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The Intersection Problem for Kite-GDDs of Type 2^u

Yonghong AN[1], **Guizhi ZHANG**[2,*]
1. *College of Continuing Education, Hulunbuir University, Inner Mongolia* 021008, *P. R. China;*
2. *Division of Science and Technology, Hulunbuir University, Inner Mongolia* 021008, *P. R. China*

Abstract The intersection problem for kite-GDDs is the determination of all pairs (T, s) such that there exists a pair of kite-GDDs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ of the same type T satisfying $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. In this paper the intersection problem for a pair of kite-GDDs of type 2^u is investigated. Let $J(u) = \{s : \exists$ a pair of kite-GDDs of type 2^u intersecting in s blocks$\}$; $I(u) = \{0, 1, \ldots, b_u - 2, b_u\}$, where $b_u = u(u-1)/2$ is the number of blocks of a kite-GDD of type 2^u. We show that for any positive integer $u \geq 4$, $J(u) = I(u)$ and $J(3) = \{0, 3\}$.

Keywords kite-GDD; group divisible design; intersection number

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1. Introduction

Let H be a simple graph and G a subgraph of H. A G-design of H ((H, G)-design in short) is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is an edge-disjoint decomposition of H into isomorphic copies (called blocks) of the graph G. If H is the complete graph K_v, we refer to such a G-design as one of order v. If G is the complete graph K_k, a K_k-design of order v is called a Steiner system $S(2, k, v)$.

The intersection problem for (H, G)-designs is the determination of all pairs (v, s) such that there exists a pair of (H, G)-designs $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ with $|X| = v$ and $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. The intersection problem for $S(2, k, v)$'s was first introduced by Kramer and Mesner [1]. A complete solution to the intersection problem for $S(2, 3, v)$'s was made by Lindner and Rosa [2]. The intersection problem for $S(2, 4, v)$'s was dealt with by Colbourn et al. [3], apart from three undecided values for $v = 25$, 28 and 37. Billington and Kreher [4] completed the intersection problem for all connected simple graphs G where the minimum of the number of vertices and the number of edges of G is not bigger than 4. Chang et al. has completely solved the triangle intersection problem for $S(2, 4, v)$ designs and a pair of disjoint $S(2, 4, v)$s (see [5, 6]). Chang et al. has completely solved the fine triangle intersection problems for kite systems [7] and $(K_4 - e)$-designs [8, 9]. The intersection problem is also considered for many other types of combinatorial structures. The interested reader may refer to [10–16].

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* Corresponding author
E-mail address: zgz_hlbr@163.com (Guizhi ZHANG)

Let K be a set of positive integers. A group divisible design K-GDD is a triple $(X,\mathcal{G},\mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set X into subsets (called groups); (2) $\mathcal{A}$ is a set of subsets of X (called blocks), each of cardinality from K, such that every 2-subset of X is either contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$ contains u_i groups of size g_i for $1 \le i \le r$, then we call $g_1^{u_1}g_2^{u_2}\cdots g_r^{u_r}$ the group type (or type) of the GDD. If $K=\{k\}$, we write a $\{k\}$-GDD as a k-GDD.

Two k-GDDs $(X,\mathcal{G},\mathcal{A}_1)$ and $(X,\mathcal{G},\mathcal{A}_2)$ are said to intersect in s blocks if $|\mathcal{A}_1\cap\mathcal{A}_2| = s$. The intersection problem for group divisible designs is to determine all pairs (T,s) such that there exists a pair of group divisible designs $(X,\mathcal{G},\mathcal{A}_1)$ and $(X,\mathcal{G},\mathcal{A}_2)$ of type T satisfying $|\mathcal{A}_1\cap\mathcal{A}_2| = s$. Butler and Hoffman [17] completely solved the intersection problem for 3-GDDs of type g^u. Zhang, Chang and Feng solved the intersection problem for 4-GDDs of type 3^u (see [18]) and the intersection problem for 4-GDDs of type 4^u (see [19]).

Let $\mathcal{H}=\{H_1,H_2,\ldots,H_m\}$ be a partition of a finite set X into subsets (called holes), where $|H_i|=n_i$ for $1\le i\le m$. Let $K_{n_1,n_2,\ldots,n_m}$ be the complete multipartite graph on X with the i-th part on H_i, and G be a subgraph of $K_{n_1,n_2,\ldots,n_m}$. A holey G-design is a triple $(X,\mathcal{H},\mathcal{B})$ such that $(X,\mathcal{B})$ is a $(K_{n_1,n_2,\ldots,n_m},G)$-design. The hole type (or type) of the holey G-design is $\{n_1,n_2,\ldots,n_m\}$. We use an "exponential" notation to describe hole types: the hole type $g_1^{u_1}g_2^{u_2}\cdots g_r^{u_r}$ denotes u_i occurrences of g_i for $1\le i\le r$. Obviously, if G is the complete graph K_k, a holey K_k-design is just a k-GDD. If G is the graph with vertices a,b,c,d and edges ab,ac,bc,cd (such a graph is called a kite), then a holey G-design is said to be a kite-GDD.

A pair of holey G-designs $(X,\mathcal{H},\mathcal{B}_1)$ and $(X,\mathcal{H},\mathcal{B}_2)$ of the same type is said to intersect in s blocks if $|\mathcal{B}_1\cap\mathcal{B}_2| = s$. In this paper we focus on the intersection problem for kite-GDDs of type 2^u. Let $J(u)=\{s : \exists$ a pair of kite-GDDs of type 2^u intersecting in s blocks$\}$. Throughout this paper we always assume that $v=2u$ with $u\ge 4$, $I(u)=\{0,1,\ldots,b_u-2,b_u\}$, where $b_u=u(u-1)/2$ is the number of blocks of a kite-GDD of type 2^u. In the following, we always denote the copy of the kite with vertices a,b,c,d and edges ab,ac,bc,cd by $[a,b,c-d]$.

As the main result of the present paper, we are to prove the following theorem.

Theorem 1.1 *For any positive integer* $u\ge 4$, $J(u)=I(u)$ *and* $J(3)=\{0,3\}$.

Obviously, $J(u)\subseteq I(u)$. We need to show that $I(u)\subseteq J(u)$.

2. Basic design constructions

We introduce the following two important construction.

Construction 2.1 ([7]) (Weighting Construction) *Suppose that* $(X,\mathcal{G},\mathcal{A})$ *is a* K*-GDD, and let* $\omega : X \longmapsto Z^+\cup\{0\}$ *be a weight function. For every block* $A\in\mathcal{A}$*, suppose that there is a pair of holey* G*-designs of type* $\{\omega(x) : x\in A\}$*, which intersect in* b_A *blocks. Then there exists a pair of holey* G*-designs of type* $\{\sum_{x\in H}\omega(x) : H\in\mathcal{G}\}$*, which intersect in* $\sum_{A\in\mathcal{A}} b_A$ *blocks.*

The following construction is simple but very useful, which is a variation in [7, Construction 2.2].

Construction 2.2 (Filling Construction) *Let m be a nonnegative integer and g_i, $a \equiv 0 \pmod m$ for $1 \le i \le s$. Suppose that there exists a pair of holey G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in b blocks. If there is a pair of holey G-designs of type $m^{g_i/m}a^1$, which intersect in b_i blocks for $1 \le i \le s-1$ and there is a pair of holey G-designs of type $m^{(g_s+a)/m}$ which intersect in b_s blocks, then there exists a pair of holey G-designs of type $m^{(\sum_{i=1}^{s} g_i+a)/m}$ intersecting in $b + \sum_{i=1}^{s} b_i$ blocks.*

Proof Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be two holey G-designs of type $\{g_1, g_2, \ldots, g_s\}$ satisfying $|\mathcal{A} \cap \mathcal{B}| = b$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ with $|G_i| = g_i, 1 \le i \le s$ and Y be any given set of length a such that $X \cap Y = \emptyset$. For $1 \le i \le s-1$, construct a pair of holey G-designs $(G_i \cup Y, \mathcal{G}_i \cup \{Y\}, \mathcal{C}_i)$ and $(G_i \cup Y, \mathcal{G}_i \cup \{Y\}, \mathcal{D}_i)$ of type $m^{g_i/m}a^1$ satisfying $|\mathcal{C}_i \cap \mathcal{D}_i| = b_i$ and construct a pair of holey G-designs $(G_s \cup Y, \mathcal{G}_s \cup \{Y\}, \mathcal{C}_s)$ and $(G_s \cup Y, \mathcal{G}_s \cup \{Y\}, \mathcal{D}_s)$ of type $m^{(g_s+a)/m}$ satisfying $|\mathcal{C}_s \cap \mathcal{D}_s| = b_s$. Then $(X \cup Y, (\bigcup_{i=1}^{s} \mathcal{G}_i) \cup \{Y\}, \mathcal{A} \cup (\bigcup_{i=1}^{s} \mathcal{C}_i))$ and $(X \cup Y, (\bigcup_{i=1}^{s} \mathcal{G}_i) \cup \{Y\}, \mathcal{B} \cup (\bigcup_{i=1}^{s} \mathcal{D}_i))$ are two holey G-designs of type $m^{(\sum_{i=1}^{s} g_i+a)/m}$. Obviously, the two holey G-designs have $b + \sum_{i=1}^{s} b_i$ common blocks. □

We quote the following result for later use.

Lemma 2.3 ([20]) *The necessary and sufficient conditions for the existence of 3-GDD and 4-GDD are as follows:*

- *A 3-GDD of type g^u exists if and only if $u \ge 3$, $(u-1)g \equiv 0 \pmod 2$, and $u(u-1)g^2 \equiv 0 \pmod 6$.*
- *A 4-GDD of type g^u exists if and only if $u \ge 4$, $(u-1)g \equiv 0 \pmod 3$, and $u(u-1)g^2 \equiv 0 \pmod{12}$, with the exception of $(g, u) \in \{(2, 4), (6, 4)\}$.*
- *A 4-GDD of type $3^u m^1$ exists if and only if either $u \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 3$, $0 \le m \le (3u-6)/2$; or $u \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 6$, $0 \le m \le (3u-3)/2$; or $u \equiv 3 \pmod 4$ and $m \equiv 3 \pmod 6$, $0 < m \le (3u-3)/2$.*

In Section 3, we examine $J(u)$ for small positive integer $u \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 19, 20\}$. In Section 4, we will examine $J(u)$ for positive integer $u \ge 12$. In Section 5, We will prove the Theorem 1.1.

3. Ingredients

Let $(X, \mathcal{G}, \mathcal{B})$ be a kite-GDD of type T. Then $(X, \mathcal{G}, \pi_s\mathcal{B})$ is also a kite-GDD of the same type T, where the π_s is a permutation of X and keep group type T the same. For example, in the following, let $\mathcal{B} = \{[0, 1, 5-4], [0, 2, 4-3], [1, 2, 3-5]\}$ and $\mathcal{G} = \{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$. Taking $\pi_0 : X \to X$ and $\pi_0 = (2\ 5)$, we have that $\pi_0\mathcal{B} = (2\ 5)\mathcal{B} = \{[0, 1, 2-4], [0, 5, 4-3], [1, 5, 3-2]\}$ and $\pi_0\mathcal{G} = (2\ 5)\mathcal{G} = \{\{0, 3\}, \{1, 4\}, \{5, 2\}\} = \mathcal{G}$. Then $(X, \mathcal{G}, \mathcal{B})$ and $(X, \mathcal{G}, \pi_s\mathcal{B})$ are a pair of kite-GDD of type 2^3. We have that $|\pi_0\mathcal{B} \cap \mathcal{B}| = 0$ and $\pi_0\mathcal{G} = \mathcal{G}$.

Lemma 3.1 *For integer $u = 3$, $J(3) = \{0, 3\}$.*

Proof Take the vertex set $X = \{0, 1, 2, 3, 4, 5\}$. Let $\mathcal{B} = \{[0, 1, 5-4], [0, 2, 4-3], [1, 2, 3-5]\}$.

Then $(X, \mathcal{G}, \mathcal{B})$ is a kite-GDD of type 2^3, where $\mathcal{G} = \{\{0,3\},\{1,4\},\{2,5\}\}$. Consider the following permutations on X. $\pi_0 = (2\ 5)$, $\pi_3 = (1)$. We have that for each $s \in \{0,3\}$, $|\pi_s\mathcal{B} \cap \mathcal{B}| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. □

Lemma 3.2 *For integer* $u = 4$, $J(4) = I(4)$.

Proof Take the vertex set $X = \{0,1,2,3,4,5,6,7\}$. Let $\mathcal{B}_1 = \{[1,7,0-5],[1,3,2-5],[3,5,4-6],[5,7,6-1],[3,6,0-2],[2,7,4-1]\}$, $\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[3,5,4-6],[5,7,6-1],[2,7,4-1]\}) \cup \{[3,5,4-2],[5,6,7-2],[1,6,4-7]\}$ and $\mathcal{B}_3 = (\mathcal{B}_1 \setminus \{[3,5,4-6],[2,7,4-1]\}) \cup \{[3,5,4-1],[2,7,4-6]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 2^4 for $i = 1,2,3$, where $\mathcal{G} = \{\{0,4\},\{1,5\},\{2,6\},\{3,7\}\}$. Consider the following permutations on X.

$$\pi_0 = (2\,6)(3\,7),\ \pi_1 = (1\,5),\ \pi_2 = (2\,6),\ \pi_3 = \pi_4 = \pi_6 = (1).$$

We have that for each $s \in I(4) \setminus \{4,6\}$, $|\pi_s\mathcal{B}_2 \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. For each $s \in \{4,6\}$, $|\pi_4\mathcal{B}_3 \cap \mathcal{B}_1| = 4$, $|\pi_6\mathcal{B}_1 \cap \mathcal{B}_1| = 6$ and $\pi_s\mathcal{G} = \mathcal{G}$. □

Lemma 3.3 *For integer* $u = 5$, $J(5) = I(5)$.

Proof Take the vertex set $X = \{0,1,\ldots,9\}$. Let $\mathcal{B}_1 = [1,9,3-4],[2,8,4-0],[3,7,8-0],[6,4,1-5],[9,5,2-6],[4,9,7-5],[6,9,8-5],[0,1,2-3],[0,3,5-6],[0,6,7-1]$, $\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[3,7,8-0],[6,9,8-5]\}) \cup \{[3,7,8-5],[6,9,8-0]\}$. $\mathcal{B}_3 = (\mathcal{B}_1 \setminus \{[0,1,2-3],[0,3,5-6],[0,6,7-1]\}) \cup \{[0,7,1-2],[0,2,3-5],[0,5,6-7]\}$, $\mathcal{B}_4 = (\mathcal{B}_1 \setminus \{[3,7,8-0],[6,9,8-5],[4,9,7-5],[0,6,7-1]\}) \cup \{[3,7,8-5],[6,9,8-0],[4,9,7-1],[0,6,7-5]\}$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus \{[3,7,8-0],[6,9,8-5]\}) \cup \{[3,7,8-5],[6,9,8-0]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 2^5 for $i = 1,2,\ldots,5$, where $\mathcal{G} = \{\{0,9\},\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$. Consider the following permutations on X.

$$\begin{aligned} &\pi_0 = (2\ 7)(3\ 6), && \pi_1 = (1\ 8)(2\ 7), && \pi_2 = (3\ 6),\\ &\pi_3 = (4\ 5), && \pi_4 = (2\ 7), && \pi_5 = \pi_6 = \pi_7 = \pi_8 = \pi_{10} = (1). \end{aligned}$$

We have that for each $s \in \{0,1,\ldots,4,10\}$, $|\pi_s\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. For each $s \in \{5,6,7,8\}$, $|\pi_s\mathcal{B}_{10-s} \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. □

Lemma 3.4 *For integer* $u = 6$, $J(6) = I(6)$.

Proof Take the vertex set $X = \{0,1,\ldots,11\}$. Let $\mathcal{B}_1 = \{[0,1,8-11],[1,2,9-11],[2,3,7-0],[3,4,8-6],[4,0,5-3],[6,10,5-2],[6,11,7-4],[8,10,7-5],[8,9,5-1],[10,0,9-7],[10,11,3-0],[0,11,2-10],[1,3,6-2],[11,4,1-10],[6,9,4-2]\}$, $\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[1,2,9-11],[10,0,9-7]\}) \cup \{[1,2,9-7],[10,0,9-11]\}$, $\mathcal{B}_3 = (\mathcal{B}_1 \setminus \{[2,3,7-0],[6,11,7-4],[8,10,7-5]\}) \cup \{[2,3,7-5],[6,11,7-0],[8,10,7-4]\}$, $\mathcal{B}_4 = (\mathcal{B}_2 \setminus \{[4,0,5-3],[6,10,5-2]\}) \cup \{[4,0,5-2],[6,10,5-3]\}$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus \{[4,0,5-3],[6,10,5-2]\}) \cup \{[4,0,5-2],[6,10,5-3]\}$, $\mathcal{B}_6 = (\mathcal{B}_3 \setminus \{[4,0,5-3],[6,10,5-2],[8,9,5-1]\}) \cup \{[4,0,5-1],[6,10,5-3],[8,9,5-2]\}$, $\mathcal{B}_7 = (\mathcal{B}_5 \setminus \{[1,2,9-11],[10,0,9-7]\}) \cup \{[1,2,9-7],[10,0,9-11]\}$, $\mathcal{B}_8 = (\mathcal{B}_6 \setminus \{[1,2,9-11],[10,0,9-7]\}) \cup \{[1,2,9-7],[10,0,9-11]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 2^6 for $i = 1,2,\ldots,6$, where $\mathcal{G} =$

$\{\{0,6\},\{1,7\},\{2,8\},\{3,9\},\{4,10\},\{5,11\}\}$. Consider the following permutations on X.

$$\pi_0=(2\ 8)(3\ 9)(5\ 11),\ \pi_1=(0\ 6)(2\ 8)(3\ 9),\ \pi_2=(2\ 8)(3\ 9),\ \pi_3=(0\ 6),\ \pi_4=(0\ 6),$$
$$\pi_5=(5\ 11),\ \pi_6=(1\ 7),\ \pi_7=\pi_8=\pi_9=\pi_{10}=\pi_{11}=\pi_{12}=\pi_{13}=\pi_{15}=(1).$$

We have that for each $s\in I(6)\setminus\{3,7,8,\dots,13\}$, $|\pi_s\mathcal{B}_1\cap\mathcal{B}_1|=s$ and $\pi_s\mathcal{G}=\mathcal{G}$. For $s=3$, we have $|\pi_3\mathcal{B}_2\cap\mathcal{B}_1|=3$ and $\pi_3\mathcal{G}=\mathcal{G}$. For each $s\in\{7,8,\dots,13\}$, $|\pi_s\mathcal{B}_{15-s}\cap\mathcal{B}_1|=s$ and $\pi_s\mathcal{G}=\mathcal{G}$. □

Lemma 3.5 *For integer $u=7$, $J(7)=I(7)$.*

Proof Take the vertex set $X=\{0,1,\dots,13\}$. Let $\mathcal{B}_1=\{[0,1,9-7],[10,13,9-5],[6,11,9-3],[2,3,8-6],[7,13,8-0],[9,4,8-12],[5,11,8-10],[1,2,7-4],[6,12,7-11],[3,5,7-10],[4,6,10-12],[1,5,10-0],[10,11,2-5],[4,0,2-6],[5,6,0-12],[13,11,0-3],[1,6,3-11],[4,12,3-13],[5,13,4-1],[11,12,1-13],[2,13,12-9]\}$.

i	A_i	C_i
2	[0,1,9-7],[10,13,9-5]	[0,1,9-5],[10,13,9-7]
3	[0,1,9-7],[10,13,9-5],[6,11,9-3]	[0,1,9-3],[10,13,9-7],[6,11,9-5]
4	[0,1,9-7],[10,13,9-5],[2,3,8-6], [7,13,8-0]	[0,1,9-5],[10,13,9-7],[2,3,8-0],[7,13,8-6]
5	[0,1,9-7],[10,13,9-5],[6,11,9-3],[2,3,8-6],[7,13,8-0]	[0,1,9-3],[10,13,9-7],[6,11,9-5],[2,3,8-0],[7,13,8-6]
6	[0,1,9-7],[10,13,9-5],[6,11,9-3], [2,3,8-6],[7,13,8-0],[9,4,8-12]	[0,1,9-3],[10,13,9-7],[6,11,9-5], [2,3,8-12],[7,13,8-6],[9,4,8-0]
7	[0,1,9-7],[10,13,9-5],[6,11,9-3],[2,3,8-6], [7,13,8-0],[9,4,8-12],[5,11,8-10]	[0,1,9-3],[10,13,9-7],[6,11,9-5],[2,3,8-0], [7,13,8-6],[9,4,8-10],[5,11,8-12]
8	[0,1,9-7],[10,13,9-5],[6,11,9-3],[2,3,8-6], [7,13,8-0],[9,4,8-12],[1,2,7-4],[6,12,7-11]	[0,1,9-3],[10,13,9-7],[6,11,9-5],[2,3,8-12], [7,13,8-6],[9,4,8-0],[1,2,7-11],[6,12,7-4]

Table 1 The blocks of kite-GDD of type 2^7

Then $(X,\mathcal{G},\mathcal{B}_i)$ is a kite-GDD of type 2^7 for $i=1,2,\dots,8$, where $\mathcal{B}_i=(\mathcal{B}_1\setminus A_i)\cup C_i$, $i=2,\dots,8$ and $\mathcal{G}=\{\{0,7\},\{1,8\},\{2,9\},\{3,10\},\{4,11\},\{5,12\},\{6,13\}\}$. Consider the following permutations on X.

$$\begin{array}{lll}
\pi_0=(2\ 9)(3\ 10)(5\ 12)(6\ 13), & \pi_1=(0\ 7)(1\ 8)(6\ 13), & \pi_2=(0\ 7)(2\ 9)(3\ 10),\\
\pi_3=(2\ 9)(5\ 12), & \pi_4=(3\ 10)(4\ 11), & \pi_5=(2\ 9)(3\ 10),\\
\pi_6=(0\ 7)(1\ 8), & \pi_7=(6\ 13), & \pi_8=(5\ 12),\\
\pi_9=(3\ 10), & \pi_{10}=(3\ 10), & \pi_{11}=(1\ 8),\\
\pi_{12}=(2\ 9), & \pi_{13}=\pi_{14}=\pi_{15}=\pi_{16}=(1), & \pi_{17}=\pi_{18}=\pi_{19}=\pi_{21}=(1).
\end{array}$$

We have that for each $s\in I(7)\setminus\{9,13,14\dots,19\}$, $|\pi_s\mathcal{B}_1\cap\mathcal{B}_1|=s$ and $\pi_s\mathcal{G}=\mathcal{G}$. For $s=9$, we have $|\pi_9\mathcal{B}_2\cap\mathcal{B}_1|=9$ and $\pi_9\mathcal{G}=\mathcal{G}$. For each $s\in\{13,14,\dots,19\}$, $|\pi_s\mathcal{B}_{21-s}\cap\mathcal{B}_1|=s$ and $\pi_s\mathcal{G}=\mathcal{G}$. □

For counting $J(u)$ for $8\le u\le 11$, we need to search for a large number of instances of kite-GDDs of type 2^u as we have done in Lemma 3.5. To reduce the computation, when $u\neq 11$, we shall first try to determine the intersection numbers of a pair of kite-GDDs of type $2^{u-\frac{h_u}{2}}{h_u}^1$

with the same group set where

$$h_u = \begin{cases} 8, & \text{if } u = 8, \\ 8, & \text{if } u = 9, \\ 10, & \text{if } u = 10. \end{cases}$$

When $u = 11$, we shall try to determine the intersection numbers of a pair of kite-GDDs of type $8^2 6^1$ with the same vertex set. These results will be listed in Lemmas 3.6–3.8.

Lemma 3.6 *Let $M_8 = \{0, 1, \ldots, 15, 22\}$ and $s \in M_8$. Then there is a pair of kite-GDDs of type $2^4 8^1$ with the same group set, which intersect in s blocks.*

Proof Take the vertex set $X = \{0, 1, \ldots, 15\}$. Let

$\mathcal{B}_1$: $[10,5,8-2]$, $[6,13,8-11]$, $[9,1,8-4]$, $[7,14,8-3]$, $[10,4,15-5]$, $[12,2,15-3]$, $[0,9,15-1]$, $[15,6,14-2]$, $[13,5,14-4]$, $[12,4,13-0]$, $[10,3,11-7]$, $[10,2,9-5]$, $[15,7,13-1]$, $[14,3,12-5]$, $[13,2,11-6]$, $[12,1,10-6]$, $[9,4,11-5]$, $[7,9,12-6]$, $[0,14,10-7]$, $[14,1,11-15]$, $[12,8,0-11]$, $[13,3,9-6]$;

$\mathcal{B}_2$: $[10,5,8-11]$, $[6,13,8-2]$, $[9,1,8-3]$, $[7,14,8-4]$, $[10,4,15-1]$, $[12,2,15-5]$, $[0,9,15-3]$, $[15,6,14-2]$, $[13,5,14-4]$, $[12,4,13-0]$, $[10,3,11-7]$, $[10,2,9-5]$, $[15,7,13-1]$, $[14,3,12-5]$, $[13,2,11-6]$, $[12,1,10-6]$, $[9,4,11-5]$, $[7,9,12-6]$, $[0,14,10-7]$, $[14,1,11-15]$, $[12,8,0-11]$, $[13,3,9-6]$.

Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type $2^4 8^1$ for each $1 \le i \le 2$, where the group set is $\mathcal{G} = \{\{8,15\}, \{9,14\}, \{10,13\}, \{11,12\}, \{0,1,\ldots,7\}\}$. Consider the following permutations on X.

$\pi_0 = (0\ 4)(2\ 6\ 7\ 3)(8\ 15)(9\ 14)(10\ 13)$, $\pi_1 = (0\ 4\ 2\ 3\ 5\ 6)(1\ 7)(8\ 15)(10\ 13)$,
$\pi_2 = (1\ 4\ 2\ 5\ 7\ 3)$, $\pi_3 = (0\ 7\ 2)(3\ 4)$,
$\pi_4 = (1\ 2)(3\ 4\ 5)$, $\pi_5 = (0\ 3\ 7)(1\ 4)$,
$\pi_6 = (4\ 5)(8\ 15)$, $\pi_7 = (1\ 2\ 5\ 4)$,
$\pi_8 = (0\ 3\ 5)$, $\pi_9 = (2\ 6\ 5)$,
$\pi_{10} = (2\ 3\ 5)$, $\pi_{11} = (0\ 7\ 1)$,
$\pi_{12} = (0\ 6)$, $\pi_{13} = (2\ 6)$,
$\pi_{14} = (3\ 7)$, $\pi_{15} = \pi_{22} = (1)$.

We have that for each $s \in M_8 \setminus \{15\}$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. For $s = 15$, we have $|\pi_{15} \mathcal{B}_2 \cap \mathcal{B}_1| = 15$ and $\pi_{15} \mathcal{G} = \mathcal{G}$. □

Lemma 3.7 *Let $M_9 = \{0, 1, \ldots, 22, 28, 30\}$ and $s \in M_9$. Then there is a pair of kite-GDDs of type $2^5 8^1$ with the same group set, which intersect in s blocks.*

Proof Take the vertex set $X = \{0, 1, \ldots, 17\}$. Let $\mathcal{B}_1$:

$[0,17,9-15]$, $[16,7,17-1]$, $[15,6,16-4]$, $[14,15,5-17]$, $[14,4,13-17]$, $[12,3,11-16]$, $[11,2,10-4]$, $[9,1,10-6]$, $[9,8,2-17]$, $[8,15,7-14]$, $[16,8,0-14]$, $[15,4,17-12]$, $[14,2,16-3]$, $[15,3,13-0]$, $[14,6,12-16]$, $[11,5,13-2]$, $[0,10,12-7]$, $[9,4,11-1]$, $[8,3,10-7]$, $[9,7,13-10]$, $[6,13,8-4]$, $[14,10,17-3]$, $[16,13,1-15]$, $[15,2,12-4]$, $[9,5,12-1]$, $[10,16,5-8]$, $[6,17,11-7]$, $[14,3,9-6]$, $[15,0,11-8]$, $[1,14,8-12]$.

$\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[11, 2, 10-4], [9, 1, 10-6]\}) \cup \{[11, 2, 10-6], [9, 1, 10-4]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type $2^5 8^1$ for each $1 \le i \le 2$, where the group set is $\mathcal{G} = \{\{8, 17\}, \{9, 16\}, \{10, 15\}, \{11, 14\}, \{12, 13\}, \{0, 1, \dots, 7\}\}$. Consider the following permutations on X.

$$\begin{array}{llll}
\pi_0 = (0\ 6\ 2\ 7\ 3)(1\ 5\ 4), & \pi_1 = (0\ 3\ 7\ 6\ 4\ 5\ 1), & \pi_2 = (0\ 6\ 7\ 1\ 2)(4\ 5), & \pi_3 = (0\ 5\ 3)(4\ 6\ 7), \\
\pi_4 = (1\ 6\ 4\ 7)(3\ 5), & \pi_5 = (0\ 4)(2\ 6\ 7\ 3), & \pi_6 = (1\ 4)(2\ 7)(5\ 6), & \pi_7 = (0\ 4\ 6\ 7\ 1), \\
\pi_8 = (1\ 6\ 7)(2\ 4), & \pi_9 = (2\ 5)(3\ 6\ 4), & \pi_{10} = (0\ 1)(4\ 5), & \pi_{11} = (2\ 5\ 7\ 6), \\
\pi_{12} = (0\ 7)(2\ 5), & \pi_{13} = (0\ 3\ 2\ 5), & \pi_{14} = (0\ 1\ 2), & \pi_{15} = (3\ 5\ 6), \\
\pi_{16} = (1\ 5\ 6), & \pi_{17} = (2\ 5\ 4), & \pi_{18} = (1\ 3), & \pi_{19} = (3\ 6), \\
\pi_{20} = (5\ 7), & \pi_{21} = (1\ 5), & \pi_{22} = (2\ 5), & \pi_{28} = \pi_{30} = (1).
\end{array}$$

We have that for each $s \in M_9 \setminus \{28\}$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. For $s = 28$, we have $|\pi_{28} \mathcal{B}_2 \cap \mathcal{B}_1| = 28$ and $\pi_{28} \mathcal{G} = \mathcal{G}$. □

Lemma 3.8 *Let $M_{10} = \{0, 1, \dots, 27, 32, 35\}$ and $s \in M_{10}$. Then there is a pair of kite-GDDs of type $2^5 10^1$ with the same group set, which intersect in s blocks.*

Proof Take the vertex set $X = \{0, 1, \dots, 19\}$. Let $\mathcal{B}_1$:

$$\begin{array}{llllll}
[0, 18, 12-5], & [11, 3, 12-19], & [10, 7, 12-6], & [17, 18, 8-13], & [15, 6, 16-4], & [14, 5, 13-3], \\
[12, 4, 13-17], & [11, 10, 2-19], & [10, 9, 17-3], & [0, 19, 11-4], & [18, 9, 19-1], & [19, 6, 17-1], \\
[16, 5, 18-1], & [17, 15, 4-19], & [16, 14, 3-19], & [15, 2, 13-18], & [1, 12, 14-8], & [11, 6, 13-0], \\
[11, 16, 9-13], & [8, 15, 10-13], & [19, 16, 8-11], & [15, 7, 18-4], & [17, 0, 14-9], & [17, 7, 16-1], \\
[16, 2, 12-8], & [15, 11, 1-10], & [4, 10, 14-19], & [19, 5, 15-0], & [0, 16, 10-3], & [18, 14, 2-17], \\
[11, 7, 14-6], & [19, 7, 13-1], & [10, 6, 18-3], & [17, 11, 5-10], & [12, 9, 15-3]. &
\end{array}$$

$\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[0, 18, 12-5], [11, 3, 12-19], [10, 7, 12-6]\}) \cup \{[0, 18, 12-6], [11, 3, 12-5], [10, 7, 12-19]\}$, Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type $2^5 10^1$ for each $1 \le i \le 2$, where the group set is $\mathcal{G} = \{\{10, 19\}, \{11, 18\}, \{12, 17\}, \{13, 16\}, \{14, 15\}, \{0, 1, \dots, 9\}\}$. Consider the following permutations on X.

$$\begin{array}{lll}
\pi_0 = (0\,7\,1)(2\,4)(3\,5\,9\,6\,8), & \pi_1 = (0\,4\,1\,3\,2)(6\,9\,8\,7), & \pi_2 = (0\,9\,6\,1\,7\,2)(4\,5\,8), \\
\pi_3 = (0\,8\,4)(2\,9\,6\,7\,3), & \pi_4 = (0\,5\,9)(1\,3)(2\,7\,8), & \pi_5 = (0\,7\,4\,2\,6\,1)(5\,8), \\
\pi_6 = (2\,7)(3\,4\,5\,8), & \pi_7 = (0\,1\,6\,7\,3)(2\,5), & \pi_8 = (0\,9\,7\,1\,6)(3\,4), \\
\pi_9 = (0\,4\,3\,1\,5\,9), & \pi_{10} = (1\,3)(2\,6\,8), & \pi_{11} = (0\,7)(1\,9\,3\,5), \\
\pi_{12} = (0\,1\,7\,6\,3), & \pi_{13} = (0\,7\,5)(1\,4), & \pi_{14} = (0\,8\,5\,2\,9), \\
\pi_{15} = (0\,5\,6\,4\,7), & \pi_{16} = (0\,9\,2\,6), & \pi_{17} = (0\,5\,2\,4), \\
\pi_{18} = (0\,5\,6\,7), & \pi_{19} = (0\,1\,9), & \pi_{20} = (5\,8\,7), \\
\pi_{21} = (0\,5\,4), & \pi_{22} = (4\,6\,7), & \pi_{23} = (1\,6),
\end{array}$$

$$\begin{array}{lll}
\pi_{24} = (1\,5), & \pi_{25} = (8\,9), & \pi_{26} = (0\,5), \\
\pi_{27} = (2\,8), & \pi_{32} = \pi_{35} = (1). &
\end{array}$$

We have that for each $s \in M_{10} \setminus \{32\}$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. For $s = 32$, $|\pi_{32} \mathcal{B}_2 \cap \mathcal{B}_1| = 32$ and $\pi_{32} \mathcal{G} = \mathcal{G}$. □

Lemma 3.9 *Let $M_{11} = \{0, 1, \ldots, 29, 40\}$ and $s \in M_{11}$. Then there is a pair of kite-GDDs of type $8^2 6^1$ with the same group set, which intersect in s blocks.*

Proof Take the vertex set $X = \{0, 1, \ldots, 21\}$. Let $\mathcal{B}$:

$$\begin{array}{lllll} [0,16,8-6], & [1,17,8-7], & [3,18,8-2], & [1,16,9-3], & [4,20,9-6], \\ [5,19,9-7], & [6,19,10-0], & [21,4,10-5], & [2,16,11-6], & [3,19,11-7], \\ [4,18,11-5], & [18,6,15-2], & [7,20,15-4], & [19,0,15-3], & [17,5,15-1], \\ [0,11,20-2], & [13,5,20-6], & [1,12,20-8], & [3,10,20-14], & [14,2,21-6], \\ [2,19,12-6], & [4,17,12-0], & [21,7,12-5], & [3,17,13-1], & [4,16,13-6], \\ [7,18,13-2], & [5,16,14-4], & [6,17,14-3], & [0,18,14-7], & [7,10,16-15], \\ [3,12,16-6], & [0,9,17-7], & [2,10,17-11], & [1,10,18-5], & [2,9,18-12], \\ [4,8,19-13], & [14,1,19-7], & [0,13,21-3], & [1,11,21-9], & [5,8,21-15]. \end{array}$$

Then $(X, \mathcal{G}, \mathcal{B})$ is a kite-GDD of type $8^2 6^1$, where the group set is $\mathcal{G} = \{\{0, 1, \ldots, 7\}, \{8, 9, \ldots, 15\}, \{16, 17, \ldots, 21\}\}$. Consider the following permutations on X.

$$\begin{array}{llll} \pi_0 = (0\ 6\ 2\ 7\ 3)(1\ 5\ 4), & \pi_1 = (0\ 6\ 7\ 1\ 2)(4\ 5), & \pi_2 = (0\ 2\ 5\ 6\ 3\ 7\ 4), & \pi_3 = (1\ 6\ 4\ 7)(3\ 5), \\ \pi_4 = (1\ 6\ 7)(2\ 4), & \pi_5 = (0\ 4)(2\ 6\ 7\ 3), & \pi_6 = (0\ 2\ 4\ 3\ 7), & \pi_7 = (2\ 5)(3\ 6\ 4), \\ \pi_8 = (0\ 4\ 3)(1\ 7), & \pi_9 = (0\ 6\ 2\ 5\ 4), & \pi_{10} = (0\ 7)(2\ 5), & \pi_{11} = (3\ 5\ 7\ 4), \\ \pi_{12} = (2\ 6\ 4\ 3), & \pi_{13} = (0\ 1)(4\ 5), & \pi_{14} = (0\ 4\ 5\ 7), & \pi_{15} = (0\ 3\ 6\ 2), \\ \pi_{16} = (0\ 1\ 2), & \pi_{17} = (2\ 5\ 4), & \pi_{18} = (0\ 4\ 6), & \pi_{19} = (0\ 6\ 2), \\ \pi_{20} = (1\ 4\ 5), & \pi_{21} = (1\ 6), & \pi_{22} = (3\ 6), & \pi_{23} = (5\ 7), \\ \pi_{24} = (0\ 7), & \pi_{25} = (1\ 5), & \pi_{26} = (4\ 5), & \pi_{27} = (17\ 20), \\ \pi_{28} = (8\ 9), & \pi_{29} = (9\ 11), & \pi_{40} = (1). & \end{array}$$

We have that for each $s \in M_{11}$, $|\pi_s \mathcal{B} \cap \mathcal{B}| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. □

Lemma 3.10 *For integer* $8 \le u \le 10$, $J(u) = I(u)$.

Proof Obviously, $J(u) \subseteq I(u)$. We need to show that $I(u) \subseteq J(u)$. For $8 \le u \le 10$, take the corresponding M_u from Lemmas 3.6 – 3.8. Let $\alpha_u \in M_u, u = 8, 9, 10$. Let

$$h_u = \begin{cases} 8, & \text{if } u = 8, \\ 8, & \text{if } u = 9, \\ 10, & \text{if } u = 10. \end{cases}$$

By Lemmas 3.6–3.8, there is a pair of kite-GDDs of type $2^{u-\frac{h_u}{2}} {h_u}^1$ $(X, \mathcal{B}_1^{(2u)})$ and $(X, \mathcal{B}_2^{(2u)})$, which intersect in α_u blocks. Here the subgraph K_{h_u} is constructed on $Y \subset X$. Let $\beta_u \in I(h_u)$. By Lemmas 3.2 and 3.3, there is a pair of kite-GDDs of type $2^{\frac{h_u}{2}}$, $(Y, \mathcal{B}_1'^{(h_u)})$ and $(Y, \mathcal{B}_2'^{(h_u)})$ with β_u common blocks. Then $(X, \mathcal{B}_1^{(2u)} \cup \mathcal{B}_1'^{(h_u)})$ and $(X, \mathcal{B}_2^{(2u)} \cup \mathcal{B}_2'^{(h_u)})$ are both kite-GDDs of type 2^u with $\alpha_u + \beta_u$ common blocks. Thus we have

$$J(u) \supseteq \{(\alpha_u + \beta_u : \alpha_u \in M_u, \beta_u \in I(h_u)\}.$$

It is readily checked that for any integer $s \in I(u)$, we have $s \in J(u)$. □

Lemma 3.11 *For integer $u = 11$, $J(11) = I(11)$.*

Proof Take the same set M_{11} as in Lemma 3.9. Let $\alpha \in M_{11}$. Then there is a pair of kite-GDDs of type $8^2 6^1$ with the same group set, which intersect in α blocks. Let γ_1 and $\gamma_2 \in I(4)$. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 intersecting in γ_i common blocks for each $i = 1, 2$. Let $\gamma_3 \in I(3)$. By Lemma 3.1, there is a pair of kite-GDDs of type 2^3 with γ_3 common blocks. Now applying Construction 2.2, we obtain a pair of kite-GDDs of type 2^{11} with $\alpha + \sum_{i=1}^{3} \gamma_i$ common blocks. Thus we have

$$J(11) \supseteq \{\alpha + \sum_{i=1}^{3} \gamma_i : \alpha \in M_{11}, \gamma_1, \gamma_2 \in I(4), \gamma_3 \in I(3)\} = I(11). \quad \square$$

Lemma 3.12 *Let $\bar{J}(3) = \{s : \exists$ a pair of kite-GDDs of type 4^3 intersecting in s blocks$\}$, $\bar{J}(3) = \{0, 1, \ldots, 10, 12\}$.*

Proof Take the vertex set $X = \{0, 1, \ldots, 11\}$. Let $\mathcal{B}_1 = \{[9,3,10-7], [8,2,10-6], [2,4,6-3], [6,5,1-10], [11,7,1-8], [0,6,11-8], [4,8,3-11], [5,8,0-10], [1,4,9-5], [7,4,0-9], [3,7,5-2], [9,11,2-7]\}$. $\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[9,3,10-7], [8,2,10-6]\}) \cup \{[9,3,10-6], [8,2,10-7]\}$, $\mathcal{B}_3 = (\mathcal{B}_1 \setminus \{[9,3,10-7], [8,2,10-6], [2,4,6-3]\}) \cup \{[9,10,3-6], [8,2,10-7], [2,4,6-10]\}$, $\mathcal{B}_4 = (\mathcal{B}_2 \setminus \{[6,5,1-10], [11,7,1-8]\}) \cup \{[6,5,1-8], [11,7,1-10]\}$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus \{[6,5,1-10], [11,7,1-8]\}) \cup \{[6,5,1-8], [11,7,1-10]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 4^3 for $i = 1, 2, 3, 4, 5$, where $\mathcal{G} = \{\{0,1,2,3\}, \{4,5,10,11\}, \{6,7,8,9\}\}$. Consider the following permutations on X.

$\pi_0 = (2\ 3)(4\ 11\ 5)(6\ 8\ 9\ 7)$, $\pi_1 = (0\ 1\ 2\ 3)(4\ 11)(6\ 7)(8\ 9)$, $\pi_2 = (0\ 3)(1\ 2)(4\ 5)(6\ 9\ 7)(10\ 11)$,
$\pi_3 = (6\ 8)(10\ 11)$, $\pi_4 = (0\ 2)(1\ 3)(4\ 5)(6\ 8)(10\ 11)$, $\pi_5 = (4\ 5)$,
$\pi_6 = (5\ 10)$, $\pi_7 = \pi_8 = \pi_9 = \pi_{10} = \pi_{12} = (1)$.

We have that for each $s \in \{0, 1, \ldots, 6, 12\}$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. For each $s \in \{7, 8, 9, 10\}$, $|\pi_s \mathcal{B}_{12-s} \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. $\square$

Lemma 3.13 *For integer $u = 18, 19$, $J(u) = I(u)$.*

Proof Start from a 3-GDD of type 3^3 from Lemma 2.3. Give each point of the GDD weight 4. By Lemma 3.12, there is a pair of kite-GDDs of type 4^3 with α common blocks, $\alpha \in \{0, 1, \ldots, 10, 12\}$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^3 with $\sum_{i=1}^{9} \alpha_i$ common blocks, where 9 is the number of blocks of the 3-GDD of type 3^3 and $\alpha_i \in \{0, 1, \ldots, 10, 12\}$ for $1 \le i \le 9$. Now for each $1 \le j \le 3$, fill in the j-th group of the resulting kite-GDDs of type 12^3 with a pair of kite-GDDs of type 2^6 with β_j common blocks, $\beta_j \in I(6)$, which exist by Lemma 3.4. By Construction 2.2 we have a pair of kite-GDDs of type 2^{18} with $\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{3} \beta_j$ common blocks, which implies

$$\begin{aligned} J(18) &\supseteq \Big\{ \sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{3} \beta_j : \alpha_i \in \{0, 1, \ldots, 10, 12\}, \beta_j \in I(6), 1 \le i \le 9, 1 \le j \le 3 \Big\} \\ &= 9 * \{0, 1, \ldots, 10, 12\} + 3 * \{0, 1, \ldots, 13, 15\} = I(18). \end{aligned}$$

Now for each $1 \le j \le 3$, fill in the j-th group of the resulting kite-GDDs of type 12^3 with a pair of kite-GDDs of type 2^7 with β_j common blocks, $\beta_j \in I(7)$, which exist by Lemma 3.5. By Construction 2.2 we have a pair of kite-GDDs of type 2^{19} with $\sum_{i=1}^{9}\alpha_i + \sum_{j=1}^{3}\beta_j$ common blocks, which implies

$$J(19) \supseteq \Big\{ \sum_{i=1}^{9}\alpha_i + \sum_{j=1}^{3}\beta_j : \alpha_i \in \{0,1,\ldots,10,12\}, \beta_j \in I(7), 1 \le i \le 9, 1 \le j \le 3 \Big\}$$
$$= 9 * \{0,1,\ldots,10,12\} + 3 * \{0,1,\ldots,19,21\} = I(19). \quad \square$$

Lemma 3.14 *For integer $u = 20$, $J(u) = I(u)$.*

Proof Start from a 4-GDD of type 5^4 from Lemma 2.3. Give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 10^4 with $\sum_{i=1}^{25}\alpha_i$ common blocks, where 25 is the number of blocks of the 4-GDD of type 5^4 and $\alpha_i \in I(4)$ for $1 \le i \le 25$. Now for each $1 \le j \le 4$, fill in the j-th group of the resulting kite-GDDs of type 10^4 with a pair of kite-GDDs of type 2^5 with β_j common blocks, $\beta_j \in I(5)$, which exist by Lemma 3.3. By Construction 2.2 we have a pair of kite-GDDs of type 2^{20} with $\sum_{i=1}^{25}\alpha_i + \sum_{j=1}^{4}\beta_j$ common blocks, which implies

$$J(20) \supseteq \Big\{ \sum_{i=1}^{25}\alpha_i + \sum_{j=1}^{4}\beta_j : \alpha_i \in I(4), \beta_j \in I(5), 1 \le i \le 25, 1 \le j \le 4 \Big\}$$
$$= 25 * \{0,1,\ldots,4,6\} + 4 * \{0,1,\ldots,8,10\} = I(20). \quad \square$$

Lemma 3.15 *Let $A_{10} = \{0,1,2,3,4,9\}$ and $s \in A_{10}$. Then there is a pair of kite-GDDs of type $2^3 4^1$ with the same group set, which intersect in s blocks.*

Proof Take the vertex set $X = \{0,1,\ldots,9\}$. Let $\mathcal{B} = \{[6,1,0-7],[1,2,7-5],[2,3,8-1],[4,9,3-7],[5,0,4-8],[9,5,1-3],[9,2,0-8],[6,3,5-8],[2,6,4-7]\}$. Then $(X,\mathcal{G},\mathcal{B})$ is a kite-GDD of type $2^3 4^1$, where the group set is $\mathcal{G} = \{\{0,3\},\{1,4\},\{2,5\},\{6,7,8,9\}\}$. Consider the following permutations on X.

$$\pi_0 = (2\,5)(6\,9\,7), \quad \pi_1 = (0\,3)(6\,8\,9), \quad \pi_2 = (0\,3),$$
$$\pi_3 = (8\,9), \quad \pi_4 = (6\,7), \quad \pi_9 = (1).$$

We have that for each $s \in A_{10}$, $|\pi_s\mathcal{B} \cap \mathcal{B}| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. $\square$

4. Working lemmas

First we need the following definition. Let s_1 and s_2 be two non-negative integers. If X and Y are two sets of non-negative integers, then $X + Y$ denotes the set $\{s_1 + s_2 : s_1 \in X, s_2 \in Y\}$. If X is a set of non-negative integers and h is some positive integer, then $h * X$ denotes the set of all non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X allowed).

Lemma 4.1 *For any integer $u \equiv 0 \pmod 3$, $u \ge 12$ and $u \ne 18$, $J(u) = I(u)$.*

Proof Let $u = 3t$ with $t \equiv 0, 1 \pmod 4$ and $t \geq 4$. Start from a 4-GDD of type 3^t from Lemma 2.3. Give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 6^t with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = 3t(t-1)/4$ is the number of blocks of the 4-GDD of type 3^t and $\alpha_i \in I(4)$ for $1 \leq i \leq b$. Now for each $1 \leq j \leq t$, fill in the j-th group of the resulting kite-GDDs of type 6^t with a pair of kite-GDDs of type 2^3 with β_j common blocks, $\beta_j \in I(3)$, which exist by Lemma 3.1. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t) &\supseteq \Big\{ \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j : \alpha_i \in I(4), \beta_j \in I(3), 1 \leq i \leq b, 1 \leq j \leq t \Big\} \\ &= b * \{0, 1, \ldots, 4, 6\} + t * \{0, 3\} = \{0, 1, \ldots, 6b-2, 6b\} + \{0, 3, 6, \ldots, 3t\} \\ &= I(3t) = I(u). \end{aligned}$$

For any $t \equiv 2, 3 \pmod 4$ and $t \geq 7$, start from a 4-GDD of type $3^{t-2}6^1$, which exists from Lemma 2.3, and give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type $6^{t-2}12^1$ with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = 3(t-2)(t+1)/4$ is the number of blocks of the 4-GDD of type $3^{t-2}6^1$ and $\alpha_i \in I(4)$, $1 \leq i \leq b$. Now for each $1 \leq j \leq t-2$, fill in the j-th group of the resulting kite-GDDs of type $6^{t-2}12^1$ with a pair of kite-GDDs of type 2^3 with β_j common blocks, $\beta_j \in I(3)$, which exist by Lemma 3.1; fill in the last group with a pair of kite-GDDs of type 2^6 with γ common blocks, $\gamma \in I(6)$, which exist by Lemma 3.4. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-2} \beta_j + \gamma$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t) &\supseteq \Big\{ \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-2} \beta_j + \gamma : \alpha_i \in I(4), \beta_j \in I(3), \gamma \in I(6), 1 \leq i \leq b, 1 \leq j \leq t-2 \Big\} \\ &= b * \{0, 1, \ldots, 4, 6\} + (t-2) * \{0, 3\} + \{0, 1, \ldots, 13, 15\} = I(3t) = I(u). \end{aligned}$$

This completes the proof. □

Lemma 4.2 *For any integer $u \equiv 1 \pmod 3$, $u \geq 13$ and $u \neq 19$, $J(u) = I(u)$.*

Proof Let $u = 3t + 1$ with $t \equiv 0, 1 \pmod 4$ and $t \geq 4$. Start from a 4-GDD of type 3^t from Lemma 2.3. Give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 6^t with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = 3t(t-1)/4$ is the number of blocks of the 4-GDD of type 3^t and $\alpha_i \in I(4)$ for $1 \leq i \leq b$. Now for each $1 \leq j \leq t$, fill in the j-th group of the resulting kite-GDDs of type 6^t with a pair of kite-GDDs of type 2^4 with β_j common blocks, $\beta_j \in I(4)$, which exist by Lemma 3.2. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t+1} with $\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t}\beta_j$ common blocks, which implies

$$J(u)=J(3t+1)\Big\{\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t}\beta_j:\alpha_i\in I(4),\beta_j\in I(4),1\le i\le b,1\le j\le t\Big\}$$
$$=b*\{0,1,\ldots,4,6\}+t*\{0,1,\ldots,4,6\}=I(3t)=I(u).$$

For any $t\equiv 2,3 \pmod 4$ and $t\ge 7$, start from a 4-GDD of type $3^{t-2}6^1$, which exists from Lemma 2.3, and give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha\in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type $6^{t-2}12^1$ with $\sum_{i=1}^{b}\alpha_i$ common blocks, where $b=3(t-2)(t+1)/4$ is the number of blocks of the 4-GDD of type $3^{t-2}6^1$ and $\alpha_i\in I(4)$, $1\le i\le b$. Now for each $1\le j\le t-2$, fill in the j-th group of the resulting kite-GDDs of type $6^{t-2}12^1$ with a pair of kite-GDDs of type 2^4 with β_j common blocks, $\beta_j\in I(4)$, which exist by Lemma 3.2; fill in the last group with a pair of kite-GDDs of type 2^7 with γ common blocks, $\gamma\in I(7)$, which exist by Lemma 3.5. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t+1} with $\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t-2}\beta_j+\gamma$ common blocks, which implies

$$J(u)=J(3t+1)\supseteq\Big\{\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t-2}\beta_j+\gamma:\alpha_i\in I(4),\beta_j\in I(4),\gamma\in I(7),1\le i\le b,1\le j\le t-2\Big\}$$
$$=b*\{0,1,\ldots,4,6\}+(t-2)*\{0,1,\ldots,4,6\}+\{0,1,\ldots,19,21\}=I(3t+1)=I(u).$$

This completes the proof. □

Lemma 4.3 *For any integer $u\equiv 2 \pmod 3$, $u\ge 14$ and $u\ne 20$, $J(u)=I(u)$.*

Proof Let $u=3t+2$ with $t\equiv 0,1 \pmod 4$ and $t\ge 4$. Start from a 4-GDD of type 3^t from Lemma 2.3. Give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha\in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 6^t with $\sum_{i=1}^{b}\alpha_i$ common blocks, where $b=3t(t-1)/4$ is the number of blocks of the 4-GDD of type 3^t and $\alpha_i\in I(4)$ for $1\le i\le b$. Now for each $1\le j\le t-1$, fill in the j-th group of the resulting kite-GDDs of type 6^t with a pair of kite-GDDs of type 2^34^1 with β_j common blocks, $\beta_j\in A(10)$, which exist by Lemma 3.15; fill in the last group with a pair of kite-GDDs of type 2^5 with γ common blocks, $\gamma\in I(5)$, which exist by Lemma 3.3. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t+2} with $\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t-1}\beta_j+\gamma$ common blocks, which implies

$$J(u)=J(3t+2)\supseteq\Big\{\sum_{i=1}^{b}\alpha_i+\sum_{j=1}^{t-1}\beta_j+\gamma:\alpha_i\in I(4),\beta_j\in A(10),\gamma\in I(5),1\le i\le b,1\le j\le t-1\Big\}$$
$$=b*\{0,1,\ldots,4,6\}+(t-1)*\{0,1,2,3,4,9\}+\{0,1,\ldots,8,10\}=I(3t+2)=I(u).$$

For any $t\equiv 2,3 \pmod 4$ and $t\ge 7$, start from a 4-GDD of type $3^{t-2}6^1$, which exists from Lemma 2.3, and give each point of the GDD weight 2. By Lemma 3.2, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha\in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type $6^{t-2}12^1$ with $\sum_{i=1}^{b}\alpha_i$ common blocks, where $b=3(t-2)(t+1)/4$

is the number of blocks of the 4-GDD of type $3^{t-2}6^1$ and $\alpha_i \in I(4)$, $1 \le i \le b$. Now for each $1 \le j \le t-2$, fill in the j-th group of the resulting kite-GDDs of type $6^{t-2}12^1$ with a pair of kite-GDDs of type $2^3 4^1$ with β_j common blocks, $\beta_j \in A(10)$, which exist by Lemma 3.15; fill in the last group with a pair of kite-GDDs of type 2^8 with γ common blocks, $\gamma \in I(8)$, which exist by Lemma 3.10. By Construction 2.2 we have a pair of kite-GDDs of type 2^{3t+2} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-2} \beta_j + \gamma$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t+2) &\supseteq \Big\{ \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-2} \beta_j + \gamma : \alpha_i \in I(4), \beta_j \in A(10), \gamma \in I(8), 1 \le i \le b, 1 \le j \le t-2 \Big\} \\ &= b * \{0, 1, \ldots, 4, 6\} + (t-2) * \{0, 1, 2, 3, 4, 9\} + \{0, 1, \ldots, 26, 28\} \\ &= I(3t+2) = I(u). \end{aligned}$$

This completes the proof. □

5. Conclusion

We prove Theorem 1.1.

Proof of Theorem 1.1 When $u \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 19, 20\}$, the conclusion follows from Lemmas 3.1–3.5, 3.10, 3.11, 3.13 and 3.14. When $u \ge 12$, combining the results of Lemmas 4.1–4.3, we complete the proof. □

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