}^{l-1}(\lambda_{j+1}-\lambda_j)\times \\
&\Big\{\int_M \mathrm{tr}(\pi_j\Upsilon_{H_0}) - \int_M (\alpha|(\bar\partial_+ + \phi)\pi_j|^2_{H_0} + (1-\alpha)|(\bar\partial_- + \phi)\pi_j|^2_{H_0})\Big\} \\
=&\int_M \mathrm{tr}(\lambda_l\cdot \mathrm{Id}_E - \sum_{j=1}^{l-1}(\lambda_{j+1}-\lambda_j)\pi_j)\Upsilon_{H_0} + \\
&\sum_{j=1}^{l-1}(\lambda_{j+1}-\lambda_j)\int_M (\alpha|(\bar\partial_+ + \phi)\pi_j|^2_{H_0} + (1-\alpha)|(\bar\partial_- + \phi)\pi_j|^2_{H_0}) \\
=&\int_M \mathrm{tr}(u_\infty\Upsilon_{H_0}) + \int_M \alpha\langle\sum_{\alpha=1}^{l-1}(\lambda_{j+1}-\lambda_j)(\mathrm{d}\Pi_j)^2(u_\infty)((\bar\partial_+ + \phi)u_\infty), (\bar\partial_+ + \phi)u_\infty\rangle_{H_0} + \\
&\int_M (1-\alpha)\langle\sum_{\alpha=1}^{l-1}(\lambda_{j+1}-\lambda_j)(\mathrm{d}\Pi_j)^2(u_\infty)((\bar\partial_- + \phi)u_\infty), (\bar\partial_- + \phi)u_\infty\rangle_{H_0},
\end{aligned}
$$

where the function $\mathrm{d}\Pi_j : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given by

$$
\mathrm{d}\Pi_j(x,y) = \begin{cases} \frac{\Pi_j(x)-P_j(y)}{x-y}, & x\neq y; \\ \Pi_j'(x), & x = y. \end{cases}
$$

It is easy to check that, if $\lambda_a \neq \lambda_b$,

$$
\sum_{j=1}^{l-1}(\lambda_{j+1}-\lambda_j)(\mathrm{d}\Pi_j)^2(\lambda_a,\lambda_b) = |\lambda_a-\lambda_b|^{-1}. \tag{4.17}
$$

Since $\mathrm{tr}(u_\infty)=0$, by (4.13) and [15, p. 793-794], we have

$$
2\pi\nu \leq -\frac{\delta}{C} < 0. \tag{4.18}
$$

Combining (4.15) with (4.18) gives

$$
\sum_{j=1}^{l-1}(\lambda_{j+1}-\lambda_j)\mathrm{rk}(E_j)(\frac{\deg_\alpha(E)}{\mathrm{rk}(E)} - \frac{\deg_\alpha(E_j)}{\mathrm{rk}(E_j)}) < 0,
$$

which contradicts the α-semi-stability of $(E,\bar\partial_+,\bar\partial_-,\phi)$. □

References

[1] M. GUALTIERI. *Generalized complex geometry*. Ann. of Math. (2), 2011, **174**(1): 75–123.

[2] C. T. SIMPSON. *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*. J. Amer. Math. Soc., 1988, **1**(4): 867–918.

[3] Shengda HU, R. MORARU, R. SEYYEDALI. *A Kobayashi-Hitchin correspondence for $I_\pm$-holomorphic bundles*. Adv. Math., 2016, **287**: 519–566.

[4] S. K. DONALDSON. *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*. Proc. London Math. Soc. (3), 1985, **50**(1): 1–26.

[5] K. UHLENBECK, S. T. YAU. *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*. Comm. Pure Appl. Math., 1986, **39**(S): suppl., S257–S293.

[6] S. KOBAYASHI. *Curvature and stability of vector bundles*. Proc. Japan Acad. Ser. A Math. Sci., 1982, **58**(4): 158–162.

[7] M. LÜBKE. *Stability of Einstein-Hermitian vector bundles*. Manuscripta Math., 1983, **42**(2-3): 245–257.

[8] L. ÁLVAREZ-CÓNSUL, O. GARCíA-PRADA. *Hitchin-Kobayashi correspondence, quivers, and vortices*. Comm. Math. Phys., 2003, **238**(1-2): 1–33.

[9] D. BARAGLIA, P. HEKMATI. *A foliated Hitchin-Kobayashi correspondence*. arXiv:1802.09699, 2018.

[10] I. BISWAS, A. JACOB, M. STEMMLER. *Existence of approximate Hermitian-Einstein structures on semistable principal bundles*. Bull. Sci. Math., 2012, **136**(7): 745–751.

[11] S. B. BRADLOW, O. GARCIA-PRADA, I. MUNDET I RIERA. *Relative Hitchin-Kobayashi correspondences for principal pairs*. Q. J. Math., 2003, **54**(2): 171–208.

[12] Jiayu LI. *Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds*. Comm. Anal. Geom., 2000, **8**(3): 445–475.

[13] R. GOTO. *Kobayashi-Hitchin correspondence of generalized holomorphic vector bundles over generalized Kähler manifolds of symplectic type*. arXiv:1903.07425, 2019.

[14] Zhi HU, Pengfei HUANG. *The Hitchin-Kobayashi correspondence for quiver bundles over generalized Kähler manifolds*. J. Geom. Anal., 2020, **30**(4): 3641–3671.

[15] Jiayu LI, Xi ZHANG. *Existence of approximate Hermitian-Einstein structures on semi-stable Higgs bundles*. Calc. Var. Partial Differential Equations, 2015, **52**(3-4): 783–795.

[16] M. LÜBKE, A. TELEMAN. *The Kobayashi-Hitchin Correspondence*. World Scientific Publishing, 1995.

[17] J. LOFTIN. *Affine Hermitian-Einstein metrics*. Asian J. Math., 2009, **13**(1): 101–130.

[18] T. MOCHIZUKI. *Kobayashi-Hitchin correspondence for analytically stable bundles*. Trans. Amer. Math. Soc., 2020, **373**(1): 551–596.

[19] I. MUNDET I RIERA. *A Hitchin-Kobayashi correspondence of Kaehler fibrations*. J. Reine Angew. Math., 2000, **528**: 41–80.

[20] M. YOSHINO. *A Kobayashi-Hitchin correspondence between Dirac-type singular mini-holomorphic bundles and HE-monopoles*. arXiv:1902.09995, 2019.

[21] Ruixin WANG, Chuanjing ZHANG. *Semi-stability for holomorphic pairs on compact Gauduchon manifolds*. Commun. Math. Stat., 2019, **7**(2): 191–206.

[22] Chuangjing ZHANG, Pan ZHANG, Xi ZHANG. *Higgs bundles over non-compact Gauduchon manifolds*. Trans. Amer. Math. Soc., 2021, **374**(5): 3735–3759.

[23] Pan ZHANG. *Canonical metrics on holomorphic bundles over compact bi-Hermitian manifolds*. J. Geom. Phys., 2019, **144**: 15–27.

[24] Changpeng PAN. *A Note on Bogomolov-Type Inequality for Semi-stable Parabolic Higgs Bundles*. Communications in Mathematics and Statistics, 2021, 1–12.

[25] S. K. DONALDSON. *Boundary value problems for Yang-Mills fields*. J. Geom. Phys., 1992, **8**(1-4): 89–122.

[26] Xi ZHANG. *Hermitian-Einstein metrics on holomorphic vector bundles over Hermitian manifolds*. J. Geom. Phys., 2005, **53**(3): 315–335.

[27] S. KOBAYASHI. *Differential Geometry of Complex Vector Bundles*. Princeton University Press, Princeton, NJ, 1987.

[28] U. BRUZZO, B. GRAÑA OTERO. *Metrics on semistable and numerically effective Higgs bundles*. J. Reine Angew. Math., 2007, **612**: 59–79.

[29] S. A. H. CARDONA. *Approximate Hermitian-Yang-Mills structures and semistability for Higgs bundles. I: generalities and the one-dimensional case*. Ann. Global Anal. Geom., 2012, **42**(3): 349–370.

[30] A. JACOB. *Existence of approximate Hermitian-Einstein structures on semi-stable bundles*. Asian J. Math., 2014, **18**(5): 859–883.

[31] Yanci NIE, Xi ZHANG. *Semistable Higgs bundles over compact Gauduchon manifolds*. J. Geom. Anal., 2018, **28**(1): 627–642.

[32] R. S. HAMILTON. *Harmonic Maps of Manifolds with Boundary*. Springer, 2006.

[33] M. E. TAYLOR. *Partial Differential Equations I. Basic Theory. Second edition*. Springer, New York, 2011.