Journal of Mathematical Research with Applications
Nov., 2021, Vol. 41, No. 6, pp. 594–606
DOI:10.3770/j.issn:2095-2651.2021.06.004
Http://jmre.dlut.edu.cn

Weighted Composition Operators on the Symmetric Fock Space

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Abstract In this paper, we study bounded (linear and anti-linear) weighted composition operators on the symmetric Fock space over a separable Hilbert space. The unitary and self-adjoint weighted composition operators are characterized completely. A class of normal weighted composition operators is considered.

Keywords symmetric Fock space; weighted composition operator; anti-linear weighted composition operator

MR(2020) Subject Classification 47B38; 47B32

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. For each positive integer n, $\otimes_s^n\mathcal{H}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$. The infinite direct sum Hilbert space

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}$$

is called the symmetric Fock space (also referred to as the Segal-Bargmann space) over $\mathcal{H}$, where $\bigotimes_s^0 \mathcal{H} = \mathbb{C}$, the complex numbers. In quantum theory, a symmetric Fock space is used to describe a Hilbert space of state for the system of a Bose field. When $\mathcal{H} = \mathbb{C}^N$, the N-dimensional Euclidean space, $\mathcal{F}(\mathbb{C}^N)$ is the classical Fock space over $\mathbb{C}^N$. Let dV be the Lebesgue measure on $\mathbb{C}^N$. It is known that

$$\mathcal{F}(\mathbb{C}^N) = \Big\{ f \text{ is analytic on } \mathbb{C}^N \Big| \|f\| = \Big(\frac{1}{\pi^N}\int_{\mathbb{C}^N} |f(z)|^2 e^{-|z|^2} \mathrm{d}V(z)\Big)^{\frac{1}{2}} < \infty \Big\}.$$

Generally, each element f in $\mathcal{F}(\mathcal{H})$ can be identified as an entire function on $\mathcal{H}$ having a power expansion of the form

$$f(z) = \sum_{n=0}^{\infty} \langle z^n, a_n\rangle \text{ for all } z \in \mathcal{H},$$

and

$$\|f\|^2 = \sum_{n=0}^{\infty} n! \|a_n\|^2,$$

Received July 1, 2020; Accepted April 6, 2021
Supported by the National Natural Science Foundation of China (Grant No. 11771401), Shanxi Scholarship Council of China (Grant No. 2020-089) and Fund Program for the Scientific Activities of Selected Returned Overseas Professionals in Shanxi Province (Grant No. 20200019).
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where $z^0 = 1$, $a_0 \in \mathbb{C}$, $z^n = z \otimes \cdots \otimes z$ and $a_n \in \otimes_s^n \mathcal{H}$ for $n \geq 1$. Furthermore, $\mathcal{F}(\mathcal{H})$ has reproducing kernels

$$K_w(z) = \exp(\langle z, w\rangle), \quad z, w \in \mathcal{H}.$$

Therefore, $\mathcal{F}(\mathcal{H})$ is identified as an analytic function space on $\mathcal{H}$ with reproducing kernels and the set $E = \operatorname{span}\{K_w \mid w \in \mathcal{H}\}$ is dense in $\mathcal{F}(\mathcal{H})$ (see [1]).

In the past thirty years, Toeplitz operators, Hankel operators and weighted composition operators on $\mathcal{F}(\mathbb{C}^N)$ have been studied intensively [2–6]. Since the study of $\mathcal{F}(\mathcal{H})$ over a separable Hilbert space $\mathcal{H}$ is related to analysis on infinite dimensional spaces, there is little known about these operators on $\mathcal{F}(\mathcal{H})$. In [7], a few basic properties of Toeplitz operators on $\mathcal{F}(\mathcal{H})$ were discussed. In [1] and [8], bounded and unbounded composition operators on $\mathcal{F}(\mathcal{H})$ were studied respectively. It seems that characterization of (weighted) composition operators on $\mathcal{F}(\mathcal{H})$ mainly depends on properties of the operators acting on reproducing kernels. Directed by this idea, in this paper, we will study some classes of bounded (linear and anti-linear) weighted composition operators on $\mathcal{F}(\mathcal{H})$. The results extend the corresponding results in $\mathcal{F}(\mathbb{C}^N)$ (see [9–13]).

Now we present the definitions of (linear and anti-linear) weighted composition operators on $\mathcal{F}(\mathcal{H})$.

Let $\psi \in \mathcal{F}(\mathcal{H})$, $\psi \neq 0$, φ be a mapping on $\mathcal{H}$ and J be a conjugation on $\mathcal{H}$. The weighted composition operator $C_{\psi,\varphi}$ and the anti-linear weighted composition operator $\mathcal{A}_{\psi,\varphi}$, $\mathcal{T}_{\psi,\varphi}$ on $\mathcal{F}(\mathcal{H})$ are defined as follows. For any $f \in \mathcal{F}(\mathcal{H})$,

$$\begin{aligned}
&(C_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)),\\
&(\mathcal{A}_{\psi,\varphi} f)(z) = \psi(z)\overline{f(J\varphi(z))}, \quad z \in \mathcal{H},\\
&(\mathcal{T}_{\psi,\varphi} f)(z) = \overline{\psi(Jz) f(\varphi(Jz))}.
\end{aligned}$$

Recall that a conjugation on a Hilbert space $\mathcal{H}$ is an anti-linear mapping which satisfies the following conditions:

$$J^2 = I;$$

$$\langle Jz, Jw\rangle = \langle w, z\rangle \text{ for all } z, w \in \mathcal{H}.$$

A linear operator A is called J-symmetric if $JAJ = A^*$. J induces a conjugation $\mathcal{J}$ on $\mathcal{F}(\mathcal{H})$, that is, for any $f \in \mathcal{F}(\mathcal{H})$,

$$(\mathcal{J} f)(z) = \overline{f(Jz)}, \quad z \in \mathcal{H}.$$

In fact, let $J^0 a = \bar{a}$ for $a \in \mathbb{C}$ and $J^n = J \otimes \cdots \otimes J$ for any positive integer n. Then for $f(z) = \sum \langle z^n, a_n\rangle \in \mathcal{F}(\mathcal{H})$,

$$(\mathcal{J} f)(z) = \overline{f(Jz)} = \overline{\sum \langle (Jz)^n, a_n\rangle} = \overline{\sum \langle J^n z^n, a_n\rangle} = \overline{\sum \langle J^n a_n, z^n\rangle} = \sum \langle z^n, J^n a_n\rangle.$$

In quantum theory, $\mathcal{J}$ is called the Boson Γ operator on $\mathcal{F}(\mathcal{H})$ for J.

It is easy to verify the following relationship between $\mathcal{A}_{\psi,\varphi}$, $\mathcal{T}_{\psi,\varphi}$ and $C_{\psi,\varphi}$.

Lemma 1.1 *Let $\psi \in \mathcal{F}(\mathcal{H})$, $\psi \neq 0$, and φ be a mapping on $\mathcal{H}$. Then*

$$\mathcal{A}_{\psi,\varphi} = C_{\psi,\varphi}\mathcal{J}, \ \mathcal{T}_{\psi,\varphi} = \mathcal{J} C_{\psi,\varphi}, \ \mathcal{T}_{\psi,\varphi} = \mathcal{A}_{\mathcal{J}\psi, J\circ\varphi\circ J}.$$

This paper is organized as follows. In Subsection 2.1, some elementary results on weighted composition operators on $\mathcal{F}(\mathcal{H})$ are discussed. In Subsection 2.2, a class of bounded weighted composition operators on $\mathcal{F}(\mathcal{H})$ is characterized. In Subsections 2.3 and 2.4, unitary and self-adjoint weighted composition operators on $\mathcal{F}(\mathcal{H})$ are characterized respectively. In Subsection 2.5, a class of normal weighted composition operators is characterized.

2. Main results and proofs

In this section, the main results and their proofs are presented. Firstly, we study some properties of bounded (linear or anti-linear) weighted composition operators on $\mathcal{F}(\mathcal{H})$. Then the unitary, self-adjoint and a class of normal weighted composition operators are characterized.

2.1. Preliminaries

In this subsection, we consider some elementary results on weighted composition operators on $\mathcal{F}(\mathcal{H})$. The letter I is used for the identity operator on either $\mathcal{H}$ or $\mathcal{F}(\mathcal{H})$.

Lemma 2.1 *Let $\psi \in \mathcal{F}(\mathcal{H})$, $\psi \neq 0$, and φ be a mapping on $\mathcal{H}$.*

(1) $\mathcal{A}_{\psi,\varphi}$ is bounded if and only if $C_{\psi,\varphi}$ is bounded.

(2) $\mathcal{A}_{\psi,\varphi}$ is isometric if and only if $C_{\psi,\varphi}$ is isometric.

(3) $\mathcal{A}_{\psi,\varphi}$ is co-isometric if and only if $C_{\psi,\varphi}$ is co-isometric.

(4) $\mathcal{A}_{\psi,\varphi}$ is anti-unitary if and only if $C_{\psi,\varphi}$ is unitary.

(5) $\mathcal{A}_{\psi,\varphi}$ is self-adjoint if and only if $C_{\psi,\varphi}$ is $\mathcal{J}$-symmetric.

(6) $\mathcal{T}_{\psi,\varphi} = \mathcal{A}_{\psi,\varphi}$ if and only if $\mathcal{J}\psi = \psi$ and $J \circ \varphi \circ J = \varphi$.

Proof Since $\mathcal{J}$ is a conjugation, $\mathcal{J}^* = \mathcal{J}$ and $\mathcal{J}^2 = I$. It follows from Lemma 1.1 that the statement (1) holds.

Again by Lemma 1.1, we have

$$\mathcal{A}^*_{\psi,\varphi}\mathcal{A}_{\psi,\varphi} = \mathcal{J}C^*_{\psi,\varphi}C_{\psi,\varphi}\mathcal{J},$$

$$\mathcal{A}_{\psi,\varphi}\mathcal{A}^*_{\psi,\varphi} = C_{\psi,\varphi}\mathcal{J}\mathcal{J}C^*_{\psi,\varphi} = C_{\psi,\varphi}C^*_{\psi,\varphi},$$

$$\mathcal{A}^*_{\psi,\varphi} = \mathcal{J}C^*_{\psi,\varphi}.$$

So $\mathcal{A}^*_{\psi,\varphi}\mathcal{A}_{\psi,\varphi} = I$ if and only if $C^*_{\psi,\varphi}C_{\psi,\varphi} = I$; $\mathcal{A}_{\psi,\varphi}\mathcal{A}^*_{\psi,\varphi} = I$ if and only if $C_{\psi,\varphi}C^*_{\psi,\varphi} = I$ and $\mathcal{A}_{\psi,\varphi} = \mathcal{A}^*_{\psi,\varphi}$ if and only if $C_{\psi,\varphi}\mathcal{J} = \mathcal{J}C^*_{\psi,\varphi}$.

The statements (2)–(5) follow from the reasoning above. (6) follows from Lemma 1.1. □

It follows from Lemmas 1.1 that the properties of $\mathcal{T}_{\psi,\varphi}$ can be obtained by the properties of $C_{\psi,\varphi}$ and $\mathcal{A}_{\psi,\varphi}$. In the following, we only consider bounded weighted composition operator $C_{\psi,\varphi}$ and anti-linear weighted composition operator $\mathcal{A}_{\psi,\varphi}$.

Lemma 2.2 *Let $\psi \in \mathcal{F}(\mathcal{H})$ and φ be a mapping on $\mathcal{H}$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$, then*

$$C^*_{\psi,\varphi}K_w = \overline{\psi(w)}K_{\varphi(w)}, \quad \mathcal{A}^*_{\psi,\varphi}K_w = \psi(w)K_{J\varphi(w)}.$$

The proof of Lemma 2.2 is routine. We omit the proof here.

For any $c \in \mathcal{H}$, let $\varphi_c(z) = z - c$ and $k_c(z) = \exp(\langle z, c\rangle - \frac{\|c\|^2}{2})$ be the normalization of K_c. Denote $U_c = C_{k_c,\varphi_c}$.

Lemma 2.3 *U_c is a unitary on $\mathcal{F}(\mathcal{H})$.*

Proof A straightforward computation shows that

$$\begin{aligned}(U_cK_w)(z) =&k_c(z)K_w(z-c) = \exp(\langle z, c\rangle - \frac{\|c\|^2}{2})\exp(\langle z-c, w\rangle)\\ =&\exp(-\frac{\|c\|^2}{2} - \langle c, w\rangle)\exp(\langle z, c+w\rangle)\\ =&\exp(-\frac{\|c\|^2}{2} - \langle c, w\rangle)K_{w+c}(z),\end{aligned}\tag{2.1}$$

which implies that K_w is in the domain of U_c and $U_cK_w \in E$. Hence E is contained in the domain of U_c, U_c is densely defined in $\mathcal{F}(\mathcal{H})$ and $U_cE \subset E$.

For any f in the domain of U_c, we have

$$\begin{aligned}\langle K_w, U_cf\rangle =&\langle K_w, k_cf\circ\varphi_c\rangle = \overline{k_c(w)f(\varphi_c(w))}\\ =&\exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)\overline{f(w-c)}\\ =&\exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)\langle K_{w-c}, f\rangle\\ =&\langle\exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)K_{w-c}, f\rangle,\end{aligned}$$

which implies that K_w is in the domain of U_c^* and

$$U_c^*K_w = \exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)K_{w-c} \in E. \tag{2.2}$$

Hence E is contained in the domain of U_c^* and $U_c^*E \subset E$.

Moreover, we have

$$\begin{aligned}U_c^*U_cK_w =&U_c^*(\exp(-\frac{\|c\|^2}{2} - \langle c, w\rangle)K_{w+c}) = \exp(-\frac{\|c\|^2}{2} - \langle c, w\rangle)U_c^*K_{w+c}\\ =&\exp(-\frac{\|c\|^2}{2} - \langle c, w\rangle)\exp(-\frac{\|c\|^2}{2} + \langle c, w+c\rangle)K_{(w+c)-c} = K_w,\\ U_cU_c^*K_w =&U_c(\exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)K_{w-c})\\ =&\exp(-\frac{\|c\|^2}{2} + \langle c, w\rangle)\exp(-\frac{\|c\|^2}{2} - \langle c, w-c\rangle)K_{(w-c)+c} = K_w.\end{aligned}$$

So $U_c^*U_c|_E = U_cU_c^*|_E = I|_E$. It follows from the density of E in $\mathcal{F}(\mathcal{H})$ that

$$U_c^*U_c = U_cU_c^* = I.$$

Thus U_c is a unitary on $\mathcal{F}(\mathcal{H})$. Combining with (2.1) and (2.2), we see that $U_c^* = U_{-c}$. □

2.2. Bounded weighted composition operators

In this subsection, we characterize a class of bounded weighted composition operators $C_{\psi,\varphi}$ and $\mathcal{A}_{\psi,\varphi}$ for ψ being a reproducing kernel in $\mathcal{F}(\mathcal{H})$.

Theorem 2.4 *Let $\psi = k_c$ and φ be a mapping on $\mathcal{H}$. Then $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$ if and only if*

$$\varphi(z) = Az + b$$

for all $z \in \mathcal{H}$, where A is a linear operator on $\mathcal{H}$ with $\|A\| \leq 1$ and $b \in \mathcal{H}$ such that

$$A^*(b + Ac) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}.$$

Moreover,

$$\|C_{\psi,\varphi}\| = \exp(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b + Ac\|^2),$$

where v is the unique vector in $\mathcal{H}$ of minimum norm satisfying $A^*(b + Ac) = (I - A^*A)^{\frac{1}{2}}v$.

Proof For $f \in \mathcal{F}(\mathcal{H})$, we have

$$\begin{aligned}(U_{-c}C_{\psi,\varphi}f)(z) =& k_{-c}(z)(C_{\psi,\varphi}f(z + c)) = k_{-c}(z)\psi(z + c)f(\varphi(z + c))\\ =& \exp(\langle z, -c\rangle - \frac{\|c\|^2}{2})\exp(\langle z + c, c\rangle - \frac{\|c\|^2}{2})f(\varphi(z + c))\\ =& f(\varphi(z + c)) = (C_{\varphi_1}f)(z),\end{aligned}$$

where $\varphi_1(z) = \varphi(z + c)$. Hence $U_{-c}C_{\psi,\varphi} = C_{\varphi_1}$.

By Lemma 2.3, we know that $C_{\psi,\varphi}$ is bounded if and only if C_{φ_1} is bounded. It follows from [1, Theorem 1.3] that C_{φ_1} is bounded if and only if $\varphi_1(z) = Az + b_1$ for all $z \in \mathcal{H}$, where A is a linear operator on $\mathcal{H}$ with $\|A\| \leq 1$ and $b_1 \in \mathcal{H}$ such that $A^*b_1 \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}$. Hence $C_{\psi,\varphi}$ is bounded if and only if

$$\varphi(z) = \varphi_1(z - c) = Az + b_1 - Ac = Az + b$$

for all $z \in \mathcal{H}$, where $b = b_1 - Ac \in \mathcal{H}$, A is a linear operator on $\mathcal{H}$ with $\|A\| \leq 1$ such that

$$A^*(b + Ac) = A^*b_1 \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}.$$

Again by Lemma 2.3 and [1, Theorem 1.3], we have

$$\|C_{\psi,\varphi}\| = \|C_{\varphi_1}\| = \exp(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b + Ac\|^2),$$

where v is the unique vector in $\mathcal{H}$ of minimum norm satisfying $A^*(b + Ac) = (I - A^*A)^{\frac{1}{2}}v$. □

By Lemma 2.1 (1) and Theorem 2.4, we have the corresponding result for anti-linear weighted composition operators on $\mathcal{F}(\mathcal{H})$.

Corollary 2.5 *Let $\psi = k_c$ and φ be a mapping on $\mathcal{H}$. Then $\mathcal{A}_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$ if and only if $\varphi(z) = Az + b$ for all $z \in \mathcal{H}$, where A is a linear operator on $\mathcal{H}$ with $\|A\| \leq 1$ and $A^*(b + Ac) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}$. Moreover, $\|\mathcal{A}_{\psi,\varphi}\| = \exp(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b + Ac\|^2)$, where v is the unique vector in $\mathcal{H}$ of minimum norm satisfying $A^*(b + Ac) = (I - A^*A)^{\frac{1}{2}}v$.*

Let $\psi(z) = K_c(z)$ and $\varphi(z) = Az + b$ such that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$. In the following subsections, we will see that many important bounded (linear and anti-linear) weighted composition operators on $\mathcal{F}(\mathcal{H})$ have the forms in Theorem 2.4 and Corollary 2.5, and the formulas below are used repeatedly.

$$\begin{aligned}
(C_{\psi,\varphi}K_w)(z) =& \psi(z)K_w(\varphi(z)) = K_c(z)K_w(Az+b)\\
=& \exp(\langle z,c\rangle)\exp(\langle Az+b,w\rangle)\\
=& \exp(\langle b,w\rangle)\exp(\langle z,c+A^*w\rangle)\\
=& \exp(\langle b,w\rangle)K_{A^*w+c}(z), \qquad (2.3)\\
(C^*_{\psi,\varphi}K_w)(z) =& \langle C^*_{\psi,\varphi}K_w, K_z\rangle = \langle K_w, C_{\psi,\varphi}K_z\rangle\\
=& \overline{\psi(w)K_z(\varphi(w))} = \overline{K_c(w)K_z(Aw+b)}\\
=& \exp(\langle c,w\rangle)\exp(\langle z,Aw+b\rangle)\\
=& \exp(\langle c,w\rangle)K_{Aw+b}(z), \qquad (2.4)\\
(C^*_{\psi,\varphi}C_{\psi,\varphi}K_w)(z) =& \langle C_{\psi,\varphi}K_w, C_{\psi,\varphi}K_z\rangle\\
=& \langle \exp(\langle b,w\rangle)K_{A^*w+c}, \exp(\langle b,z\rangle)K_{A^*z+c}\rangle\\
=& \exp(\langle b,w\rangle+\langle z,b\rangle)K_{A^*w+c}(A^*z+c)\\
=& \exp(\langle b,w\rangle+\langle z,b\rangle)\exp(\langle A^*z+c,A^*w+c\rangle)\\
=& \exp(\langle A^*z,A^*w\rangle+\langle z,Ac+b\rangle+\langle Ac+b,w\rangle+\langle c,c\rangle), \qquad (2.5)\\
(C_{\psi,\varphi}C^*_{\psi,\varphi}K_w)(z) =& \langle C^*_{\psi,\varphi}K_w, C^*_{\psi,\varphi}K_z\rangle\\
=& \langle \overline{\psi(w)}K_{\varphi(w)}, \overline{\psi(z)}K_{\varphi(z)}\rangle\\
=& \psi(z)\overline{\psi(w)}K_{\varphi(w)}(\varphi(z))\\
=& \exp(\langle z,c\rangle+\langle c,w\rangle)K_{Aw+b}(Az+b)\\
=& \exp(\langle z,c\rangle+\langle c,w\rangle)\exp(\langle Az+b,Aw+b\rangle)\\
=& \exp(\langle Az,Aw\rangle+\langle z,A^*b+c\rangle+\langle A^*b+c,w\rangle+\langle b,b\rangle), \qquad (2.6)\\
(\mathcal{A}_{\psi,\varphi}K_w)(z) =& \psi(z)\overline{K_w(J\varphi(z))} = \exp(\langle z,c\rangle+\langle w,J(Az+b)\rangle)\\
=& \exp(\langle z,c\rangle+\langle Az+b,Jw\rangle)\\
=& \exp(\langle z,A^*Jw+c\rangle+\langle w,Jb\rangle)\\
=& K_{Jb}(w)K_{A^*Jw+c}(z), \qquad (2.7)\\
(\mathcal{A}^*_{\psi,\varphi}K_w)(z) =& \langle \mathcal{A}^*_{\psi,\varphi}K_w, K_z\rangle = \langle \mathcal{A}_{\psi,\varphi}K_z, K_w\rangle = (\mathcal{A}_{\psi,\varphi}K_z)(w)\\
=& K_{Jb}(z)K_{A^*Jz+c}(w)\\
=& K_c(w)K_{JAw+Jb}(z), \qquad (2.8)\\
\mathcal{A}^*_{\psi,\varphi}\mathcal{A}_{\psi,\varphi}K_w =& \mathcal{A}^*_{\psi,\varphi}(K_{Jb}(w)K_{A^*Jw+c}) = \overline{K_{Jb}(w)}(\mathcal{A}^*_{\psi,\varphi}K_{A^*Jw+c})\\
=& K_w(Jb)K_c(A^*Jw+c)K_{JA(A^*Jw+c)+Jb}\\
=& K_w(Jb)K_c(A^*Jw+c)K_{JAA^*Jw+JAc+Jb}, \qquad (2.9)\\
\mathcal{A}_{\psi,\varphi}\mathcal{A}^*_{\psi,\varphi}K_w =& \mathcal{A}_{\psi,\varphi}(K_c(w)K_{JAw+Jb}) = \overline{K_c(w)}(\mathcal{A}_{\psi,\varphi}K_{JAw+Jb})\\
=& K_w(c)K_{Jb}(JAw+Jb)K_{A^*J(JAw+Jb)+c}\\
=& K_w(c)K_{Jb}(JAw+Jb)K_{A^*Aw+A^*b+c}. \qquad (2.10)
\end{aligned}$$

2.3 Unitary weighted composition operators

In this subsection, we completely characterize the unitary (linear and anti-linear) weighted composition operators on $\mathcal{F}(\mathcal{H})$.

Proposition 2.6 *Let $\psi(z) = k_c(z)$ and φ be a mapping on $\mathcal{H}$ such that $C_{\psi,\varphi}$ is bounded. Then $C_{\psi,\varphi}$ is isometric if and only if there exists a co-isometric operator A on $\mathcal{H}$ such that*

$$\varphi(z) = Az - Ac.$$

Proof It follows from the proof of Theorem 2.4 that

$$U_{-c}C_{\psi,\varphi} = C_{\varphi_1}, \quad \varphi_1(z) = \varphi(z + c).$$

So, by Lemma 2.3,

$$C^*_{\varphi_1}C_{\varphi_1} = C^*_{\psi,\varphi}U^*_{-c}U_{-c}C_{\psi,\varphi} = C^*_{\psi,\varphi}C_{\psi,\varphi}.$$

Hence $C_{\psi,\varphi}$ is isometric if and only if C_{φ_1} is isometric. By [1, Proposition 5.1], C_{φ_1} is isometric if and only if there exists a co-isometric operator A on $\mathcal{H}$ such that $\varphi_1(z) = Az$. Therefore, $C_{\psi,\varphi}$ is isometric if and only if there exists a co-isometric operator A on $\mathcal{H}$ such that

$$\varphi(z) = \varphi_1(z - c) = Az - Ac. \quad \Box$$

By Lemma 2.1 (2) and Proposition 2.6, we have the following result for anti-linear weighted composition operators.

Corollary 2.7 *Let $\psi(z) = k_c(z)$ and φ be a mapping on $\mathcal{H}$ such that $\mathcal{A}_{\psi,\varphi}$ is bounded. Then $\mathcal{A}_{\psi,\varphi}$ is isometric if and only if there exists a co-isometric operator A on $\mathcal{H}$ such that $\varphi(z) = Az - Ac$.*

Theorem 2.8 *Let $\psi \in \mathcal{F}(\mathcal{H})$ and φ be a mapping on $\mathcal{H}$ such that $C_{\psi,\varphi}$ is bounded.*

(1) $C_{\psi,\varphi}$ is co-isometric if and only if there exists an isometric operator A on $\mathcal{H}$ and a vector $b \in \mathcal{H}$ such that

$$\varphi(z) = Az + b,\ \psi(z) = \psi(0)K_{-A^*b}(z),\ |\psi(0)|^2 \exp(\|b\|^2) = 1. \tag{2.11}$$

(2) $C_{\psi,\varphi}$ is a unitary if and only if there exists a unitary operator A on $\mathcal{H}$ and a vector $b \in \mathcal{H}$ such that (2.11) holds.

Proof (1) Suppose that $C_{\psi,\varphi}$ is co-isometric. Then $C_{\psi,\varphi}C^*_{\psi,\varphi} = I$. Thus

$$K_w(z) = (C_{\psi,\varphi}C^*_{\psi,\varphi}K_w)(z) = \overline{\psi(w)}\psi(z)K_{\varphi(w)}(\varphi(z)). \tag{2.12}$$

Let $w = 0$. Then we have

$$\overline{\psi(0)}\psi(z)K_{\varphi(0)}(\varphi(z)) = 1.$$

Let $z = 0$. Then

$$\overline{\psi(0)}\psi(0)K_{\varphi(0)}(\varphi(0)) = 1.$$

Hence

$$|\psi(0)|^2 \exp(\|\varphi(0)\|^2) = 1$$

and

$$\psi(z) = \psi(0)\exp(\|\varphi(0)\|^2 - \langle\varphi(z), \varphi(0)\rangle). \tag{2.13}$$

Taking (2.13) into (2.12), we obtain

$$\begin{aligned}\exp(\langle z, w\rangle) &= |\psi(0)|^2 \exp(2\|\varphi(0)\|^2)\exp(-\langle\varphi(z), \varphi(0)\rangle - \langle\varphi(0), \varphi(w)\rangle + \langle\varphi(z), \varphi(w)\rangle) \\ &= \exp(\langle\varphi(z) - \varphi(0), \varphi(w) - \varphi(0)\rangle).\end{aligned}$$

Hence

$$\langle\varphi(z) - \varphi(0), \varphi(w) - \varphi(0)\rangle = \langle z, w\rangle.$$

Let $Az = \varphi(z) - \varphi(0)$. Then $\langle Az, Aw\rangle = \langle z, w\rangle$ for all $z, w \in \mathcal{H}$. So A is an isometric operator on $\mathcal{H}$, and

$$\varphi(z) = Az + b$$

with $b = \varphi(0)$. It follows form (2.13) that

$$\psi(z) = \psi(0)K_{-A^*b}(z).$$

Suppose that A is an isometric operator on $\mathcal{H}$, $b \in \mathcal{H}$ such that ψ, φ satisfy the condition (2.11). Let $c = -A^*b$. It follows from (2.6) that

$$\begin{aligned}(C_{\psi,\varphi}C^*_{\psi,\varphi}K_w)(z) &= |\psi(0)|^2 \exp(\langle Az, Aw\rangle + \langle z, A^*b - A^*b\rangle + \langle A^*b - A^*b, w\rangle + \langle b, b\rangle) \\ &= K_w(z).\end{aligned}$$

Hence $C_{\psi,\varphi}C^*_{\psi,\varphi} = I$ and $C_{\psi,\varphi}$ is co-isometric.

(2) Suppose that $C_{\psi,\varphi}$ is a unitary. Then $C_{\psi,\varphi}$ is isometric and co-isometric. It follows from (1) that there exists an isometric operator A on $\mathcal{H}$ and a vector $b \in \mathcal{H}$ such that ψ, φ satisfy the condition (2.11).

Let $c = -A^*b$. It follows from (2.5) that

$$\begin{aligned}\exp(\langle z, w\rangle) &= K_w(z) = (C^*_{\psi,\varphi}C_{\psi,\varphi}K_w)(z) \\ &= |\psi(0)|^2 \exp(\langle A^*z, A^*w\rangle + \langle z, -AA^*b + b\rangle + \langle -AA^*b + b, w\rangle + \langle -A^*b, -A^*b\rangle) \\ &= |\psi(0)|^2 \exp(\langle b, w\rangle + \langle z, b\rangle)\exp(\langle A^*(z - b), A^*(w - b)\rangle) \\ &= \exp(-\langle b, b\rangle + \langle b, w\rangle + \langle z, b\rangle)\exp(\langle A^*(z - b), A^*(w - b)\rangle).\end{aligned}$$

Hence

$$\langle z - b, w - b\rangle = \langle A^*(z - b), A^*(w - b)\rangle$$

for all $z, w \in \mathcal{H}$. So A^* is an isometric operator. Therefore, A is a unitary.

Suppose A is a unitary on $\mathcal{H}$, $b \in \mathcal{H}$ such that ψ, φ satisfy the condition (2.11). Let $c = -A^*b$. It follows from (2.5) and (2.6) that

$$\begin{aligned}&(C^*_{\psi,\varphi}C_{\psi,\varphi}K_w)(z) \\ &= |\psi(0)|^2 \exp(\langle A^*z, A^*w\rangle + \langle -AA^*b + b, w\rangle + \langle z, -AA^*b + b\rangle + \langle -A^*b, -A^*b\rangle) \\ &= |\psi(0)|^2 \exp(\langle z, w\rangle + \langle b, b\rangle) = K_w(z),\end{aligned}$$

$$
\begin{aligned}
&(C_{\psi,\varphi}C^*_{\psi,\varphi}K_w)(z)\\
&\quad=|\psi(0)|^2\exp(\langle Az,Aw\rangle+\langle z,A^*b-A^*b\rangle+\langle A^*b-A^*b,w\rangle+\langle b,b\rangle)=K_w(z).
\end{aligned}
$$

Hence $C_{\psi,\varphi}C^*_{\psi,\varphi}=C^*_{\psi,\varphi}C_{\psi,\varphi}=I$ and $C_{\psi,\varphi}$ is a unitary. $\square$

For anti-linear weighted composition operators on $\mathcal{F}(\mathcal{H})$, we have the following conclusions.

Corollary 2.9 *Let $\psi\in\mathcal{F}(\mathcal{H})$ and φ be a mapping on $\mathcal{H}$ such that $\mathcal{A}_{\psi,\varphi}$ is bounded.*

(1) $\mathcal{A}_{\psi,\varphi}$ is co-isometric if and only if there exists an isometric operator A on $\mathcal{H}$ and a vector $b\in\mathcal{H}$ such that the condition (2.11) holds.

(2) $\mathcal{A}_{\psi,\varphi}$ is an anti-unitary if and only if there exists a unitary operator A on $\mathcal{H}$ and a vector $b\in\mathcal{H}$ such that the condition (2.11) holds.

2.4. Self-adjoint weighted composition operators

In this subsection, we give a complete characterization of (linear and anti-linear) self-adjoint weighted composition operators on $\mathcal{F}(\mathcal{H})$.

Theorem 2.10 *Let $\psi\in\mathcal{F}(\mathcal{H})$, $\psi\neq0$ and φ be a mapping on $\mathcal{H}$ such that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$. Then $C_{\psi,\varphi}$ is self-adjoint if and only if there exists a self-adjoint operator A on $\mathcal{H}$ with $\|A\|\le1$ and a vector $b\in\mathcal{H}$ such that*

$$
\begin{gathered}
A(I+A)b\in\operatorname{ran}(I-A^2)^{\frac12},\\
\varphi(z)=Az+b,\ \psi(z)=\psi(0)K_b(z),\ \psi(0)\text{ is a nonzero real number.}
\end{gathered}\tag{2.14}
$$

Proof Suppose that $C_{\psi,\varphi}$ is a self-adjoint operator. Then $C_{\psi,\varphi}=C^*_{\psi,\varphi}$. So we have

$$
\begin{aligned}
\psi(z)\exp(\langle\varphi(z),w\rangle)=&\psi(z)K_w(\varphi(z))=(C_{\psi,\varphi}K_w)(z)=(C^*_{\psi,\varphi}K_w)(z)\\
=&\overline{\psi(w)}K_{\varphi(w)}(z)=\overline{\psi(w)}\exp(\langle z,\varphi(w)\rangle).
\end{aligned}\tag{2.15}
$$

Let $w=0$. We have

$$
\psi(z)=\overline{\psi(0)}\exp(\langle z,\varphi(0)\rangle).\tag{2.16}
$$

Since $\psi\neq0$, $\psi(0)\neq0$. Let $z=0$ in (2.16). We have $\psi(0)=\overline{\psi(0)}$, which implies that $\psi(0)$ is a real number. Let $b=\varphi(0)$. Then $\psi(z)=\psi(0)K_b(z)$.

Taking (2.16) into (2.15), we obtain

$$
\psi(0)\exp(\langle z,\varphi(0)\rangle+\langle\varphi(z),w\rangle)=\psi(0)\exp(\langle\varphi(0),w\rangle+\langle z,\varphi(w)\rangle).
$$

So $\exp(\langle z,\varphi(w)-\varphi(0)\rangle)=\exp(\langle\varphi(z)-\varphi(0),w\rangle)$. It follows that

$$
\langle z,\varphi(w)-\varphi(0)\rangle=\langle\varphi(z)-\varphi(0),w\rangle.
$$

Let $Az=\varphi(z)-\varphi(0)$. Then $\langle z,Aw\rangle=\langle Az,w\rangle$ for all $z,w\in\mathcal{H}$. So A is a self-adjoint operator on $\mathcal{H}$ and $\varphi(z)=Az+b$ with $b=\varphi(0)$.

Since $C_{\psi,\varphi}$ is bounded, it follows from Theorem 2.4 that $\|A\|\le1$ and

$$
A(I+A)b=A^*(b+Ab)\in\operatorname{ran}(I-A^*A)^{\frac12}=\operatorname{ran}(I-A^2)^{\frac12}.
$$

Suppose that A is a self-adjoint operator on $\mathcal{H}$ with $\|A\| \leq 1$ and $b \in \mathcal{H}$ such that the condition (2.14) holds. Let $c = b$. Then by (2.3) and (2.4), we have

$$(C_{\psi,\varphi}K_w)(z) = \psi(0)\exp(\langle b, w\rangle)K_{A^*w+b}(z),$$
$$(C^*_{\psi,\varphi}K_w)(z) = \psi(0)\exp(\langle b, w\rangle)K_{Aw+b}(z).$$

It follows from $A = A^*$ that $C_{\psi,\varphi}K_w = C^*_{\psi,\varphi}K_w$. Hence $C_{\psi,\varphi}$ is self-adjoint. □

Theorem 2.11 *Let $\psi \in \mathcal{F}(\mathcal{H})$, $\psi \neq 0$ and φ be a mapping on $\mathcal{H}$ such that $\mathcal{A}_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$. Then $\mathcal{A}_{\psi,\varphi}$ is self-adjoint if and only if there exists a J-symmetric operator A on $\mathcal{H}$ with $\|A\| \leq 1$ and a vector $b \in \mathcal{H}$ such that*

$$\begin{aligned} &JA(I + JA)Jb \in \operatorname{ran}(I - (JA)^2)^{\frac{1}{2}}, \\ &\varphi(z) = Az + b, \;\; \psi(z) = \psi(0)K_{Jb}(z). \end{aligned} \tag{2.17}$$

Proof Suppose that $\mathcal{A}_{\psi,\varphi}$ is a self-adjoint operator. Then $\mathcal{A}_{\psi,\varphi} = \mathcal{A}^*_{\psi,\varphi}$. So we have

$$\begin{aligned} \psi(z)\exp(\langle\varphi(z), Jw\rangle) =& \psi(z)\exp(\langle w, J\varphi(z)\rangle) = \psi(z)\overline{\exp(\langle J\varphi(z), w\rangle)} \\ =& \psi(z)\overline{K_w(J\varphi(z))} = (\mathcal{A}_{\psi,\varphi}K_w)(z) \\ =& (\mathcal{A}^*_{\psi,\varphi}K_w)(z) = \psi(w)K_{J\varphi(w)}(z) \\ =& \psi(w)\exp(\langle z, J\varphi(w)\rangle). \end{aligned} \tag{2.18}$$

Let $w = 0$. We have

$$\psi(z) = \psi(0)\exp(\langle z, J\varphi(0)\rangle). \tag{2.19}$$

Since $\psi \neq 0$, $\psi(0) \neq 0$. Let $b = \varphi(0)$. Then

$$\psi(z) = \psi(0)K_{Jb}(z).$$

Taking (2.19) into (2.18), we obtain

$$\psi(0)\exp(\langle z, J\varphi(0)\rangle + \langle\varphi(z), Jw\rangle) = \psi(0)\exp(\langle w, J\varphi(0)\rangle + \langle z, J\varphi(w)\rangle).$$

So $\exp(\langle z, J(\varphi(w) - \varphi(0))\rangle) = \exp(\langle w, J(\varphi(z) - \varphi(0))\rangle)$. It follows that

$$\langle z, J(\varphi(w) - \varphi(0))\rangle = \langle w, J(\varphi(z) - \varphi(0))\rangle.$$

Let $Bz = J(\varphi(z) - \varphi(0))$. Then $\langle z, Bw\rangle = \langle w, Bz\rangle$. So B is an anti-linear self-adjoint operator on $\mathcal{H}$. Let $A = JB$. Then

$$A^* = JAJ, \quad \varphi(z) = Az + b$$

with $b = \varphi(0)$.

Since $\mathcal{A}_{\psi,\varphi}$ is bounded, it follows from Corollary 2.5 that $\|A\| \leq 1$ and

$$JA(I + JA)Jb = A^*(b + AJb) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}} = \operatorname{ran}(I - (JA)^2)^{\frac{1}{2}}.$$

Suppose that A is a J-symmetric operator on $\mathcal{H}$ with $\|A\| \leq 1$ and $b \in \mathcal{H}$ such that the condition (2.17) holds. Then we have

$$(\mathcal{A}_{\psi,\varphi}K_w)(z) = \psi(z)\overline{K_w(J\varphi(z))} = \psi(0)\exp(\langle z, Jb\rangle)\overline{\exp(\langle J(Az + b), w\rangle)}$$

$$
\begin{aligned}
&= \psi(0)\exp(\langle z, Jb\rangle)\exp(\langle w, J(Az+b)\rangle),\\
(\mathcal{A}^*_{\psi,\varphi}K_w)(z) &= \psi(w)\overline{K_z(J\varphi(w))} = \psi(0)\exp(\langle w, Jb\rangle)\overline{\exp(\langle J(Aw+b), z\rangle)}\\
&= \psi(0)\exp(\langle w, Jb\rangle)\exp(\langle z, J(Aw+b)\rangle).
\end{aligned}
$$

It follows from $A = JA^*J$ that $\mathcal{A}_{\psi,\varphi}K_w = \mathcal{A}^*_{\psi,\varphi}K_w$. Hence $\mathcal{A}_{\psi,\varphi}$ is self-adjoint. $\square$

Remark 2.12 By Lemma 2.1, we know that $\mathcal{A}_{\psi,\varphi}$ is a bounded self-adjoint operator on $\mathcal{F}(\mathcal{H})$ if and only if $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$ and

$$\mathcal{J}C_{\psi,\varphi}\mathcal{J} = C^*_{\psi,\varphi}.$$

That is, $\mathcal{A}_{\psi,\varphi}$ is bounded and self-adjoint on $\mathcal{F}(\mathcal{H})$ if and only if $C_{\psi,\varphi}$ is bounded and $\mathcal{J}$-symmetric on $\mathcal{F}(\mathcal{H})$. Therefore, Theorem 2.11 is also the characterization for bounded weighted composition operators $C_{\psi,\varphi}$ to be $\mathcal{J}$-symmetric on $\mathcal{F}(\mathcal{H})$.

Since an anti-linear operator is a conjugation if and only if the operator is both unitary and self-adjoint, we obtain the following result by Corollary 2.9 (2) and Theorem 2.11.

Corollary 2.13 *Let $\psi \in \mathcal{F}(\mathcal{H})$ and φ be a mapping on $\mathcal{H}$ such that $\mathcal{A}_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$. Then $\mathcal{A}_{\psi,\varphi}$ is a conjugation if and only if there exists a J-symmetric unitary operator A on $\mathcal{H}$ and a vector $b \in \mathcal{H}$ such that*

$$
\begin{gathered}
(I + JA)Jb = 0,\\
\varphi(z) = Az + b,\ \ \psi(z) = \psi(0)K_{Jb}(z),\ \ |\psi(0)|^2\exp(\|b\|^2) = 1.
\end{gathered}
\tag{2.20}
$$

2.5. Normal weighted composition operators

In this subsection, we characterize a class of normal weighted composition operators on $\mathcal{F}(\mathcal{H})$.

Theorem 2.14 *Let $\psi(z) = K_c(z)$ and φ be a mapping on $\mathcal{H}$ such that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}(\mathcal{H})$.*

(1) $C_{\psi,\varphi}$ is normal if and only if there exists a normal operator A on $\mathcal{H}$ with $\|A\| \le 1$ and a vector $b \in \mathcal{H}$ such that

$$
\begin{gathered}
\varphi(z) = Az + b,\\
A^*(b + Ac) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}, \langle c, c\rangle = \langle b, b\rangle, Ac + b = A^*b + c.
\end{gathered}
\tag{2.21}
$$

(2) $\mathcal{A}_{\psi,\varphi}$ is normal if and only if there exists an operator A on $\mathcal{H}$ with $\|A\| \le 1$, $JAA^*J = A^*A$ and a vector $b \in \mathcal{H}$ such that

$$
\begin{gathered}
\varphi(z) = Az + b,\\
A^*(b + Ac) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}}, \langle c, c\rangle = \langle b, b\rangle, J(Ac + b) = A^*b + c.
\end{gathered}
\tag{2.22}
$$

Proof (1) By Theorem 2.4, we know that

$$\varphi(z) = Az + b,\ \ A^*(b + Ac) \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}},$$

where A is a linear operator on $\mathcal{H}$ with $\|A\| \le 1$ and $b \in \mathcal{H}$.

$C_{\psi,\varphi}$ is normal if and only if

$$(C^*_{\psi,\varphi}C_{\psi,\varphi}K_w)(z) = (C_{\psi,\varphi}C^*_{\psi,\varphi}K_w)(z)$$

for all $z, w \in \mathcal{H}$. It follows from (2.5) and (2.6) that

$$(C_{\psi,\varphi}^* C_{\psi,\varphi} K_w)(z) = \exp(\langle A^* z, A^* w\rangle + \langle Ac + b, w\rangle + \langle z, Ac + b\rangle + \langle c, c\rangle),$$

$$(C_{\psi,\varphi} C_{\psi,\varphi}^* K_w)(z) = \exp(\langle Az, Aw\rangle + \langle z, A^* b + c\rangle + \langle A^* b + c, w\rangle + \langle b, b\rangle).$$

So $C_{\psi,\varphi}$ is normal if and only if

$$\begin{aligned}&\langle A^* z, A^* w\rangle + \langle Ac + b, w\rangle + \langle z, Ac + b\rangle + \langle c, c\rangle \\ &\quad = \langle Az, Aw\rangle + \langle z, A^* b + c\rangle + \langle A^* b + c, w\rangle + \langle b, b\rangle.\end{aligned} \tag{2.23}$$

Let $z = w = 0$ in (2.23). We have $\langle c, c\rangle = \langle b, b\rangle$. Taking this equation into (2.23) and let $w = 0$, we have

$$\langle z, Ac + b\rangle = \langle z, A^* b + c\rangle$$

for all $z \in \mathcal{H}$, which implies that

$$Ac + b = A^* b + c.$$

Taking these equations into (2.23), we obtain

$$\langle A^* z, A^* w\rangle = \langle Az, Aw\rangle$$

for all $z, w \in \mathcal{H}$, which implies that A is normal. The necessary conditions are completed.

Let A be a normal operator on $\mathcal{H}$ with $\|A\| \le 1$ and $b \in \mathcal{H}$ such that the condition (2.21) holds. Then equation (2.23) is true, which implies that $C_{\psi,\varphi}$ is normal.

(2) By Corollary 2.5, we know that

$$\varphi(z) = Az + b, \;\; A^*(b + Ac) \in \operatorname{ran}(I - A^* A)^{\frac{1}{2}},$$

where A is a linear operator on $\mathcal{H}$ with $\|A\| \le 1$ and $b \in \mathcal{H}$.

$\mathcal{A}_{\psi,\varphi}$ is normal if and only if

$$(\mathcal{A}_{\psi,\varphi}^* \mathcal{A}_{\psi,\varphi} K_w)(z) = (\mathcal{A}_{\psi,\varphi} \mathcal{A}_{\psi,\varphi}^* K_w)(z)$$

for all $z, w \in \mathcal{H}$. It follows from (2.9) and (2.10) that

$$\begin{aligned}(\mathcal{A}_{\psi,\varphi}^* \mathcal{A}_{\psi,\varphi} K_w)(z) &= \exp(\langle Jb, w\rangle + \langle A^* Jw + c, c\rangle + \langle z, JAA^* Jw + JAc + Jb\rangle) \\ &= \exp(\langle J(Ac + b), w\rangle + \langle c, c\rangle + \langle z, JAA^* Jw\rangle + \langle z, J(Ac + b)\rangle), \\ (\mathcal{A}_{\psi,\varphi} \mathcal{A}_{\psi,\varphi}^* K_w)(z) &= \exp(\langle c, w\rangle + \langle JAw + Jb, Jb\rangle + \langle z, A^* Aw + A^* b + c\rangle) \\ &= \exp(\langle A^* b + c, w\rangle + \langle Jb, Jb\rangle + \langle z, A^* Aw\rangle + \langle z, A^* b + c\rangle).\end{aligned}$$

So $\mathcal{A}_{\psi,\varphi}$ is normal if and only if

$$\begin{aligned}&\langle J(Ac + b), w\rangle + \langle c, c\rangle + \langle z, JAA^* Jw\rangle + \langle z, J(Ac + b)\rangle \\ &\quad = \langle A^* b + c, w\rangle + \langle Jb, Jb\rangle + \langle z, A^* Aw\rangle. + \langle z, A^* b + c\rangle.\end{aligned} \tag{2.24}$$

Let $z = w = 0$ in (2.24). We have

$$\langle c, c\rangle = \langle Jb, Jb\rangle = \langle b, b\rangle.$$

Taking the equation above into (2.24) and then let $w = 0$, we have

$$\langle J(Ac+b), z\rangle = \langle A^*b + c, z\rangle$$

for all $z \in \mathcal{H}$, which implies that

$$J(Ac+b) = A^*b + c.$$

Taking these equations into (2.24), we obtain

$$JAA^*J = A^*A.$$

The necessary condition is proved.

Let A be an operator on $\mathcal{H}$ with $\|A\| \le 1$, $JAA^*J = A^*A$ and $b \in \mathcal{H}$ such that the condition (2.22) holds. Then equation (2.24) is true, which implies that $\mathcal{A}_{\psi,\varphi}$ is normal. □

Acknowledgements We thank the referees for their time and comments.

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