Journal of Mathematical Research with Applications
Nov., 2021, Vol. 41, No. 6, pp. 579–593
DOI:10.3770/j.issn:2095-2651.2021.06.003
Http://jmre.dlut.edu.cn

Coefficient Related Problem Studies for New Subclass of Bi-Univalent Functions Defined by (s, t)-Derivative Operator and Quasi-Subordination

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Abstract In this paper we introduce and investigate a new generalized class of bi-univalent functions defined by using (s, t)-derivative operator and quasi-subordination. We obtain the estimates of the first two coefficients $|a_2|, |a_3|$ and general coefficient $|a_n|$ $(n \geq 4)$ by using Faber polynomial expansion for the new class and some of its subclasses. And then we solve Fekete-Szegö probelm for the newly defined classes.

Keywords bi-univalent function; (s, t)-derivative; quasi-subordination; coefficient estimate; Fekete-Szegö problem; Faber polynomial expansion

MR(2020) Subject Classification 30C45; 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$.

For two analytic functions f and g, the function f is subordinate to g in $\mathbb{D}$, written as follows

$$f(z) \prec g(z), \quad z \in \mathbb{D},$$

if there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$ such that

$$f(z) = g(\omega(z)).$$

Furthermore, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

In 1970, Robertson [1] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is quasi-subordinate to g in $\mathbb{D}$, written as follows

$$f(z) \prec_q g(z), \quad z \in \mathbb{D},$$

Received July 30, 2020; Accepted January 5, 2021
Supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (Grant No. 2020MS01010) and the Higher-School Science Foundation of Inner Mongolia Autonomous Region (Grant No. NJZY19211).
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if there exists an analytic function h with $|h(z)| \le 1$ such that $\frac{f(z)}{h(z)}$ is analytic in $\mathbb{D}$ and

$$\frac{f(z)}{h(z)} \prec g(z), \quad z \in \mathbb{D}$$

that is, there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$ such that

$$f(z) = h(z)g(\omega(z)).$$

Observe that when $h(z) = 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{D}$. Also notice that if $\omega(z) = z$, then $f(z) = h(z)g(z)$ and f is said to be majorized by g, written as $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [2–6] for works related to quasi-subordination.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. It is a well known fact that every function $f \in \mathcal{S}$ has an inverse functions f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{D}$$

and

$$f(f^{-1}(\omega)) = \omega, \quad |\omega| < r_0(f),\ r_0(f) \ge \frac{1}{4}.$$

In fact, according to the Kobe One-Quarter Theorem [7], the inverse function $f^{-1}$is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots = \omega + \sum_{n=2}^{\infty} b_n\omega^n. \quad (1.2)$$

Let Σ denote the class of all bi-univalent functions in $\mathbb{D}$ given by the Taylor-Maclaurin series expansion in (1.1). Coefficient estimate problem of bi-univalent function was widely researched in the literature. In 1967, Lewin [8] first introduced the class Σ and studied the estimate for the coefficient $|a_2|$ of functions in Σ, and obtained that $|a_2| \le 1.51$. Subsequently, Branan and Clunie [9] improved Lewin's result to $|a_2| \le \sqrt{2}$ and later Netanyahu [10] proved that $|a_2| \le 4/3$. Kedzierawski [11] proved the Brannan-Clunie conjecture for bi-starlike functions. In 1984, Tan [12] obtained that $|a_2| < 1.485$, which is the best known estimate for bi-univalent functions in Σ. Brannan and Taha [13] also investigated certain subclasses of bi-univalent functions and found the non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$. In recent years, many researchers have been devoted to various subclasses of the bi-univalent functions and obtained the estimates on the initial coefficients $|a_2|$ and $|a_3|$. The interest on estimates for the initial coefficients $|a_2|, |a_3|$ of the bi-univalent functions keeps on by some researchers (see, for example, Srivastava et al. [14], Frasin and Aouf [15], Hayami and Owa [16], Xu et al. [17], and others [18–24]). Quite recently, only few works also determined the Fekete-Szegö problem (i.e., estimate for the upper bound of $|a_3 - \mu a_2^2|$) for some subclasses of bi-univalent functions, for example [25–30]. In the meantime, the estimate on the general coefficients $|a_n|$ $(n \ge 4)$ of bi-univalent functions has attracted the attention of some researchers. By using the Faber polynomial coefficient expansions Jahangiri and Hamidi [31] obtained bounds for the coefficient $|a_n|$ of bi-univalent functions in certain subclass of Σ with a given gap series condition. Since then, some of authors considered and studied the bound of general coefficient $|a_n|$ for bi-univalent functions in certain subclasses of Σ, for example [32–36]. The estimate on the general coefficients $|a_n|$ $(n \geq 4)$ of bi-univalent functions is still an open problem.

Although many subclasses of bi-univalent functions have already been introduced and some coefficient estimates have been studied, our focus is not only to further extend the bi-univalent functions class, but also to study the above coefficient estimate problems and Fekete-Szegö problem of the new classes of bi-univalent functions.

We begin by recalling the definition details of the following (s,t)-derivative operator (defined by Chakrabarti and Jagannathan [37], see also [38]), which will be used in this paper.

Definition 1.1 *Let the function $f \in \mathcal{A}$ be given by (1.1) and $0 < t < s \leq 1$. The (s,t)-derivative of the function f is defined as*

$$(D_{s,t}f)(z) = \begin{cases} \frac{f(sz)-f(tz)}{(s-t)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

According to the above definition, we have

$$(D_{s,t}f)(z) = 1 + \sum_{n=2}^{\infty}[n]_{s,t}a_n z^{n-1}$$

where the symbol $[n]_{s,t}$ denotes the (s,t)-number or twin-basic number $[n]_{s,t} = \frac{s^n-t^n}{s-t}$.

Note that by putting $s = 1$, the (s,t)-derivative reduces to the Jackson t-derivative given by [39]

$$(D_t f)(z) = \begin{cases} \frac{f(z)-f(tz)}{(1-t)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

And, for $f \in \mathcal{A}$ given by (1.1), we have

$$(D_t f)(z) = 1 + \sum_{n=2}^{\infty}[n]_t a_n z^{n-1}$$

where $[n]_t = \frac{1-t^n}{1-t}$.

Also, by taking $t \to 1^-$, we have $[n]_t \to n$. So $(D_t f)(z)$ reduces to $f'(z)$ for $f \in \mathcal{A}$.

Now by using (s,t)-derivative operator and quasi-subordination we introduce a generalization class of analytic and bi-univalent functions.

Definition 1.2 *Let $0 \leq \lambda < 1, \gamma \in \mathbb{C}\backslash\{0\}$. A function $f(z) \in \sum$ given by (1.1) is said to be in the class $\mathcal{Q}^q_{\sum}(\gamma,\lambda,s,t,\varphi)$ if the following conditions are satisfied*

$$\frac{1}{\gamma}\Big(\frac{z(D_{s,t}f)(z)}{(1-\lambda)f(z)+\lambda z(D_{s,t}f)(z)} - 1\Big) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D} \tag{1.4}$$

and

$$\frac{1}{\gamma}\Big(\frac{\omega(D_{s,t}g)(\omega)}{(1-\lambda)g(\omega)+\lambda \omega(D_{s,t}g)(\omega)} - 1\Big) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D} \tag{1.5}$$

where $g(\omega) = f^{-1}(\omega)$ is defined by (1.2).

Remark 1.3 There are some suitable choices of λ, s, t, γ which would provide the following subclasses of the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$.

(1) By taking $\lambda = 0$ in Definition 1.2, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ reduces to the class $\mathcal{Q}^q_{\sum}(\gamma, s, t, \varphi)$ which satisfies

$$\frac{1}{\gamma}(\frac{z(D_{s,t}f)(z)}{f(z)} - 1) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D}$$

and

$$\frac{1}{\gamma}(\frac{\omega(D_{s,t}g)(\omega)}{g(\omega)} - 1) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D}.$$

(2) By taking $s = 1$ in Definition 1.2, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ reduces to the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, t, \varphi)$ which satisfies

$$\frac{1}{\gamma}(\frac{z(D_t f)(z)}{(1-\lambda)f(z) + \lambda z(D_t f)(z)} - 1) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D}$$

and

$$\frac{1}{\gamma}(\frac{\omega(D_t g)(\omega)}{(1-\lambda)g(\omega) + \lambda\omega(D_t g)(\omega)} - 1) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D}.$$

(3) By taking $\lambda = 0$ and $s = 1$ in Definition 1.2, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ reduces to the class $\mathcal{Q}^q_{\sum}(\gamma, t, \varphi)$ which satisfies

$$\frac{1}{\gamma}(\frac{z(D_t f)(z)}{f(z)} - 1) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D}$$

and

$$\frac{1}{\gamma}(\frac{\omega(D_t g)(\omega)}{g(\omega)} - 1) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D}.$$

(4) By taking $s = 1$ and $t \to 1^-$ in Definition 1.2, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ reduces to the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, \varphi)$ which satisfies

$$\frac{1}{\gamma}(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D}$$

and

$$\frac{1}{\gamma}(\frac{\omega g'(\omega)}{(1-\lambda)g(\omega) + \lambda\omega g'(\omega)} - 1) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D}.$$

Specially, for $\gamma = 1$, if quasi-subordination is reduced to subordination, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, \varphi)$ reduces to the class $\mathcal{M}^{\lambda}_{\sum}(\varphi)$ introduced by Altinkaya et al. [40].

(5) By taking $\lambda = 0, s = 1$ and $t \to 1^-$ in Definition 1.2, the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ reduces to the class $\mathcal{S}^q_{\sum}(\gamma, \varphi)$ which satisfies

$$\frac{1}{\gamma}(\frac{zf'(z)}{f(z)} - 1) \prec_q \varphi(z) - 1, \quad z \in \mathbb{D}$$

and

$$\frac{1}{\gamma}(\frac{\omega g'(\omega)}{g(\omega)} - 1) \prec_q \varphi(\omega) - 1, \quad \omega \in \mathbb{D}.$$

Specially, for $\gamma = 1$, the class $\mathcal{S}^q_{\sum}(\gamma, \varphi)$ reduces to the class $\mathcal{S}^q_{\sum}(\varphi)$ introduced by Vyas and Kant [41].

The object of this paper is to study two kinds of coefficient estimate problems and Fekete-Szegö problem for the class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ and some of its subclasses. Our results are new in this direction and they give birth to many corollaries.

In order to derive our main results, we have to recall here the following lemma.

Lemma 1.4 ([42]) *If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for each n, where $\mathcal{P}$ is the family of all function p analytic in $\mathbb{D}$ for which* Re $p(z) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *for $z \in \mathbb{D}$.*

2. Main Results

In the sequel, it is assumed that $\varphi(z)$ is an analytic function with positive real part in $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Then function $\varphi(z)$ has the Taylor series expansion of the form

$$\varphi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots, \quad \xi_1 > 0. \tag{2.1}$$

Suppose that ψ and ϕ are analytic in the unit disk $\mathbb{D}$ with $|\psi(z)| < 1, |\phi(\omega)| < 1$, and suppose that

$$\psi(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \quad \phi(\omega) = l_0 + l_1 \omega + l_2 \omega^2 + \cdots. \tag{2.2}$$

2.1. Coefficient estimates problem

In this section, we obtain the coefficient estimates for the function class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$.

For this purpose, we need to use the Faber polynomial expansions of inverse functions. For the function $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed by [43, 44]

$$g(\omega) = f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} \frac{1}{n} K^{-n}_{n-1}(a_2, a_3, \ldots, a_n)\omega^n, \tag{2.3}$$

where

$$\begin{aligned} K^{-n}_{n-1} =& K^{-n}_{n-1}(a_2, a_3, \ldots, a_n) = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(-2n+2)!(n-3)!} a_2^{n-3} a_3 + \\ & \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(-2n+4)!(n-5)!} a_2^{n-5} [a_5 + (n-2) a_3^2] + \\ & \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-5} [a_6 + (-2n+5) a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see [45]). In particular, the first three terms of K^{-n}_{n-1} are

$$K^{-2}_1 = -2a_2, K^{-3}_2 = 3(2a^2 - a_3), K^{-4}_3 = -4(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, for $n \geq 1$ and $\alpha \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, an expansion of K^{α}_{n-1} is given by [43]

$$K^{\alpha}_{n-1} = \alpha a_n + \frac{\alpha(\alpha-1)}{2} E^2_{n-1} + \frac{\alpha!}{(\alpha-3)!3!} E^3_{n-1} + \cdots + \frac{\alpha!}{(\alpha-n+1)!(n-1)!} E^{n-1}_{n-1},$$

where $E^{\alpha}_{n-1} = E^{\alpha}_{n-1}(a_2, a_3, \dots, a_n)$ are homogeneous polynomial explicated in [46]

$$E^{\alpha}_{n-1}(a_2, a_3, \dots, a_n) = \sum_{n=2}^{\infty} \frac{\alpha!}{j_1 \cdots j_{n-1}} a_2^{j_1} \cdots a_n^{j_{n-1}} \quad \text{for } \alpha \le n-1,$$

and the sum is taken over all nonnegative integers $j_1, \dots, j_{n-1}$ satisfying

$$\begin{cases} j_1 + j_2 + \cdots + j_{n-1} = \alpha, \\ j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = n-1. \end{cases}$$

It is clear that $E^{n-1}_{n-1}(a_2, a_3, \dots, a_n) = a_2^{n-1}$.

Consequently, for function $f \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ of the form (1.1), we can obtain

$$\frac{1}{\gamma}\Big(\frac{z(D_{s,t}f)(z)}{(1-\lambda)f(z) + \lambda z(D_{s,t}f)(z)} - 1\Big) = \sum_{n=2}^{\infty} \frac{F_{n-1}(A_2, A_3, \dots, A_n)}{\gamma} z^{n-1}, \tag{2.4}$$

$$\frac{1}{\gamma}\Big(\frac{\omega(D_{s,t}g)(\omega)}{(1-\lambda)g(\omega) + \lambda \omega(D_{s,t}g)(\omega)} - 1\Big) = \sum_{n=2}^{\infty} \frac{G_{n-1}(B_2, B_3, \dots, B_n)}{\gamma} \omega^{n-1}, \tag{2.5}$$

where

$$F_{n-1}(A_2, A_3, \dots, A_n) = ([n]_{s,t}a_n - A_n) + \sum_{j=1}^{n-2} K_j^{-1}(A_2, A_3, \dots, A_{j+1})([n-j]_{s,t}a_{n-j} - A_{n-j}),$$

$$G_{n-1}(B_2, B_3, \dots, B_n) = ([n]_{s,t}b_n - B_n) + \sum_{j=1}^{n-2} K_j^{-1}(B_2, B_3, \dots, B_{j+1})([n-j]_{s,t}b_{n-j} - B_{n-j})$$

with $A_n = [1 + ([n]_{s,t} - 1)\lambda]a_n, B_n = [1 + ([n]_{s,t} - 1)\lambda]b_n$.

In addition, for analytic functions $u(z) = c_1 z + c_2 z^2 + \cdots$, $v(\omega) = d_1\omega + d_2\omega^2 + \cdots$ and analytic function $\varphi \in \mathcal{A}$ of the form (2.1), we can get

$$\varphi(u(z)) - 1 = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \xi_k E_n^k(c_1, c_2, \dots, c_n) z^n, \varphi(v(\omega)) - 1 = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \xi_k E_n^k(d_1, d_2, \dots, d_n)\omega^n.$$

Now by using Faber polynomial expansions, we prove our first main result which provides an estimate for the general coefficients $|a_n|$ of functions in $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ subject to a given gap series condition.

Theorem 2.1 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ be given by (1.1). If $a_i = 0$ $(2 \le i \le n-1)$, then*

$$|a_n| \le \frac{2|\gamma|}{([n]_{s,t} - 1)(1-\lambda)} \min\Big\{ \sum_{i=0}^{n-1} |h_i| \Big(\sum_{k=1}^{n-i-1} |E^k_{n-i-1}(c_1, c_2, \dots, c_{n-i-1})| \Big), \\ \sum_{i=0}^{n-1} |l_i| \Big(\sum_{k=1}^{n-i-1} |E^k_{n-i-1}(d_1, d_2, \dots, d_{n-i-1})| \Big) \Big\}. \tag{2.6}$$

Proof Since $f \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$, then there exist two Schwarz functions $u(z) = c_1 z + c_2 z^2 + \cdots$, $v(\omega) = d_1\omega + d_2\omega^2 + \cdots$ and analytic functions ψ, φ defined by (2.2) such that

$$\frac{1}{\gamma}\Big(\frac{z(D_{s,t}f)(z)}{(1-\lambda)f(z) + \lambda z(D_{s,t}f)(z)} - 1\Big) = \psi(z)[\varphi(u(z)) - 1]$$

and

$$\frac{1}{\gamma}\Big(\frac{\omega(D_{s,t}g)(\omega)}{(1-\lambda)g(\omega)+\lambda\omega(D_{s,t}g)(\omega)}-1\Big)=\phi(\omega)[\varphi(v(\omega))-1],$$

where

$$\psi(z)[\varphi(u(z))-1]=\sum_{n=1}^{\infty}\Big[\sum_{i=0}^{n}h_i\Big(\sum_{k=1}^{n-i}\xi_k E_{n-i}^{k}(c_1,c_2,\ldots,c_{n-i})\Big)\Big]z^n, \tag{2.7}$$

$$\phi(\omega)[\varphi(v(\omega))-1]=\sum_{n=1}^{\infty}\Big[\sum_{i=0}^{n}l_i\Big(\sum_{k=1}^{n-i}\xi_k E_{n-i}^{k}(d_1,d_2,\ldots,d_{n-i})\Big)\Big]\omega^n. \tag{2.8}$$

Comparing the corresponding coefficients of (2.4) and (2.7), for any $n\geq 2$ we have

$$\begin{aligned}&([n]_{s,t}a_n-A_n)+\sum_{j=1}^{n-2}K_j^{-1}(A_2,A_3,\ldots,A_{j+1})([n-j]_{s,t}a_{n-j}-A_{n-j})\\&=\gamma\sum_{i=0}^{n-1}h_i\Big(\sum_{k=1}^{n-i-1}\xi_k E_{n-i-1}^{k}(c_1,c_2,\ldots,c_{n-i-1})\Big)\end{aligned} \tag{2.9}$$

and similarly, from (2.5) and (2.8) we have

$$\begin{aligned}&([n]_{s,t}b_n-B_n)+\sum_{j=1}^{n-2}K_j^{-1}(B_2,B_3,\ldots,B_{j+1})([n-j]_{s,t}b_{n-j}-B_{n-j})\\&=\gamma\sum_{i=0}^{n-1}l_i\Big(\sum_{k=1}^{n-i-1}\xi_k E_{n-i-1}^{k}(d_1,d_2,\ldots,d_{n-i-1})\Big).\end{aligned} \tag{2.10}$$

For $a_i=0$ $(2\leq i\leq n-1)$, we get $A_i=0$, $b_i=B_i=0$ $(2\leq i\leq n-1)$ and $b_n=-a_n$. Hence

$$([n]_{s,t}-1)(1-\lambda)a_n=\gamma\sum_{i=0}^{n-1}h_i\Big(\sum_{k=1}^{n-i-1}\xi_k E_{n-i-1}^{k}(c_1,c_2,\ldots,c_{n-i-1})\Big) \tag{2.11}$$

and

$$-([n]_{s,t}-1)(1-\lambda)a_n=\gamma\sum_{i=0}^{n-1}l_i\Big(\sum_{k=1}^{n-i-1}\xi_k E_{n-i-1}^{k}(d_1,d_2,\ldots,d_{n-i-1})\Big). \tag{2.12}$$

Finally, by taking the moduli in both sides of (2.11) and (2.12) and using Lemma 1.4, we get the desired estimate on $|a_n|$ as asserted in (2.6). This evidently completes the proof of Theorem 2.1. $\square$

Example 2.2 Let the function $f(z)\in\mathcal{S}_{\sum}^{q}(\varphi)$ be given by (1.1). If $a_i=0$ $(2\leq i\leq n-1)$, then

$$\begin{aligned}|a_n|\leq\frac{2}{n-1}\min\Big\{&\sum_{i=0}^{n-1}|h_i|\Big(\sum_{k=1}^{n-i-1}|E_{n-i-1}^{k}(c_1,c_2,\ldots,c_{n-i-1})|\Big),\\&\sum_{i=0}^{n-1}|l_i|\Big(\sum_{k=1}^{n-i-1}|E_{n-i-1}^{k}(d_1,d_2,\ldots,d_{n-i-1})|\Big)\Big\}.\end{aligned}$$

Proof Let function f be given by (1.1). We have

$$\frac{zf'(z)}{f(z)}-1=a_2z+(2a_3-a_2^2)z^2+\cdots+[(n-1)a_n+$$

$$\sum_{j=1}^{n-2} K_j^{-1}(a_2,a_3,\ldots,a_{j+1})(n-j-1)a_{n-j}]z^n+\cdots.$$

And for its inverse map $g=f^{-1}$ given by (1.2), we have

$$\begin{aligned}\frac{\omega g'(\omega)}{g(\omega)}-1=&b_2\omega+(2b_3-b_2^2)\omega^2+\cdots+[(n-1)b_n+\\&\sum_{j=1}^{n-2} K_j^{-1}(b_2,b_3,\ldots,b_{j+1})(n-j-1)b_{n-j}]\omega^n+\cdots.\end{aligned}$$

Since $f\in\mathcal{S}^q_{\sum}(\varphi)$, there exist two Schwarz functions $u(z)=\sum_{n=1}^{\infty}c_nz^n$, $v(\omega)=\sum_{n=1}^{\infty}d_n\omega^n$ and analytic functions ψ,φ defined by (2.2) such that

$$\frac{zf'(z)}{f(z)}-1=\psi(z)[\varphi(u(z))-1]$$

and

$$\frac{\omega g'(\omega)}{g(\omega)}-1=\phi(\omega)[\varphi(v(\omega))-1]$$

where $\psi(z)[\varphi(u(z))-1]$, $\phi(\omega)[\varphi(v(\omega))-1]$ are defined by (2.7) and (2.8).

Using arguments similar to those in the proof of Theorem 2.1, we can obtain the estimate result of $|a_n|$. This completes the proof of Example 2.2. □

By taking special values of parameters λ,s,t in Theorem 2.1, we easily obtain the following results.

Corollary 2.3 *Let the function $f(z)\in\mathcal{Q}^q_{\sum}(\gamma,s,t,\varphi)$ be given by (1.1). If $a_i=0$ $(2\le i\le n-1)$, then*

$$\begin{aligned}|a_n|\le&\frac{2|\gamma|}{[n]_{s,t}-1}\min\Big\{\sum_{i=0}^{n-1}|h_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(c_1,c_2,\ldots,c_{n-i-1})|\Big),\\&\sum_{i=0}^{n-1}|l_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(d_1,d_2,\ldots,d_{n-i-1})|\Big)\Big\}.\end{aligned}$$

Corollary 2.4 *Let the function $f(z)\in\mathcal{Q}^q_{\sum}(\gamma,\lambda,t,\varphi)$ be given by (1.1). If $a_i=0$ $(2\le i\le n-1)$, then*

$$\begin{aligned}|a_n|\le&\frac{2|\gamma|}{([n]_t-1)(1-\lambda)}\min\Big\{\sum_{i=0}^{n-1}|h_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(c_1,c_2,\ldots,c_{n-i-1})|\Big),\\&\sum_{i=0}^{n-1}|l_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(d_1,d_2,\ldots,d_{n-i-1})|\Big)\Big\}.\end{aligned}$$

Corollary 2.5 *Let the function $f(z)\in\mathcal{Q}^q_{\sum}(\gamma,\lambda,\varphi)$ be given by (1.1). If $a_i=0$ $(2\le i\le n-1)$, then*

$$\begin{aligned}|a_n|\le&\frac{2|\gamma|}{(n-1)(1-\lambda)}\min\Big\{\sum_{i=0}^{n-1}|h_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(c_1,c_2,\ldots,c_{n-i-1})|\Big),\\&\sum_{i=0}^{n-1}|l_i|\Big(\sum_{k=1}^{n-i-1}|E^k_{n-i-1}(d_1,d_2,\ldots,d_{n-i-1})|\Big)\Big\}.\end{aligned}$$

Our next main result provides estimates for the initial coefficients $|a_2|$ and $|a_3|$ of functions in $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ with no gap series restrictions imposed.

Theorem 2.6 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ be given by (1.1). Then*

$$|a_2| \leq \min\{\frac{|\gamma|\sqrt{2(h_0^2 + l_0^2)}}{|[2]_{s,t} - 1|(1-\lambda)}, \sqrt{\frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda|(1-\lambda)}}\}, \tag{2.13}$$

$$\begin{aligned} |a_3| \leq & \frac{|\gamma|}{1-\lambda} \min\{\frac{2|\gamma|(h_0^2 + l_0^2)}{([2]_{s,t} - 1)^2(1-\lambda)} + \frac{2(|h_0| + |l_0|) + (|h_1| + |l_1|)}{|[3]_{s,t} - 1|}, \\ & \frac{|C(\lambda, s, t)|(2|h_0| + |h_1|) + |D(\lambda, s, t)|(2|l_0| + |l_1|)}{|([3]_{s,t} - 1)([3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda)|}\} \end{aligned} \tag{2.14}$$

where $C(\lambda, s, t) = 2[3]_{s,t} - [2]_{s,t} - 1 - ([2]_{s,t} - 1)^2\lambda, D(\lambda, s, t) = ([2]_{s,t} - 1)[1 + ([2]_{s,t} - 1)\lambda]$.

Proof Putting $n = 2$ and $n = 3$ in (2.9) and (2.10), respectively, we obtain

$$([2]_{s,t} - 1)(1-\lambda)a_2 = \gamma h_0 \xi_1 c_1 \tag{2.15}$$

$$\begin{aligned} & ([3]_{s,t} - 1)(1-\lambda)a_3 - ([2]_{s,t} - 1)(1-\lambda)[1 + ([2]_{s,t} - 1)\lambda]a_2^2 \\ & = \gamma[(h_0 c_2 + h_1 c_1)\xi_1 + h_0 c_1^2 \xi_2] \end{aligned} \tag{2.16}$$

and

$$-([2]_{s,t} - 1)(1-\lambda)a_2 = \gamma l_0 \xi_1 d_1 \tag{2.17}$$

$$\begin{aligned} & ([3]_{s,t} - 1)(1-\lambda)(2a_2^2 - a_3) - ([2]_{s,t} - 1)(1-\lambda)[1 + ([2]_{s,t} - 1)\lambda]a_2^2 \\ & = \gamma[(l_0 d_2 + l_1 d_1)\xi_1 + l_0 d_1^2 \xi_2]. \end{aligned} \tag{2.18}$$

From (2.15) and (2.17), we obtain

$$a_2^2 = \frac{\gamma^2 \xi_1^2 (h_0^2 c_1^2 + l_0^2 d_1^2)}{2([2]_{s,t} - 1)^2(1-\lambda)^2}. \tag{2.19}$$

Also, from (2.16) and (2.18), we find

$$a_2^2 = \frac{\gamma[\xi_1(h_0 c_2 + h_1 c_1 + l_0 d_2 + l_1 d_1) + \xi_2(h_0 c_1^2 + l_0 d_1^2)]}{2(1-\lambda)([3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda)}. \tag{2.20}$$

For the coefficients of the Schwarz functions $u(z)$ and $v(\omega)$ we have $|c_n| \leq 1$ and $|d_n| \leq 1$ (see [7]). Taking the moduli in both sides of (2.19) and (2.20), and applying Lemma 1.4, we get

$$|a_2| \leq \frac{|\gamma|\sqrt{2(h_0^2 + l_0^2)}}{|[2]_{s,t} - 1|(1-\lambda)}$$

and

$$|a_2| \leq \sqrt{\frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda|(1-\lambda)}}$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.13).

Next, in order to find the bound on $|a_3|$, by subtracting (2.18) from (2.16), we obtain

$$a_3 = a_2^2 + \frac{\gamma[\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)]}{2([3]_{s,t} - 1)(1-\lambda)}. \tag{2.21}$$

Thus, upon substituting the value of a_2^2 from (2.19) into (2.21), it follows that

$$a_3 = \frac{\gamma^2\xi_1^2(h_0^2c_1^2 + l_0^2d_1^2)}{2([2]_{s,t} - 1)^2(1-\lambda)^2} + \frac{\gamma[\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)]}{2([3]_{s,t} - 1)(1-\lambda)}$$

which yields

$$|a_3| \le \frac{2\gamma^2(h_0^2 + l_0^2)}{([2]_{s,t} - 1)^2(1-\lambda)^2} + \frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_{s,t} - 1|(1-\lambda)}. \tag{2.22}$$

On the other hand, upon substituting the value of a_2^2 from (2.20) into (2.21), we obtain

$$\begin{aligned} a_3 =& \frac{\gamma[\xi_1(h_0c_2 + h_1c_1 + l_0d_2 + l_1d_1) + \xi_2(h_0c_1^2 + l_0d_1^2)]}{2([3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda)(1-\lambda)} + \\ & \frac{\gamma[\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)]}{2([3]_{s,t} - 1)(1-\lambda)}. \end{aligned}$$

It follows that

$$\begin{aligned} |a_3| \le & \frac{|\gamma|}{|([3]_{s,t} - 1)([3]_{s,t} - [2]_{s,t} - ([2]_{s,t} - 1)^2\lambda)|(1-\lambda)} \\ & [|2([3]_{s,t} - 1) - ([2]_{s,t} - 1)[1 + ([2]_{s,t} - 1)\lambda]|(2|h_0| + |h_1|) + \\ & |([2]_{s,t} - 1)[1 + ([2]_{s,t} - 1)\lambda]|(2|l_0| + |l_1|)]. \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we get the desired estimate on the coefficient $|a_3|$ as asserted in (2.14). This evidently completes the proof of Theorem 2.6. □

By taking special values of parameters λ, s, t in Theorem 2.6, we easily obtain the following results.

Corollary 2.7 *Let the function $f(z) \in \mathcal{Q}_{\sum}^{q}(\gamma, s, t, \varphi)$ be given by (1.1). Then*

$$|a_2| \le \min\{\frac{|\gamma|\sqrt{2(h_0^2 + l_0^2)}}{|[2]_{s,t} - 1|}, \sqrt{\frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_{s,t} - [2]_{s,t}|}}\}$$

$$\begin{aligned} |a_3| \le& \min\{\frac{2\gamma^2(h_0^2 + l_0^2)}{([2]_{s,t} - 1)^2} + \frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_{s,t} - 1|}, \\ & \frac{|\gamma|[|2[3]_{s,t} - [2]_{s,t} - 1|(2|h_0| + |h_1|) + |[2]_{s,t} - 1|(2|l_0| + |l_1|)]}{|([3]_{s,t} - 1)([3]_{s,t} - [2]_{s,t})|}\}. \end{aligned}$$

Corollary 2.8 *Let the function $f(z) \in \mathcal{Q}_{\sum}^{q}(\gamma, \lambda, t, \varphi)$ be given by (1.1). Then*

$$|a_2| \le \min\{\frac{|\gamma|\sqrt{2(h_0^2 + l_0^2)}}{|[2]_t - 1|(1-\lambda)}, \sqrt{\frac{|\gamma|[2(|h_0| + |l_0|) + (|h_1| + |l_1|)]}{|[3]_t - [2]_t - ([2]_t - 1)^2\lambda|(1-\lambda)}}\}$$

$$\begin{aligned} |a_3| \le& \frac{|\gamma|}{1-\lambda}\min\{\frac{2|\gamma|(h_0^2 + l_0^2)}{([2]_t - 1)^2(1-\lambda)} + \frac{2(|h_0| + |l_0|) + (|h_1| + |l_1|)}{[3]_t - 1}, \\ & \frac{|2[3]_t - [2]_t - 1 - ([2]_t - 1)^2\lambda|(2|h_0| + |h_1|) + ([2]_t - 1)[1 + ([2]_t - 1)\lambda](2|l_0| + |l_1|)}{([3]_t - 1)|[3]_t - [2]_t - ([2]_t - 1)^2\lambda|}. \end{aligned}$$

Corollary 2.9 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, \varphi)$ be given by (1.1). Then*

$$|a_2| \leq \frac{1}{1-\lambda}\min\{|\gamma|\sqrt{2(h_0^2+l_0^2)}, \sqrt{|\gamma|[2(|h_0|+|l_0|)+(|h_1|+|l_1|)]}\}$$

$$|a_3| \leq \frac{|\gamma|}{1-\lambda}\min\{\frac{2|\gamma|(h_0^2+l_0^2)}{1-\lambda}+\frac{2(|h_0|+|l_0|)+(|h_1|+|l_1|)}{2}, \frac{(3-\lambda)(2|h_0|+|h_1|)+(1+\lambda)(2|l_0|+|l_1|)}{2(1-\lambda)}\}.$$

Remark 2.10 For $\lambda = 0$, $s = 1$, $\gamma = 1$ and $t \to 1^-$ in Theorem 2.6, we obtain the bounds on $|a_2|$ and $|a_3|$ which are the improved results [41, Corollary 2.6] obtained by Vyas et al.

2.2. Fekete-Szegö problem

In this section, we obtain Fekete-Szegö problem for the function class $\mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$.

Theorem 2.11 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$ be given by (1.1). Then for any number $\mu \in \mathbb{C}$ and $[3]_{s,t} > 1$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2([3]_{s,t}-1)(1-\lambda)}, & 0 \leq |M(\mu)| \leq \frac{1}{2([3]_{s,t}-1)(1-\lambda)}, \\ [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)| \geq \frac{1}{2([3]_{s,t}-1)(1-\lambda)}. \end{cases} \tag{2.24}$$

For any number $\mu \in \mathbb{C}$ and $[3]_{s,t} < 1$

$$|a_3 - \mu a_2^2| \leq \begin{cases} [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & 0 \leq |M(\mu)| \leq \frac{1}{2(1-[3]_{s,t})(1-\lambda)}, \\ \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2(1-[3]_{s,t})(1-\lambda)}, & |M(\mu)| \geq \frac{1}{2(1-[3]_{s,t})(1-\lambda)}, \end{cases} \tag{2.25}$$

where

$$M(\mu) = \frac{\gamma h_0 l_0 \xi_1^2(1-\mu)}{(1-\lambda)[2\gamma h_0 l_0([3]_{s,t}-[2]_{s,t}-([2]_{s,t}-1)^2\lambda)\xi_1^2-([2]_{s,t}-1)^2(1-\lambda)(l_0+h_0)(\xi_2-\xi_1)]}.$$

Proof Since $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, s, t, \varphi)$, then there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = 0 = v(0), |u(z)| < 1, |v(\omega)| < 1$ and analytic functions ψ, ϕ defined by (2.2) such that

$$\frac{1}{\gamma}(\frac{z(D_{s,t}f)(z)}{(1-\lambda)f(z)+\lambda z(D_{s,t}f)(z)} - 1) = \psi(z)[\varphi(u(z)) - 1] \tag{2.26}$$

and

$$\frac{1}{\gamma}(\frac{\omega(D_{s,t}g)(\omega)}{(1-\lambda)g(\omega)+\lambda\omega(D_{s,t}g)(\omega)} - 1) = \phi(\omega)[\varphi(v(\omega)) - 1]. \tag{2.27}$$

Define the functions p_1 and p_2 in $\mathcal{P}$ given by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$p_2(\omega) = \frac{1+v(\omega)}{1-v(\omega)} = 1 + q_1\omega + q_2\omega^2 + \cdots.$$

It follows

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}(p_2 - \frac{p_1^2}{2})z^2 + \cdots, \tag{2.28}$$

$$v(\omega)=\frac{p_2(\omega)-1}{p_2(\omega)+1}=\frac{1}{2}q_1\omega+\frac{1}{2}(q_2-\frac{q_1^2}{2})\omega^2+\cdots. \tag{2.29}$$

Using (2.1), (2.2), (2.28) and (2.29), it is evident that

$$\psi(z)[\varphi(u(z))-1]=\frac{1}{2}h_0\xi_1p_1z+[\frac{1}{2}h_1\xi_1p_1+\frac{1}{2}h_0\xi_1p_2+\frac{1}{4}h_0(\xi_2-\xi_1)p_1^2]z^2+\cdots, \tag{2.30}$$

$$\phi(\omega)[\varphi(v(\omega))-1]=\frac{1}{2}l_0\xi_1q_1\omega+[\frac{1}{2}l_1\xi_1q_1+\frac{1}{2}l_0\xi_1q_2+\frac{1}{4}l_0(\xi_2-\xi_1)q_1^2]\omega^2+\cdots. \tag{2.31}$$

Using (2.4) and (2.30) in (2.26) and comparing the coefficient of z and z^2, we get

$$([2]_{s,t}-1)(1-\lambda)a_2=\frac{1}{2}h_0\xi_1p_1, \tag{2.32}$$

$$\begin{aligned}&([3]_{s,t}-1)(1-\lambda)a_3-([2]_{s,t}-1)(1-\lambda)[1+([2]_{s,t}-1)\lambda]a_2^2\\&=\frac{1}{2}h_1\xi_1p_1+\frac{1}{2}h_0\xi_1p_2+\frac{1}{4}h_0(\xi_2-\xi_1)p_1^2.\end{aligned} \tag{2.33}$$

Similarly using (2.5) and (2.31) in (2.27) and comparing the coefficient of ω and ω^2, we get

$$-([2]_{s,t}-1)(1-\lambda)a_2=\frac{1}{2}l_0\xi_1q_1, \tag{2.34}$$

$$\begin{aligned}&([3]_{s,t}-1)(1-\lambda)(2a_2^2-a_3)-([2]_{s,t}-1)(1-\lambda)[1+([2]_{s,t}-1)\lambda]a_2^2\\&=\frac{1}{2}l_1\xi_1q_1+\frac{1}{2}l_0\xi_1q_2+\frac{1}{4}l_0(\xi_2-\xi_1)q_1^2.\end{aligned} \tag{2.35}$$

From (2.33) and (2.35), we get

$$a_3=a_2^2+\frac{\gamma[\frac{1}{2}(h_1p_1-l_1q_1)\xi_1+\frac{1}{2}(h_0p_2-l_0q_2)\xi_1+\frac{1}{4}(\xi_2-\xi_1)(h_0p_1^2-l_0q_1^2)]}{2([3]_{s,t}-1)(1-\lambda)}, \tag{2.36}$$

$$a_2^2=\frac{\gamma[\frac{1}{2}(h_1p_1+l_1q_1)\xi_1+\frac{1}{2}(h_0p_2+l_0q_2)\xi_1+\frac{1}{4}(\xi_2-\xi_1)(h_0p_1^2+l_0q_1^2)]}{2\{([3]_{s,t}-1)(1-\lambda)-([2]_{s,t}-1)(1-\lambda)[1+([2]_{s,t}-1)\lambda]\}}. \tag{2.37}$$

Using (2.32) and (2.34), we obtain

$$h_0p_1^2+l_0q_1^2=\frac{4([2]_{s,t}-1)^2(1-\lambda)^2(l_0+h_0)}{\gamma^2h_0l_0\xi^2}a_2^2. \tag{2.38}$$

From (2.35)–(2.37), we get

$$\begin{aligned}a_3-\mu a_2^2=&\frac{\gamma\xi_1}{2}[(M(\mu)+\frac{1}{2([3]_{s,t}-1)(1-\lambda)})(h_1p_1+h_0p_2)+\\&(M(\mu)-\frac{1}{2([3]_{s,t}-1)(1-\lambda)})(l_1q_1+l_0q_2)]+\frac{(\xi_2-\xi_1)(h_0p_1^2-l_0q_1^2)}{8([3]_{s,t}-1)(1-\lambda)}\end{aligned} \tag{2.39}$$

where

$$M(\mu)=\frac{\gamma h_0l_0\xi_1^2(1-\mu)}{(1-\lambda)[2\gamma h_0l_0([3]_{s,t}-[2]_{s,t}-([2]_{s,t}-1)^2\lambda)\xi_1^2-([2]_{s,t}-1)^2(1-\lambda)(l_0+h_0)(\xi_2-\xi_1)]}.$$

By taking the moduli on both sides of (2.39) and applying Lemma 1.4, we finally obtain (2.24) and (2.25). This evidently completes the proof of Theorem 2.11. □

By taking special values of parameters λ,s,t in Theorem 2.11, we easily obtain the following results.

Corollary 2.12 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, s, t, \varphi)$ be given by (1.1). Then for any number $\mu \in \mathbb{C}$ and $[3]_{s,t} > 1$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2([3]_{s,t}-1)}, & 0 \leq |M(\mu)| \leq \frac{1}{2([3]_{s,t}-1)}, \\ [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)| \geq \frac{1}{2([3]_{s,t}-1)} \end{cases}$$

For any number $\mu \in \mathbb{C}$ and $[3]_{s,t} < 1$

$$|a_3 - \mu a_2^2| \leq \begin{cases} [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & 0 \leq |M(\mu)| \leq \frac{1}{2(1-[3]_{s,t})}, \\ \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2(1-[3]_{s,t})}, & |M(\mu)| \geq \frac{1}{2(1-[3]_{s,t})} \end{cases}$$

where

$$M(\mu) = \frac{\gamma h_0 l_0 \xi_1^2(1-\mu)}{2\gamma h_0 l_0([3]_{s,t} - [2]_{s,t} - ([2]_{s,t}-1)^2)\xi_1^2 - ([2]_{s,t}-1)^2(l_0+h_0)(\xi_2-\xi_1)}.$$

Corollary 2.13 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, t, \varphi)$ be given by (1.1). Then for any number $\mu \in \mathbb{C}$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2([3]_t-1)(1-\lambda)}, & 0 \leq |M(\mu)| \leq \frac{1}{2([3]_t-1)(1-\lambda)}, \\ [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)| \geq \frac{1}{2([3]_t-1)(1-\lambda)} \end{cases}$$

where

$$M(\mu) = \frac{\gamma h_0 l_0 \xi_1^2(1-\mu)}{(1-\lambda)[2\gamma h_0 l_0([3]_t - [2]_t - ([2]_t-1)^2\lambda)\xi_1^2 - ([2]_t-1)^2(1-\lambda)(l_0+h_0)(\xi_2-\xi_1)]}.$$

Corollary 2.14 *Let the function $f(z) \in \mathcal{Q}^q_{\sum}(\gamma, \lambda, \varphi)$ be given by (1.1). Then for any number $\mu \in \mathbb{C}$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\gamma|\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{4(1-\lambda)}, & 0 \leq |M(\mu)| \leq \frac{1}{4(1-\lambda)}, \\ [2|\gamma|\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)| \geq \frac{1}{4(1-\lambda)} \end{cases}$$

where

$$M(\mu) = \frac{\gamma h_0 l_0 \xi_1^2(1-\mu)}{(1-\lambda)^2[2\gamma h_0 l_0 \xi_1^2 - (l_0+h_0)(\xi_2-\xi_1)]}.$$

Acknowledgements We thank the referees for their time and comments.

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