Journal of Mathematical Research with Applications
Nov., 2021, Vol. 41, No. 6, pp. 565–578
DOI:10.3770/j.issn:2095-2651.2021.06.002
Http://jmre.dlut.edu.cn

Unicyclic Graphs with Five Laplacian Eigenvalues Different from 0 and 1

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Abstract Let U be a unicyclic graph of order n, and $m_U(1)$ the multiplicity of Laplacian eigenvalue 1 of U. It is well-known that 0 is a simple Laplacian eigenvalue of connected graph. This means that if U has five Laplacian eigenvalues different from 0 and 1, then $m_U(1) = n-6$. In this paper, we completely characterize all the unicyclic graphs with $m_U(1) = n-6$.

Keywords unicyclic graph; Laplacian eigenvalue; multiplicity

MR(2020) Subject Classification 05C50

1. Introduction

Throughout this paper we consider finite undirected simple graphs of order n. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $A(G)$ be the adjacency matrix of G. We denote by $d(v_i)$ the degree of v_i in G. Let $D(G)$ be the diagonal matrix of the degrees of G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric, positive semidefinite matrix. It is not difficult to find that the row sum of $L(G)$ is 0, and so the smallest eigenvalue is equivalent to 0. For convenience, we always assume that the Laplacian eigenvalues of G are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. In [1], it is well known that $\mu_{n-1} > 0$ if and only if G is connected and hence is called algebraic connectivity of G. The multiplicity of μ_i is denoted by $m_G(\mu)$, and the number of Laplacian eigenvalues in an interval I is denoted by $m_G I$. The Laplacian spectrum of G is a multiple set of Laplacian eigenvalues together with their multiplicities. We denote $\mathrm{Spec}_L(G) = \{\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_r^{k_r}\}$ where $\mu_1, \mu_2, \ldots, \mu_{r-1}$ and μ_r are r distinct Laplacian eigenvalues and $m_G(\mu_i) = k_i$ is the multiplicity of μ_i $(1 \leq i \leq r)$ and $\sum_{i=1}^{r} k_i = n$.

For a graph G of order n, a vertex of degree one is called a pendant vertex, and we write $p(G)$ for the number of pendant vertices of G. A vertex of G is quasipendant vertex if it is adjacent to a pendant vertex, and we write $q(G)$ for the number of quasipendant vertices of G. Let $r(G)$ be the number of inner vertices of G. Let

$$V_P = \{v \in V(G) \mid v \text{ is a pendent vertex }\},$$

Received September 4, 2020; Accepted January 3, 2021
Supported by the National Natural Science Foundation of China (Grant No. 11961041) and Excellent postgraduates of Gansu Provincial Department of Education "Star of innovation" Foundation (Grant No. 2021CXZX-594).
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$$V_Q = \{v \in V(G) \mid v \text{ is a quasipendant vertex }\}$$

and

$$V_R = V(G) \setminus (V_P \cup V_Q).$$

Then V_R is the set of the inner vertices of G which are not pendent vertices and quasipendant vertices. Obviously, $|V_P| = p(G)$, $|V_Q| = q(G)$ and $|V_R| = r(G) = n - p(G) - q(G)$. Let $L_R(G)$ be the principal submatrix of $L(G) - E_n$ that corresponds to the inner vertices of G, where E_n is an identity matrix of order n. The nullity of $L_R(G)$ denoted by $\nu(L_R(G))$, and C_n $(n \geq 3)$ always represents the cycle. The diameter of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any two vertices of G. Meanwhile, the girth of G, denoted by g, is the length of the shortest cycle in G. A unicyclic graph is a connected graph with the same number of edges and vertices. We denote by $\mathcal{U}(n, g)$ the set of all connected unicyclic graphs with girth g $(g \geq 3)$ on n vertices. For graph theoretic notations and terminologies not defined here, we refer the readers to [2].

In the past two decades, connected graphs with few distinct eigenvalues have been investigated for several graph matrices since such graphs always have pretty combinatorial properties. This problem was perhaps first raised by Doob [3]. Since then, a lot of publications (see [4–10]) have focused on graphs with fewer eigenvalues. In fact, a graph with fewer eigenvalues means that it has large multiplicity on some eigenvalues. Thus, characterizing graphs with largest multiplicity of eigenvalues is important for graphs with few distinct eigenvalues.

The multiplicity of Laplacian eigenvalue of graphs has attracted plenty of attention. Faria [11] proved that $m_G(1)$ is bounded by $p(G)-q(G)$, that is, $m_G(1) \geq p(G)-q(G)$, which is also called Faria's inequality. Then Andrade et al. [12] presented a unified approach on the Faria's inequality for the Laplacian and signless Laplacian spectra. In [13], it was shown that for a tree T with order n, if an integer $\lambda > 1$ is a Laplacian eigenvalue of T, then $m_T(\lambda) = 1$ and λ divides n. Additionally, Guo, Feng and Zhang [14] characterized all trees with $n - 6 \leq m_T(1) \leq n$. Also Barik, Lal and Pati [15] investigated the multiplicities of Laplacian eigenvalue 1 of a graph.

Base on above, we consider $m_U(1) = n - 6$ for a unicyclic graph U on $n \geq 7$ vertices in this paper, i.e., the unicyclic graph has five Laplcian eigenvalues different from 0 and 1 since 0 is a simple Laplacian eigenvalue of a connected graph, and obtain the following result:

Theorem 1.1 *Let U be a unicyclic graph on $n \geq 7$ vertices. Then $m_U(1) = n - 6$ if and only if U is isomorphic to one of $H_3^1(a, b, c)$ $(a \geq 1, b \geq 1, c \geq 1)$, $H_3^2(a, c)$ $(a \geq 0, c \geq 1)$, $H_3^3(a, c)$ $(a \geq 1, c \geq 1)$, $H_3^4(a, b)$ $(a \geq 0, b \geq 1)$, $H_3^5(a, b)$ $(a \geq 1, b \geq 1)$, $H_4^1(a, b)$ $(a \geq 1, b \geq 1)$, $H_4^2(a, b)$ $(a \geq 0, b \geq 1)$, $H_5^1(a, c)$ $(a \geq 0, c \geq 0)$, and $H_6^1(a, b)$ $(a \geq 0, b \geq 0)$. All of these graphs are shown in Figure 1.*

Moreover, we also present some unicyclic graphs with $m_U(1) = n - 7$ (see Lemma 3.1).

2. Preliminaries

In this section, we introduce some lemmas which will be useful for the proof of main results.

Lemma 2.1 ([2]) *If e is an edge of the graph G and $G' = G - e$, then*

$$\mu_1(G) \geq \mu_1(G') \geq \cdots \geq \mu_{n-1}(G) \geq \mu_{n-1}(G') \geq \mu_n(G) = \mu_n(G') = 0.$$

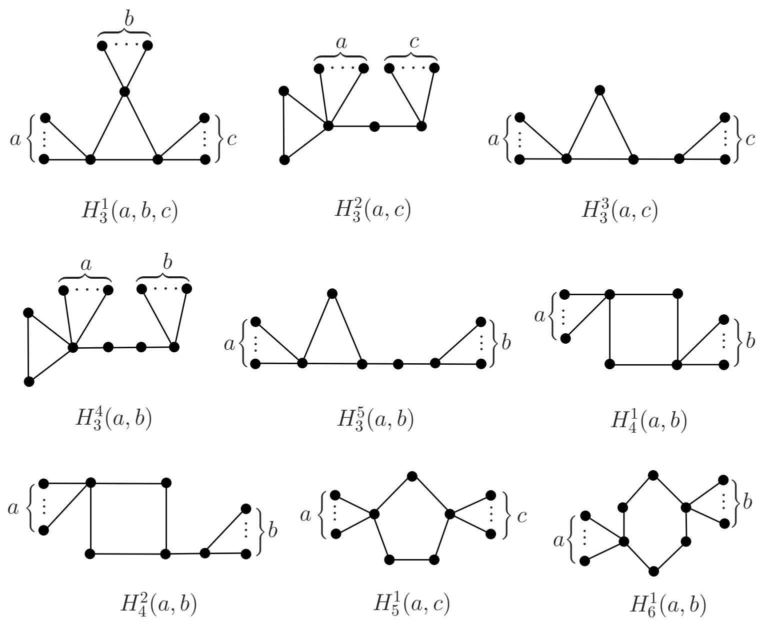

Figure 1 Graphs $H_3^1 \sim H_6^1$

Lemma 2.2 ([13]) *Let v be a pendant vertex of $\tilde{G}$ and let $G = \tilde{G}\backslash v$. Then the Laplacian eigenvalues of G interlace the Laplacian eigenvalues of $\tilde{G}$.*

Lemma 2.3 ([13]) *Let G be a connected graph with p pendant vertices and q quasipendant vertices. Then $m_G(1) = p - q + \nu(L_R(G))$.*

Lemma 2.4 ([16]) *Let C_n be a cycle of order n. Then $Spec_L(C_n) = \{2 - 2\cos\frac{2\pi j}{n} | j = 0, 1, \ldots, n-1\}$.*

Lemma 2.5 ([17]) *If G is a connected graph with a cutpoint v, then $\mu_{n-1}(G) \leq 1$, where equality holds if and only if v is adjacent to every vertex of G.*

Let $Gu : vH$ be the graph obtained from G and H by joining a vertex u of G to a vertex v of H. In particular, if $H = P_2(= vw)$, we denote by $Gu : vw$ for short.

Lemma 2.6 ([18]) *Let H be a graph, and S_k a star on $k \geq 3$ vertices. Set $G = Hu : vS_k$.*

(1) If v is a pendant vertex of S_k, then $m_G(1) = m_H(1) + k - 3$;

(2) If v is the center of S_k, then $m_G(1) = m_{Hu:vw}(1) + k - 2$.

Lemma 2.7 ([12]) *Let G be a connected graph on n vertices. If $r(G) = 0$, that is, any internal vertex of G is also a quasi-pendant vertex, then $m_G(1) = p(G) - q(G)$.*

Lemma 2.8 *Let D_n be the following determinant of order n.*

$$D_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}_{n\times n}$$

Then

$$\det D_n = \begin{cases} (-1)^{\frac{n}{3}}, & \textit{if } n \equiv 0 \pmod 3; \\ (-1)^{\frac{n-1}{3}}, & \textit{if } n \equiv 1 \pmod 3; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof Applying Laplacian Expansion Theorem in the first column of D_n yields

$$D_n = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}_{(n-1)\times(n-1)} + \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= D_{n-1} - D_{n-2} = -D_{n-3}$$

From the recurrence relation above we get

$$D_n = (-1)^i D_{n-3i}. \tag{2.1}$$

For a fixed n, we by induction on i prove that the (2.1) holds. When $i = 1$, $D_n = (-1)^1 D_{n-3}$. Clearly, (2.1) is true. Assume that the holds for $i < k$. If $i = k$, then

$$D_n = (-1)^{k-1} D_{n-3(k-1)} = (-1)^{k-1}(-1)D_{n-3k} = (-1)^k D_{n-3k}.$$

If $n \equiv 0$ (mod 3), then there exists a k, such that $n = 3k$, $D_n = D_{3k} = (-1)^k = (-1)^{\frac{n}{3}}$; Similarly, if $n \equiv 1 \pmod 3$, it draws $D_n = (-1)^{\frac{n-1}{3}}$; otherwise, $D_n = 0$. The proof is completed. □

Lemma 2.9 *Let M_n^m be the following determinant and $m \in R$.*

$$M_n^m = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & m \end{vmatrix}_{n\times n}$$

Then

$$M_n^m = \begin{cases} -2((-1)^{\frac{n-3}{3}}+1) & \textit{if } n \equiv 0 \pmod 3; \\ m\cdot(-1)^{\frac{n-1}{3}}-2 & \textit{if } n \equiv 1 \pmod 3; \\ (m-2)\cdot(-1)^{\frac{n-2}{3}}-2 & \textit{otherwise.} \end{cases}$$

Proof Applying Laplacian Expansion Theorem in the last column of M_n^m yields

$$\begin{aligned} M_n^m &= mD_{n-1} + (-1)^{n+n}\cdot \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}_{n-1} + (-1)^{n+2}\cdot \begin{vmatrix} -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ -1 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}_{n-1} \\ &= mD_{n-1} + (-1)^{n+1}\cdot \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{vmatrix}_{n-2} + (-1)^{2n-1}\cdot \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}_{n-2} + \\ &\quad (-1)^{2n+3}\cdot \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}_{n-2} + (-1)^{n+3}\cdot \begin{vmatrix} -1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}_{n-2} \\ &= mD_{n-1} - 2D_{n-2} - 2 \end{aligned}$$

If $n \equiv 1 \pmod 3$, then by Lemma 2.7 we get $D_{n-1} = (-1)^{\frac{n-1}{3}}$ and $D_{n-2} = 0$. Thus, $M_n^m = (-1)^{\frac{n-1}{3}}m - 2$. Similarly, if $n \equiv 2 \pmod 3$, then $D_{n-1} = D_{n-2} = (-1)^{\frac{n-2}{3}}$, it therefore follows $M_n^m = (-1)^{\frac{n-2}{3}}(m-2) - 2$; if $n \equiv 0 \pmod 3$, then $D_{n-1} = 0$ and $D_{n-2} = (-1)^{\frac{n-3}{3}}$. Hence $M_n^m = -2((-1)^{\frac{n-3}{3}} + 1)$. □

Lemma 2.10 *Let U be a unicyclic graph on $n \geq 4$ vertices. If $U \cong S_n^3$ (see Figure 2), then*

$$m_U(1) = n - 3.$$

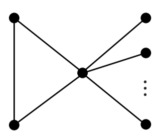

Figure 2 Graph S_n^3

Proof If $U \cong S_n^3$, one can get $p(S_n^3) = n - 3$ and $q(S_n^3) = 1$. Let $L_R(S_n^3)$ be the principal submatrix of $L(S_n^3) - E_n$ that corresponds to the inner vertices of S_n^3, then

$$\det(L_R(S_n^3)) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = D_2.$$

It follows from Lemma 2.8 that $\det(L_R(S_n^3)) = 0$, which implies $\nu(L_R(S_n^3)) = 1$. Hence, from Lemma 2.3 we get $m_{S_n^3}(1) = n - 3 - 1 + 1 = n - 3$. $\Box$

Lemma 2.11 *Let U be a unicyclic graph on $n \geq 7$ vertices. If U is isomorphic to one of those graphs $C_3^1(a,b)$ $(a \geq 1, b \geq 1)$, $C_3^2(a,b)$ $(a \geq 0, b \geq 1)$ and $C_4(a,b)$ $(a \geq 0, b \geq 0)$, shown in Figure 3, then $m_U(1) = n - 5$.*

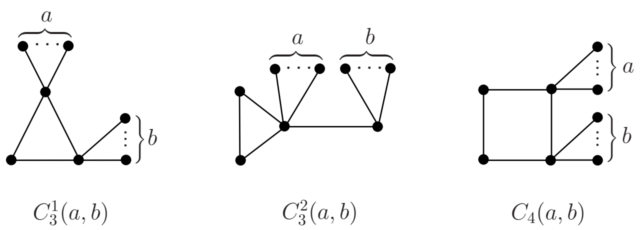

Figure 3 Graphs $C_3^1(a,b)$, $C_3^2(a,b)$ and $C_4(a,b)$

Proof If U has the form $C_3^1(a,b)$ on $n = a + b + 3$ vertices, where $a \geq 1, b \geq 1$. Clearly, $C_3^1(a,b)$ has only one inner vertex with degree 2, which implies $\det(L_R(C_3^1(a,b))) = 1$, and so $\nu(L_R(C_3^1(a,b))) = 0$. Therefore, it follows from Lemma 2.3 that $m_{C_3^1(a,b)}(1) = n - 3 - 2 = n - 5$.

If U has the form $C_3^2(a,b)$ on $n = a+b+4$ vertices, where $a \geq 1, b \geq 1$, then $\det(L_R(C_3^2(a,b))) = D_2 = 0$, it leads to $\nu(L_R(C_3^2(a,b))) = 1$. By Lemma 2.3, we have $m_{C_3^2(a,b)}(1) = n-4-2+1 = n-5$. Moreover, $a = 0$ implies $U \cong C_3^2(0,b)$ $(b \geq 1)$ with $n = b + 4$ vertices. And then, we obtain

$$\det(L_R(C_3^2(0,b))) = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = M_3^2.$$

Thus, it follows from Lemma 2.9 that $\det(L_R(C_3^2(0,b))) = -4$, which indicates $\nu(L_R(C_3^2(0,b))) = 0$. Therefore, by Lemma 2.3 we have $m_{C_3^2(0,b)}(1) = n - 4 - 1 = n - 5$. By the similar method as above, one can obtain $m_{C_4(a,b)}(1) = n - 5$ where $a \geq 0, b \geq 0$.

Sum up the above, we complete the proof. $\Box$

3. Proof of the main result

Before proving Theorem 1.1, we give some useful lemmas which needs to be used in the following.

Lemma 3.1 *Let U be a unicyclic graph on $n \geq 7$ vertices, if U is one of $U_3^2(a,b,c)$ $(a \geq 0, b \geq 1, c \geq 1)$, $U_3^3(a,b,c)$ $(a \geq 1, b \geq 1, c \geq 1)$, $U_3^4(a,b)$ $(a \geq 1, b \geq 1)$, $U_3^5(a,b,0,d)$ $(a \geq 1, b \geq 1, c \geq 0, d \geq 1)$, $U_4^1(a,b,c,0)$ $(a \geq 1, b \geq 1, c \geq 1, d \geq 0)$, $U_4^2(a,b)$ $(a \geq 1, b \geq 1)$, $U_4^3(a,b,c)$ $(a \geq 1, b \geq 0, c \geq 1)$ and $U_5^1(a,b,c)$ $(a \geq 1, b \geq 1, c \geq 0)$, then $m_U(1) = n - 7$ (see*

Figure 4).

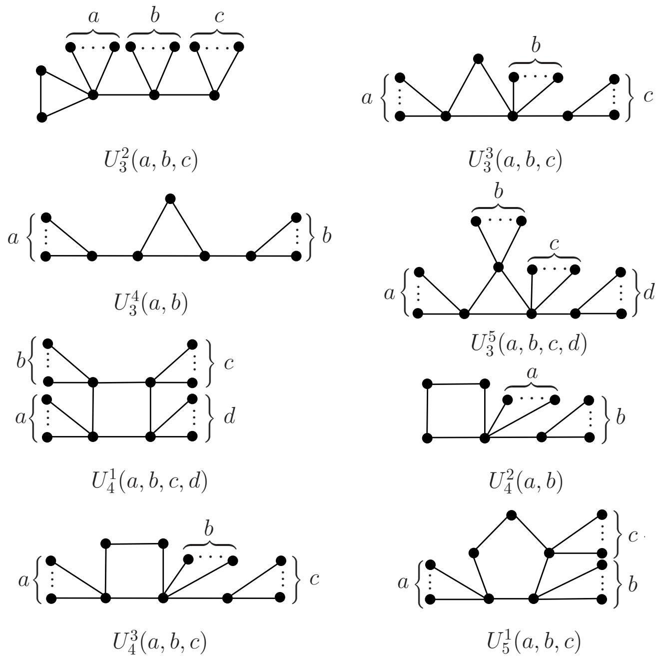

Figure 4 Some related graphs

Proof Let $U \in \mathcal{U}(n,g)$ be a unicylic graph and let $L_R(U)$ be the principal submatrix of $L(U)-E_n$ corresponding to inner vertices of U.

If $U \cong U_3^2(a,b,c)$ $(a \ge 1, b \ge 1, c \ge 1)$, then $n = a+b+c+5$ and

$$\det(L_R(U_3^2(a,b,c))) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = D_2$$

It follows from Lemma 2.8 that $\det(L_R(U_3^2(a,b,c))) = 0$, which implies $\nu(L_R(U_3^2(a,b,c))) = 1$. From Lemma 2.3 we have $m_{U_3^2(a,b,c)}(1) = a+b+c-3+\nu(L_R(U_3^2(a,b,c))) = n-7$. In addition, $a = 0$ implies $U \cong U_3^2(0,b,c)$ $(b \ge 1, c \ge 1)$ with $n = b+c+5$, then

$$\det(L_R(U_3^2(0,b,c))) = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = M_3^2.$$

It is deduced from Lemma 2.9 that $\det(L_R(U_3^2(0,b,c))) = -4$, which means $\nu(L_R(U_3^2(0,b,c))) = 0$. By Lemma 2.3 we have $m_{U_3^2(0,b,c)}(1) = b+c-2 = n-7$. Thus $m_{U_3^2(a,b,c))}(1) = n-7$ where $a \ge 0$, $b \ge 1$, $c \ge 1$. With the same argument, we can obtain $m_{U_3^4(a,b)}(1) = n-7$ where $a \ge 1$, $b \ge 1$.

If $U \cong U_3^3(a,b,c)$ $(a \ge 1, b \ge 1, c \ge 1)$, then $n = a+b+c+4$. It is easy to see that the inner vertex of $U_3^3(a,b,c)$ is the unique vertex with degree 2, and so, $\nu(L_R(U_3^3(a,b,c))) = 0$. Thus,

$m_{U_3^3(a,b,c)}(1) = a + b + c - 3 = n - 7$. It can be shown in a similar way that

$$m_{U_3^5(a,b,0,d)}(1) = a + b + d - 3 = n - 7$$

where $a \geq 1$, $b \geq 1$, $d \geq 1$ and $m_{U_4^1(a,b,c,0)} = n - 7$ where $a \geq 1$, $b \geq 1$, $c \geq 1$.

If $U \cong U_4^2(a,b)$ $(a \geq 1, b \geq 1)$, then $n = a + b + 5$ and

$$\det(L_R(U_4^2(a,b))) = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = D_3$$

It follows from Lemma 2.8 that $\det(L_R(U_4^2(a,b))) = -1$, which leads to $\nu(L_R(U_4^2(a,b))) = 0$. So we have $m_{U_4^2(a,b)}(1) = n - 7$. In the same way, we can get $U_5^1(a,b,c) = n - 7$ where $a \geq 1$, $b \geq 1$, $c \geq 0$.

If $U \cong U_4^3(a,b,c)$ $(a \geq 1, b \geq 1, c \geq 1)$, then $n = a + b + c + 5$. Clearly, we can find that $\det(L_R(U_4^3(a,b,c)))$ is D_2. Then by Lemma 2.8, one can get $\det(L_R(U_4^3(a,b,c))) = 0$, that is, $\nu(L_R(U_4^3(a,b,c))) = 1$. Therefore,

$$m_{U_4^3(a,b,c)}(1) = a + b + c - 3 + \nu(L_R(U_4^3(a,b,c))) = n - 7.$$

Moreover, $b = 0$ means $U \cong U_4^3(a,0,c)$ $(a \geq 1, c \geq 1)$ with $n = a + c + 5$ and

$$\det(L_R(U_4^3(a,0,c))) = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

By direct calculation, $\det(L_R(U_4^3(a,0,c))) = -1$, so we have $\nu(L_R(U_4^3(a,0,c))) = 0$. Thus, $m_{U_4^3(a,0,c)}(1) = n - 7$ by Lemma 2.3 again. $\square$

Lemma 3.2 *Let $U \in \mathcal{U}(n,g)$ be a unicyclic graph on $n \geq 7$ vertices. If $m_U(1) = n - 6$, then U has no P_8 as its induced subgraph.*

Proof Assume that G contains P_8 as its induced subgraph. Then by direct calculation we get

$$\mathrm{Spec}_L(P_8) = \{0, 0.1522, 0.5858, 1.2346, 2, 2.7654, 3.4142, 3.8478\}. \tag{3.1}$$

It is not difficult to find that $G_0 = P_8 \cup (n-8)K_1$ is a spanning subgraph of U, and U can be obtained from G_0 by adding $n - 7$ edges: $e_1, e_2, \ldots, e_{n-7}$ (say). Let $U_i = G_0 + \{e_1, \ldots, e_i\}$ be a spanning subgraph of U for $i = 1, 2, \ldots, n-7$. Obviously, $U \cong U_{n-7}$. Then we apply Lemma 2.1 repeatedly to get

$$\mu_j(U) \geq \mu_j(U_{n-8}) \geq \mu_j(U_{n-9}) \geq \cdots \geq \mu_j(U_1) \geq \mu_j(P_8), \tag{3.2}$$

for $j = 1, 2, \ldots, 8$.

By (3.1) we know that $m_{P_8}(1,n] = 5$, it therefore follows from (3.2) that $m_U(1,n] \geq 5$. In addition, by Lemma 2.5 we get $\mu_{n-1}(U) < 1$, and together with $m_U(0) = 1$ we have $m_U[0,1) \geq 2$. Thus,

$$m_U(1) = n - m_U[0,1) - m_U(1,n] \leq n - 7$$

which contradicts $m_U(1) = n - 6$, and so we complete the proof. □

By Lemma 2.4 we can obtain that $m_{C_n}(1) \le 2$, and if 1 is an eigenvalue of C_n, it implies that $2 - 2\cos\frac{2\pi j}{n} = 1$ for some j. Then we can deduce $\cos\frac{2\pi j}{n} = \frac{1}{2}$, and further get $j = \frac{n}{6}$ or $j = \frac{5n}{6}$, i.e., $n = 6t$ for $t \ge 1$. Hence, $m_{C_n}(1) = 2$ if and only if $n = 6t$ for $t \ge 1$.

It is worth mentioning that if U is a cycle, then $m_U(1) = n - 6$, then $n - 6 = 2$, i.e., $n = 8$, but $6 \nmid 8$, a contradiction. So we always suppose that $U \ncong C_n$ in what follows.

Let $U \in \mathcal{U}(n, g)$ be a unicyclic graph with $m_U(1) = n - 6$. If either $\mathrm{diam}(U) \ge 6$ or $g \ge 7$, then U must contain P_8 as its induced subgraph. Thus, from Lemma 3.2 we have the following corollary.

Corollary 3.3 *Let $U \in \mathcal{U}(n, g)$ be a unicyclic graph on $n \ge 7$ vertices. If $m_U(1) = n - 6$, then $\mathrm{diam}(U) \le 5$ and $g \le 6$.*

Lemma 3.4 *Let $U \in \mathcal{U}(n, 3)$ be a unicyclic graph on $n \ge 7$ vertices. Then $m_U(1) = n-6$ if and only if U is one of $H_3^1(a,b,c)$ $(a \ge 1, b \ge 1, c \ge 1)$, $H_3^2(a,c)$ $(a \ge 0, c \ge 1)$, $H_3^3(a,c)$ $(a \ge 1, c \ge 1)$, $H_3^4(a,b)$ $(a \ge 0, b \ge 1)$ and $H_3^5(a,b)$ $(a \ge 1, b \ge 1)$ (see Figure 1).*

Proof Let $U \in \mathcal{U}(n, 3)$. Then $\mathrm{diam}(U) \le 5$ by Corollary 3.3. For the sake of clarity, we here discuss it by the diameter of U below.

Case 1. $\mathrm{diam}(U) \le 3$.

When $\mathrm{diam}(U) = 2$, one can find that $U \cong S_n^3$, it is clearly impossible since $m_{S_n^3}(1) = n - 3$ by Lemma 2.10; when $\mathrm{diam}(U) = 3$, U has one of forms $C_3^2(a,b)$ $(a \ge 0, b \ge 1)$ and $H_3^1(a,b,c)$ with $n = a+b+c+3$. Then by Lemma 2.11, we can see that $m_{C_3^2(a,b)}(1) = n - 5$, which is a contradiction. Hence, U has just the form of $H_3^1(a,b,c)$ (see Figure 1). Clearly, $a \ge 1$, $b \ge 1$ and $c \ge 1$ since if one or two of a, b, c equal(s) zero, then U has one of the forms S_n^3 and $C_3^1(a,b)$. From Lemmas 2.10 and 2.11, it is also impossible. We notice that $H_3^1(a,b,c)$ has no inner vertex. So, it follows from Lemma 2.7 that

$$m_{H_3^1(a,b,c)}(1) = p(H_3^1(a,b,c)) - q(H_3^1(a,b,c)) = a + b + c - 3 = n - 6.$$

Case 2. $\mathrm{diam}(U) = 4$.

When $\mathrm{diam}(U) = 4$, assume that U contains I_1 as its induced subgraph (see Figure 5). Then by simple computation we get

$$\mathrm{Spec}_L(I_1) = \{0, 0.3065,\ 0.3820,\ 1.6703,\ 2.6180,\ 3,\ 3.3297,\ 4.6935\}.$$

By the same reasoning as P_8, we get $m_U[0,1) \ge 2$ and $m_U(1,n] \ge 5$. It follows

$$m_U(1) = n - m_{U[0,1)} - m_U(1,n] \le n - 7$$

it is a contradiction. For the same reasoning as I_1, one can prove that U does not contain I_2 as its induced subgraph from Table 1. Thus, U has one of forms $U_3^2(a,b,c)$ $(a \ge 0, b \ge 0, c \ge 1)$, $U_3^3(a,b,c)$ $(a \ge 1, b \ge 0, c \ge 1)$ and $U_3^5(a,b,c,d)$ $(a \ge 1, b \ge 1, c \ge 0, d \ge 1)$. According to Lemma 3.1, one can easily get $m_{U_3^2(a,b,c)}(1) = n - 7$ with $a \ge 0, b \ge 1$ and $c \ge 1$, $m_{U_3^3(a,b,c)}(1) = n - 7$ with $a \ge 1, b \ge 1$ and $c \ge 1$, and $m_{U_3^5(a,b,0,d)}(1) = n - 7$ with $a \ge 1, b \ge 1$ and $d \ge 1$, which are all impossible since $m_U(1)=n-6$. Furthermore, if $U\cong U_3^5(a,b,c,d)$ with $a\ge 1,b\ge 1,c\ge 1$ and $d\ge 1$, then $m_{U_3^5(a,b,c,d)}(1)=n-8$ by Lemma 2.3, it is also impossible. Consequently, U has one of the forms $U_3^2(a,0,c)$ with $a\ge 0,c\ge 1$ and $U_3^3(a,0,c)$ with $a\ge 1,c\ge 1$.

For convenience, we denote by $U_3^2(a,0,c)=H_3^2(a,c)$ and $U_3^3(a,0,c)=H_3^3(a,c)$ (see Figure 1) now. For the graph $H_3^2(a,c)$ $(a\ge 1,c\ge 1)$, it can be obtained from S_{a+3}^3 and S_{c+2} by joining the center of S_{a+3}^3 to a pendant vertex of S_{c+2}. Thus, it follows from Lemma 2.6 (1) that $m_{H_3^2(a,c)}(1)=m_{S_{a+3}^3}(1)+c+2-3=n-6$. Moreover, $a=0$ implies $U\cong H_3^2(0,c)$, it can be obtained from C_3 and S_{c+2} by joining an arbitrary vertex of C_3 to a pendant vertex of S_{c+2}. It therefore follows from Lemma 2.6 (1) that $m_{H_3^2(0,c)}(1)=c-1=n-6$. For the graph $H_3^3(a,c)$, it is deduced from Lemma 2.3 that $m_{H_3^3(a,c)}(1)=n-6$.

Case 3. $\mathrm{diam}(U)=5$.

If $\mathrm{diam}(U)=5$, by similar reasoning as I_1, one can prove that U does not contain I_3, I_4, I_5, I_6, I_7, I_8 and I_9 as its induced subgraphs in terms of Table 1. Therefore, U has one of forms $H_3^4(a,b)$ with $a\ge 0$, $b\ge 1$, $H_3^5(a,b)$ with $a\ge 1$, $b\ge 1$ and $U_3^4(a,b)$ with $a\ge 1$, $b\ge 1$.

If $U\cong U_3^4(a,b)$ $(a\ge 1,b\ge 1)$, it follows from Lemma 3.1 that $m_{U_3^4}(1)=n-7$, which contradicts $m_U(1)=n-6$. Further, if $U\cong H_3^4(a,b)$ $(a\ge 1,b\ge 1)$, then $n=a+b+6$ and

$$L_R(H_3^4(a,b))=\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}$$

By direct calculation, we can get $\nu(L_R(H_3^4(a,b)))=2$. It follows from Lemma 2.3 that

$$m_{H_3^4(a,b)}(1)=a+b-2+\nu(L_R(H_3^4(a,b)))=n-6.$$

If $U\cong H_3^4(0,b)$ $(b\ge 1)$ with $n=b+6$, then

$$L_R(H_3^4(0,b))=\begin{pmatrix}1&-1&-1&0&0\\-1&1&-1&0&0\\-1&-1&2&-1&0\\0&0&-1&1&-1\\0&0&0&-1&1\end{pmatrix}$$

It follows $\nu(L_R(H_3^4(0,b)))=1$. By Lemma 2.3 again, we have $m_{H_3^4(0,b)}(1)=n-6$.

If $U\cong H_3^5(a,b)$ $(a\ge 1,b\ge 1)$ with $n=a+b+5$, then

$$L_R(H_3^5(a,b))=\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$$

Similarly, we can get $\nu(L_R(H_3^5(a,b)))=1$, which leads to $m_{H_3^5(a,b)}(1)=a+b-1=n-6$ by Lemma 2.3.

Conversely, from Lemma 2.3 the conclusion holds. The proof is completed. □

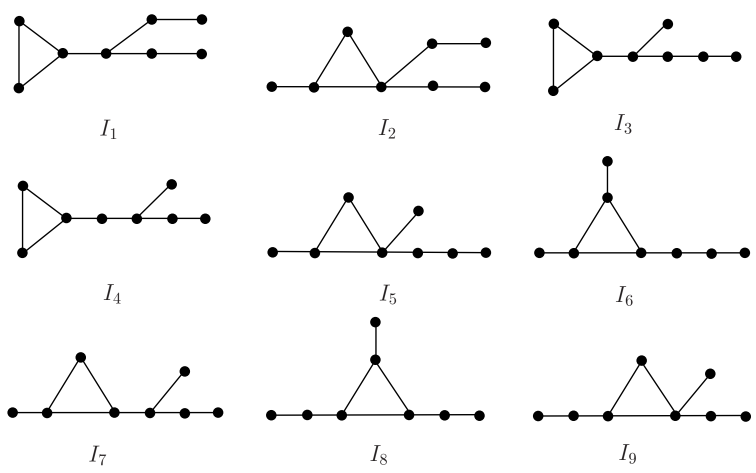

Figure 5 Graphs I_i $(1 \leq i \leq 9)$

I_1	0	0.3065	0.3820	1.6703	2.6180	3.0000	3.3297	4.6935
I_2	0	0.3820	0.4280	1.2285	2.2799	2.6180	3.8123	5.2513
I_3	0	0.2243	0.5858	1.4108	2.7237	3.0000	3.4142	4.6412
I_4	0	0.2137	0.6177	1.4977	2.3537	3.0000	3.8408	4.4763
I_5	0	0.2593	0.7150	1.3232	1.5891	3.1143	3.8086	5.1905
I_6	0	0.2434	0.6972	1.1798	2.0000	3.1386	4.3028	4.4383
I_7	0	0.2588	0.6436	1.1385	2.1603	3.1943	3.8943	4.7103
I_8	0	0.3004	0.4915	1.3204	2.2391	2.8258	4.3623	4.4605
I_9	0	0.2955	0.5979	1.1449	2.3295	2.4734	3.9635	5.1952

Table 1 The Laplacian spectra of I_i $(1 \leq i \leq 9)$

Lemma 3.5 *Let $U \in \mathcal{U}(n,4)$ be a unicyclic graph on $n \geq 7$ vertices. Then $m_U(1) = n-6$ if and only if $U \cong H_4^1(a,b)$ $(a \geq 1, b \geq 1)$, or $U \cong H_4^2(a,b)$ $(a \geq 0, b \geq 1)$ (see Figure 1).*

Proof Let U be a unicyclic graph of $\mathcal{U}(n,4)$ with order $n \geq 7$. Then $\mathrm{diam}(U) \leq 5$ by Corollary 3.3. According to Table 2, we can deduce that U does not contain I_{10}, I_{11}, I_{12} and I_{13} (see Figure 6) as its induced subgraphs by the same argument as I_1. Then U must have one of forms $U_4^1(a,b,c,d)$ $(a \geq 0, b \geq 0, c \geq 0, d \geq 0)$, $U_4^2(a,b)$ $(a \geq 1, b \geq 1)$, $U_4^3(a,b,c)$ $(a \geq 1, b \geq 0, c \geq 1)$ and $H_4^2(a,b)$ $(a \geq 0, b \geq 1)$ (see Figures 1 and 4).

When U has the form of $U_4^1(a,b,c,d)$, if $a \geq 1, b \geq 1, c \geq 1$ and $d \geq 1$, then $r(U_4^1(a,b,c,d)) = 0$. It follows from Lemma 2.7 that

$$m_{U_4^1(a,b,c,d)}(1) = p(U_4^1(a,b,c,d)) - q(U_4^1(a,b,c,d)) = a+b+c+d-4 = n-8,$$

which contradicts $m_U(1) = n-6$; if one of a,b,c,d is equal to zero, without loss of generality, we may assume $d = 0$, then $U \cong U_4^1(a,b,c,0)$ $(a \geq 1, b \geq 1, c \geq 1)$. By Lemma 3.1, one can get

$m_{U_4^1(a,b,c,0)}(1) = n-7$, it is also a contradiction. In addition, we see that $U \ncong C_4(a,b)$ $(a \geq 0, b \geq 0)$ due to $m_{C_4(a,b)}(1) = n-5$ in terms of Lemma 2.11. Thus, U has just the form $H_4^1(a,b)$ $(a \geq 1, b \geq 1)$. It follows from Lemma 2.3 that $m_{H_4^1(a,b)} = n-6$.

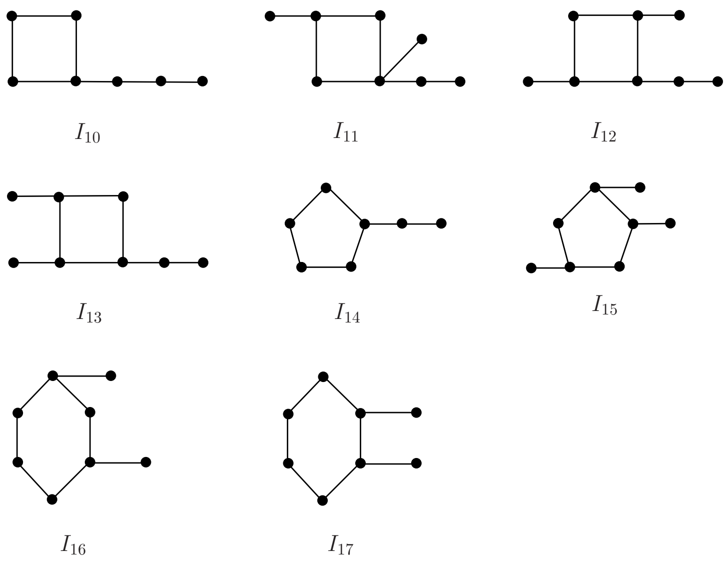

Figure 6 Graphs I_i $(10 \leq i \leq 17)$

I_{10}	0	0.2765	1.3323	2.0000	2.5219	3.2920	4.5772	
I_{11}	0	0.3581	0.6918	1.2843	2.0000	2.4091	3.8877	5.3689
I_{12}	0	0.3636	0.5858	1.3478	2.0000	3.2222	3.4142	5.0664
I_{13}	0	0.3432	0.6639	1.1805	2.2491	2.9045	3.5994	5.0594
I_{14}	0	0.3820	1.3820	1.5858	2.6180	3.6180	4.4142	
I_{15}	0	0.4915	0.6228	1.3204	1.7261	2.8258	4.3623	4.6511
I_{16}	0	0.4679	0.7369	1.4843	1.6527	3.1826	3.8794	4.5962

Table 2 The Laplacian spectra of I_i $(10 \leq i \leq 16)$

When U has the form of $U_4^2(a,b)$ $(a \geq 1, b \geq 1)$ or $U_4^3(a,b,c)$ $(a \geq 1, b \geq 0, c \geq 1)$, then by Lemma 3.1 we find that it is impossible as $m_{U_4^2(a,b)}(1) = m_{U_4^3(a,b,c)}(1) = n-7$.

When U has the form of $H_4^2(a,b)$, if $a \geq 1, b \geq 1$, then

$$L_R(H_4^2(a,b)) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

By direct calculation, one can deduce $\nu(L_R(H_4^2(a,b))) = 1$, and so, it follows from Lemma 2.3 that $m_{H_4^2(a,b))}(1) = a+b-2+\nu(L_R(H_4^2(a,b))) = n-6$; if $a=0$, then $U \cong H_4^2(0,b)$. Clearly, $H_4^2(0,b)$ can be obtained from C_4 and S_{b+1} by joining an arbitrary vertex of C_4 to the center of S_{b+1}. It therefore follows from Lemma 2.6 (2) that $m_{H_4^2(b)}(1) = m_{C_4u:vw}(1) + b + 1 - 2 = n-6$.

Conversely, it follows by the discussion above. The proof is completed. □

Lemma 3.6 *Let $U \in \mathcal{U}(n,5)$ be a unicyclic graph on $n \geq 7$ vertices. Then $m_U(1) = n-6$ if and only if $U \cong H_5^1(a,c)$ $(a \geq 0, c \geq 0)$ (see Figure 1).*

Proof Let U be a graph of $\mathcal{U}(n,5)$ with order $n \geq 7$. Then by Corollary 3.3 $\mathrm{diam}(U) \leq 5$. Using the same argument as I_1, we can obtain that U does not contain I_{14} and I_{15} as its induced subgraphs from Table 2. Therefore, U has the form of $U_5^1(a,b,c)$. By Lemma 3.1, one can obtain that $m_{U_5^1(a,b,c)}(1) = n-7$ if $a \geq 1$, $b \geq 1$, $c \geq 0$. Thus, U should only take the form of $H_5^1(a,c)$ $(a \geq 0, c \geq 0)$. According to the symmetry, we may assume that $a = 0$ and $c \geq 1$, then $U \cong H_5^1(0,c)$ with $n = c+5$, and so

$$\det(L_R(H_5^1(0,c))) = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = D_4$$

In light of Lemma 2.8, we have $\det(L_R(H_5^1(0,c))) = -1$, which implies $\nu(L_R(H_5^1(0,c))) = 0$. Therefore, $m_{H_5^1(0,c)}(1) = c-1+\nu(L_R(H_5^1(0,c))) = n-6$. If $a \geq 1$ and $c \geq 1$, adopting the same way as above, we obtain $m_{H_5^1(a,c)}(1) = a+c-2+\nu(L_R(H_5^1(a,c))) = n-6$.

Conversely, it follows by Lemma 2.3. The proof is completed. □

Lemma 3.7 *Let $U \in \mathcal{U}(n,6)$ be a unicyclic graph on $n \geq 7$ vertices. Then $m_U(1) = n-6$ if and only if $U \cong H_6^1(a,b)$ $(a \geq 0, b \geq 0)$ (see Figure 1).*

Proof Let U be a unicyclic graph of $\mathcal{U}(n,6)$ with order $n \geq 7$. Then $\mathrm{diam}(U) \leq 5$ by Corollary 3.3. It can be shown in the same way as I_1, U does not contain I_{16} as its induced subgraph by Table 2. Besides, from Lemma 3.2, U cannot include I_{17} as its induced subgraph either because P_8 is a subgraph of I_{17}. Therefore, U can only take the form of $H_6^1(a,b)$. If $a = 0$ or $b = 0$, without loss of generality, we assume $a = 0$, then $U \cong H_6^1(0,b)$ with $n = b+6$, which can be obtained from C_6 by attaching pendant vertices at any vertex of C_6, and

$$\det(L_R(H_6^1(0,b))) = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = D_5$$

It follows from Lemma 2.8 that $\det(L_R(H_6^1(0,b))) = 0$, which implies $\nu(L_R(H_6^1(0,b))) = 1$, and so $m_{H_6^1(0,b)}(1) = n-6$; if $a \geq 1$ and $b \geq 1$, then $U \cong H_6^1(a,b)$ with $n = a+b+6$. It also follows from Lemma 2.3 that

$$m_{H_6^1(a,b)}(1) = a+b-2+\nu(L_R(U_6^1(a,b))) = n-6.$$

Conversely, it is obvious by the discussion above. The proof is completed. □

Proof of Theorem 1.1 Let $U \in \mathcal{U}(n,g)$ be a unicyclic graph on $n \geq 7$ vertices. If U satisfies $g \geq 7$, then U must contain P_8 as its subgraph due to $U \ncong C_n$, which obviously contradicts Lemma 3.2. So we have $3 \leq g \leq 6$. Furthermore, together with Lemmas 3.4–3.7, the proof therefore follows. $\square$

Acknowledgements The authors would like to thank the referees for their valuable comments which lead to an improvement of the original manuscript.

References

[1] M. FIEDLER. *Algebraic connectivity of trees*. Czechoslovak Math. J., 1987, **37**(4): 660–670.

[2] D. CVETKOVIĆ, P. ROWLINSON, S. K. SIMIĆ. *An Introduction to the Theory of Graph Spectra*. Cambridge University Press, Cambridge, 2010.

[3] M. DOOB. *Graphs with a small number of distinct eigenvalues*. Ann. New York Acad. Sci., 2010, **175**: 104–110.

[4] W. G. BRIDGES, R. A. MENA. *Multiplicative cones–a family of three eigenvalue graphs*. Aequationes Math., 1981, **22**(2-3): 208–214.

[5] Ximing CHENG, A. L. GAVRILYUK, G. R. W. GREAVES, et al. *Biregular graphs with three eigenvalues*. European J. Combin., 2016, **56**: 57–80.

[6] M. MUZYCHUK, M. KLIN. *On graphs with three eigenvalues*. Discrete Math., 1998, **189**(1-3): 191–207.

[7] P. ROWLINSON. *On graphs with just three distinct eigenvalues*. Linear Algebra Appl., 2016, **507**: 462–473.

[8] E. R. VAN DAM. *Regular graphs with four eigenvalues*. Linear Algebra Appl., 1995, **226/228**: 139–162.

[9] E. R. VAN DAM, E. SPENCE. *Small regular graphs with four eigenvalues*. Discrete Math., 1998, **189**(1-3): 233–257.

[10] E. R. VAN DAM. *Nonregular graphs with three eigenvalues*. J. Combin. Theory Ser. B, 1998, **73**(2): 101–118.

[11] I. FARIA. *Permanental roots and the star degree of a graph*. Linear Algebra Appl., 1985, **64**: 255–265.

[12] E. ANDRADE, D. M. CARDOSO, G. PASTÉN, et al. *On the Faria's inequality for the Laplacian and signless Laplacian spectra: A unified approach*. Linear Algebra Appl., 2015, **472**: 81–96.

[13] R. GRONE, R. MERRIS, V. S. SUNDER. *The Laplacian spectrum of a graph*. SIAM J. Matrix Anal. Appl., 1990, **11**(2): 218–238.

[14] Jingming GUO, Lin FENG, Jiongming. ZHANG. *On the multiplicity of laplacian eigenvalues of graphs*. Czechoslovak Math. J., 2010, **60**(3): 689–698.

[15] S. BARIK, A. K. LAL, S. PATI. *On trees with Laplacian eigenvalue one*. Linear Multilinear Algebra, 2008, **56**(6): 597–610.

[16] A. E. BROUWER, W. H. HAEMERS. *Spectra of Graphs*. Springer, New York, 2012.

[17] S. KIRKLAND. *A bound on algebra connectivity of a graph in terms of the number cutpoints*. Linear Multilinear Algebra, 2000, **47**(1): 93–103.

[18] Lu LU, Qiongxiang HUANG, Xueyi HUANG. *On graphs with distance Laplacian spectral radius of multiplicity $n-3$*. Linear Algebra Appl., 2017, **530**(1): 485–499.