Journal of Mathematical Research with Applications
Nov., 2021, Vol. 41, No. 6, pp. 607–614
DOI:10.3770/j.issn:2095-2651.2021.06.005
Http://jmre.dlut.edu.cn

A Refined Regularity Criterion for 3D Liquid Crystal Equations Involving Horizontal Velocity

Xiaoli CHEN*, **Xiujuan ZHA**
School of Mathematics and Statistics, Jiangxi Normal University, Jiangxi 330022*, P. R. China*

Abstract This note investigates the global regularity of 3D liquid crystal equations in terms of the vertical derivative of u_h. More precisely, we prove that if the vertical derivative of the horizontal velocity component u_h satisfies $\partial_3 u_h \in L^p(0,T;\mathrm{R}^3)$ with $\frac{2}{p}+\frac{3}{q}\le\frac{3}{2}$, $2\le p\le\infty$, then the local strong solution (u,d) can be smoothly extended beyond $t=T$.

Keywords liquid crystal flow; regularity criterion; local strong solution

MR(2020) Subject Classification 35Q35; 76D03

1. Introduction

In this paper, we study the three dimensional liquid crystal equations:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi = \nu\Delta u - \lambda\nabla\cdot(\nabla d\otimes\nabla d), & (t,x)\in\mathrm{R}^+\times\mathrm{R}^3,\\ \partial_t d + u\cdot\nabla d = \gamma(\Delta d - f(d)), & (t,x)\in\mathrm{R}^+\times\mathrm{R}^3,\\ \mathrm{div} u = 0, & (t,x)\in\mathrm{R}^+\times\mathrm{R}^3,\\ u|_{t=0}=u_0, d|_{t=0}=d_0, \end{cases} \tag{1.1}$$

where u is the velocity field, π is the scalar pressure and d represents the macroscopic molecular orientation field of the liquid crystal materials. The (i,j)-th entry of $\nabla d\otimes\nabla d$ is given by $\nabla_{x_i} d\otimes\nabla_{x_j} d$ for $1\le i,j\le 3$. Here we take $f(d)$ to be the gradient of a scale of function $F(d)$,

$$f(d)=\nabla F(d).$$

Here $F(d)=\frac{1}{\eta^2}(|d|^2-1)^2$. Without loss of generality, we take $\nu=\lambda=\eta=\gamma=1$, since ν,λ,γ,η are positive constants.

It is well-known that Ericksen and Leslie [1,2] established the hydrodynamic theory of liquid crystal in 1960s. Lin [3] first introduced the above liquid crystal flow (1.1). Later Lin and Liu [4] obtained the global existence theorem for weak solution and local well-posedness for the strong solution to the system (1.1). In 2008, Fan and Guo [5] showed that if u satisfies one of the following conditions:

$$u\in L^s(0,T;\dot{M}_{p,q}(\mathrm{R}^3)) \text{ with } \frac{2}{s}+\frac{3}{p}=1,\quad p\ge 3,\ p\ge q\ge 1,$$

Received October 18, 2020; Accepted June 26, 2021
Supported by the National Natural Science Foundation of China (Grant Nos. 11961032; 11971209) and the Natural Science Foundation of Jiangxi Province (Grant No. 20191BAB201003).
* Corresponding author
E-mail address: littleli_chen@163.com (Xiaoli CHEN); 365268389@qq.com (Xiujuan ZHA)

or

$$\nabla u \in L^s(0,T;\dot{M}_{p,q}(\mathrm{R}^3)) \text{ with } \frac{2}{s}+\frac{3}{p}=2,\ \ p\ge\frac{3}{2},\ p\ge q\ge 1,$$

then (u,d) can be extended beyond $t=T$. Later Liu, Zhao and Cui [6] obtained the regularity criterion to the system (1.1) under the assumption that $\partial_3 u\in L^\beta(0,T;L^\alpha)$ with $\frac{2}{\beta}+\frac{3}{\alpha}\le 1, \alpha>3$. Recently, Wei, Li and Yao [7] proved that if the weak solution (u,d) satisfies

$$u_3,\nabla d\in L^\beta(0,T;L^\alpha(\mathrm{R}^3)),\ \text{with } \frac{2}{\beta}+\frac{3}{\alpha}\le\frac{3}{4}+\frac{1}{\alpha},\ \ \alpha>\frac{10}{3},$$

then (u,b) can be extended beyond $t=T$. Qian [8] proved that if

$$\int_0^T \|u_3(\tau)\|_{L^\alpha}^\beta+\|\omega_3\|_{L^a}^b+\|\partial_3 u_h(\tau)\|_{L^a}^b\mathrm{d}\tau\le M,\ \ \text{for some } M>0, \tag{1.2}$$

with $\frac{3}{\alpha}+\frac{2}{\beta}=1, \alpha\in(3,\infty], \frac{3}{a}+\frac{2}{b}=2, a\in(\frac{3}{2},3]$. Then the local strong solution (u,d) to (1.1) is regular, here $\omega_3=\partial_1 u_2-\partial_2 u_1$. Later Qian [9] showed that if

$$\int_0^T \|\nabla u_h(\tau)\|_{L^p}^q+\|\partial_3\nabla d\|_{L^p}^q\mathrm{d}\tau\le\infty,\ \frac{3}{p}+\frac{2}{q}=2,\ \frac{3}{2}<p\le\infty, \tag{1.3}$$

then the local solution (u,d) to (1.1) can be extended smoothly beyond $t=T$. In 2020, Zhao, Wang and Wang [10] removed the condition on ∇d and proved that if

$$u_h\in L^s(0,T;L^q(\mathrm{R}^3)),\ \frac{2}{s}+\frac{3}{q}\le\frac{1}{2},\ \ 6\le q\le\infty, \tag{1.4}$$

then the local solution (u,d) to (1.1) can be extended smoothly beyond $t=T$. Recently, Yuan and Li [11] proved if

$$\nabla u_h\in L^q(0,T;L^p(\mathrm{R}^3)),\ \frac{2}{q}+\frac{3}{p}\le\frac{3}{2},\ \ 2\le p\le\infty, \tag{1.5}$$

the local strong solution (u,d) to (1) can be extended smoothly beyond $t=T$. Motivated by [7], [10] and [11], the aim of this paper is to get rid of the condition on ∇d in (1.3). We can show the local strong solution (u,d) to the 3D liquid crystal flow (1.1) is regular in terms of $\partial_3 u_h$. Our main result is stated as follows.

Theorem 1.1 *Let $(u_0,d_0)\in H^2(\mathrm{R}^3)\times H^3(\mathrm{R}^3)$ satisfy $\nabla\cdot u_0=0$. Assume that $(u,d$ is a local strong solution $(0,T)$ for some $T>0$ to the 3D liquid crystal Eq. (1.1) with the initial value (u_0,d_0). If u satisfies the following condition*

$$\int_0^T \|\partial_3 u_h\|_{L^p}^q\mathrm{d}\tau\le M,\ \frac{3}{p}+\frac{2}{q}\le\frac{3}{2},\ \ p\ge 2. \tag{1.6}$$

Then (u,d) can be smoothly extended beyond $t=T$.

Remark 1.2 The condition (1.6) is only part of (1.3), therefore Theorem 1.1 improves the corresponding one in [9]. Meanwhile our result is partially better than that in [8]. On the other hand, $\partial_3 u_h$ is one component of $\nabla u_h=(\partial_1 u_h,\partial_2 u_h,\partial_3 u_h)$. So our condition is weaker than (1.3). Therefore, Theorem 1.1 also refines the result in [11]. Our condition is also weaker than (1.4), Theorem 1.1 improves the result in [10].

Remark 1.3 When $d = 0$, the equations of the liquid crystal flows become the incompressible Navier-Stokes equations, Theorem 1.1 improves the result in [12] by getting rid of the assumption $u \in L^\infty(0,T;L^3)$.

The rest of this paper is organized as follows. Some crucial lemmas are provided in Section 2. The proof of Theorem 1.1 can be found in Section 3.

2. Some crucial lemmas

We first recall the following Sobolev inequality in R^3. The first lemma can be found in [13].

Lemma 2.1 *The following inequality*

$$\Big|\int_{\mathrm{R}^3} fgh\mathrm{d}x\Big| \le C\|g\|_{L^2}^{\frac{q-2}{q}}\|\partial_1 g\|_{L^2}^{\frac{1}{q}}\|\partial_2 g\|_{L^2}^{\frac{1}{q}}\|f\|_{L^2}^{\frac{q-1}{q}}\|\partial_3 f\|_{L^{\frac{2}{3-q}}}^{\frac{1}{q}}\|h\|_{L^2} \tag{2.1}$$

holds for $2 \le q \le 3$.

Lemma 2.2 *Let* $(u,d) \in H^2(\mathrm{R}^3) \times H^3(\mathrm{R}^3)$ *be a strong solution to the liquid crystal flow (1.1) in* $(0,T)\times\mathrm{R}^3$*. If* u *satisfies*

$$\partial_3 u_h \in L^p(0,T;L^q(\mathrm{R}^3)),\ \frac{3}{p}+\frac{2}{q}\le\frac{3}{2},\quad p\ge 2, \tag{2.2}$$

then

$$\partial_3 d \in L^\infty(0,T;L^3(\mathrm{R}^3)),\quad \nabla|\partial_3 d|^{\frac{3}{2}} \in L^2(0,T;L^2(\mathrm{R}^3)).$$

Proof Applying ∂_3 to Eq. $(1.1)_2$, and multiplying it by $|\partial_3 d|\partial_3 d$ to give rise to

$$\begin{aligned}\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 d\|_{L^3}^3 + \frac{8}{9}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 &= -\int_{R^3}\partial_3(u\cdot\nabla d)|\partial_3 d|\partial_3 d\mathrm{d}x - \int_{R^3}\partial_3[(|d|^2 d - d)]|\partial_3 d|\partial_3 d\mathrm{d}x\\ &= A_1 + A_2. \end{aligned}\tag{2.3}$$

To estimate A_1, we decompose A_1 into the following two terms.

$$\begin{aligned}A_1 &= -\int_{R^3}\sum_{i=1}^3 \partial_3(u_i\partial_i d)|\partial_3 d|\partial_3 d\mathrm{d}x\\ &= -\int_{R^3}\sum_{i=1}^3\partial_3 u_i\partial_i d|\partial_3 d|\partial_3 d\mathrm{d}x - \int_{\mathrm{R}^3} u\cdot\nabla\partial_3 d|\partial_3 d|\partial_3 d\mathrm{d}x\\ &= -\int_{R^3}\sum_{i=1}^3\partial_3 u_i\partial_i d|\partial_3 d|\partial_3 d\mathrm{d}x\\ &= -\int_{R^3}\sum_{i=1}^2\partial_3 u_i\partial_i d|\partial_3 d|\partial_3 d\mathrm{d}x - \int_{R^3}\partial_3 u_3\partial_3 d|\partial_3 d|\partial_3 d\mathrm{d}x\\ &= A_{11} + A_{12}.\end{aligned}$$

Thanks to the Hölder and the Young inequality, we deduce

$$\begin{aligned}A_{11} &\le \int_{R^3}|\partial_3 u_h||\nabla d||\partial_3 d|^{\frac{1}{2}}|\partial_3 d|^{\frac{3}{2}}\mathrm{d}x\\ &\le \|\partial_3 u_h\|_{L^{p_1}}\|\nabla d\|_{L^{p_2}}\||\partial_3 d|^{\frac{1}{2}}\|_{L^6}\||\partial_3 d|^{\frac{3}{2}}\|_{L^6}\end{aligned}$$

$$
\begin{aligned}
&\le \|\partial_3 u_h\|_{L^{p_1}}\|\nabla d\|_{L^{p_2}}\|\partial_3 d\|_{L^3}^{\frac{1}{2}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}\\
&\le \frac{1}{18}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 + C\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2\|\partial_3 d\|_{L^3},
\end{aligned}
\tag{2.4}
$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{3}=1$. We infer from Lemma 2.1 and (3.1)

$$
\begin{aligned}
A_{12} &= -\int_{R^3}\partial_3 u_3\partial_3 d|\partial_3 d|\partial_3 d\mathrm{d}x = \int_{R^3}\sum_{i=1}^{2}\partial_i u_i|\partial_3 d|^3\mathrm{d}x\\
&= -\int_{R^3}\sum_{i=1}^{2}u_i\partial_i(|\partial_3 d|^3)\mathrm{d}x \le C\int_{R^3}|u_h||\partial_3 d|^2|\nabla|\partial_3 d||\mathrm{d}x\\
&\le C\int_{R^3}|u_h||\partial_3 d|^{\frac{3}{2}}|\nabla|\partial_3 d|^{\frac{3}{2}}|\mathrm{d}x\\
&\le C\|u_h\|_{L^2}^{\frac{2(p-1)}{3p-2}}\|\partial_3 u_h\|_{L^p}^{\frac{p}{3p-2}}\||\partial_3 d|^{\frac{3}{2}}\|_{L^2}^{\frac{p-2}{3p-2}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^{\frac{2p}{3p-2}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}\\
&\le \frac{1}{18}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 + C\|u_h\|_{L^2}^{\frac{4(p-1)}{p-2}}\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\||\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2\\
&\le \frac{1}{18}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 + C\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\partial_3 d\|_{L^3}^3.
\end{aligned}
\tag{2.5}
$$

From the proof of Lemma 2.1 in [10], we can get

$$
A_2 \le \frac{1}{9}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 + C\|\partial_3 d\|_{L^3}^3,
$$

which along with (2.3)–(2.5) yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 d\|_{L^3}^3 + \|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2 \le C(1+\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}})\|\partial_3 d\|_{L^3}^3 + C\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2\|\partial_3 d\|_{L^3}, \tag{2.6}
$$

which implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 d\|_{L^3}^2 \le C(1+\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}})\|\partial_3 d\|_{L^3}^2 + C\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2.
$$

The Gronwall inequality guarantees

$$
\|\partial_3 d\|_{L^3}^3 \le C\|\partial_3 d_0\|_{L^3}^3\exp\Big\{\int_0^T(1+\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}})\mathrm{d}\tau\Big\} + \int_0^T\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2\mathrm{d}\tau, \tag{2.7}
$$

where

$$
\int_0^T\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2\mathrm{d}\tau \le \|\partial_3 u_h\|_{L_t^{p_3}L_x^{p_1}}^2\|\nabla d\|_{L_t^{p_4}L_x^{p_2}}^2 \tag{2.8}
$$

is true for $\frac{2}{p_3}+\frac{2}{p_4}=1$.

The inequality (1.6) says

$$
\|\partial_3 u_h\|_{L_t^{p_3}L_x^{p_1}} < \infty, \text{ for } \frac{3}{p_1}+\frac{2}{p_3}=\frac{3}{2}. \tag{2.9}
$$

Thanks to (3.1), we get

$$
\nabla d \in L_t^\infty L_x^2 \cap L_t^2\dot{H}_x^1.
$$

Interpolation inequality implies

$$
\|\nabla d\|_{L_t^{p_4}L_x^{p_2}} < \infty \tag{2.10}
$$

holds for $\frac{3}{p_2}+\frac{2}{p_4}=\frac{3}{2}$. In terms of (2.7), (2.8)–(2.10), one has $\partial_3 d\in L^\infty(0,T;L^3)$. This along with

$$\|\partial_3 d\|_{L^3}^3+\int_0^T\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2\mathrm{d}\tau\leq C\int_0^T(1+\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}})\|\partial_3 d\|_{L^3}^3\mathrm{d}\tau+ C\int_0^T\|\partial_3 u_h\|_{L^{p_1}}^2\|\nabla d\|_{L^{p_2}}^2\|\partial_3 d\|_{L^3}\mathrm{d}\tau\leq C$$

helps us to obtain the desired result. □

3. Proof of Theorem 1.1

In this section, we will present the proof of Theorem 1.1.

Proof Step 1. L^2-estimates.

The L^2-energy estimate is standard, which can be found in [7] and [10],

$$\begin{aligned}&\|u\|_{L^\infty(0,T;L^2)}+\|u\|_{L^2(0,T;H^1)}+\|d\|_{L^\infty(0,T;H^1)}+\|d\|_{L^2(0,T;H^2)}\\&\leq C(\|u_0\|_{L^2(R^3)},\|d_0\|_{H^1(R^3)}).\end{aligned}\tag{3.1}$$

Step 2. H^1-estimates of $(u,\nabla d)$. Multiplying $(1.1)_1$ by $-\Delta u$ and integrating over R^3 to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2&=\int_{R^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x+\int_{R^3}\nabla\cdot(\nabla d\otimes\nabla d)\cdot\Delta u\mathrm{d}x\\&=I+II.\end{aligned}\tag{3.2}$$

Taking Δ on both sides of $(1.1)_2$, multiplying the resulting equation by Δd and integrating over R^3 gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta d\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2&=-\int_{R^3}\Delta[(u\cdot\nabla)d]\cdot\Delta d\mathrm{d}x-\int_{R^3}\Delta f(d)\cdot\Delta d\mathrm{d}x\\&=III+IV.\end{aligned}\tag{3.3}$$

We decompose the first nonlinear term I into

$$\begin{aligned}I=&\int_{R^3}\sum_{i,j,k=1}^{3}u_i\partial_i u_j\partial_{kk}u_j\mathrm{d}x=\int_{R^3}\sum_{i=1}^{2}\sum_{j,k=1}^{3}u_i\partial_i u_j\partial_{kk}u_j\mathrm{d}x+\\&\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_3\partial_3 u_j\partial_{kk}u_j\mathrm{d}x+\int_{R^3}\sum_{k=1}^{3}u_3\partial_3 u_3\partial_{kk}u_3\mathrm{d}x\\=&I_1+I_2+I_3.\end{aligned}\tag{3.4}$$

We infer from the fact $\mathrm{div}u=0$ and integrating by parts that

$$\begin{aligned}I_2=&\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_3\partial_k u_3\partial_k u_j\mathrm{d}x+\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_k u_3\partial_3\partial_k u_j\mathrm{d}x+\\&\int_{R^3}\sum_{i,j=1}^{2}\sum_{k=1}^{3}u_i\partial_i\partial_k u_j\partial_k u_j\mathrm{d}x,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}I_3 &= -\int_{R^3}\sum_{k=1}^{3}\partial_k u_3\partial_3 u_3\partial_k u_3\mathrm{d}x - \int_{R^3}\sum_{k=1}^{3}u_3\partial_k\partial_3 u_3\partial_k u_3\mathrm{d}x\\
&= -\frac{1}{2}\int_{R^3}\sum_{k=1}^{3}u_3\partial_3(\partial_k u_3)^2\mathrm{d}x - \int_{R^3}\sum_{k=1}^{3}\partial_k u_3\partial_3 u_3\partial_k u_3\mathrm{d}x\\
&= -\frac{1}{2}\int_{R^3}\sum_{k=1}^{3}\partial_3 u_3(\partial_k u_3)^2\mathrm{d}x = \frac{1}{2}\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}\partial_j u_j(\partial_k u_3)^2\mathrm{d}x\\
&= -\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_j\partial_k u_3\partial_k u_3\mathrm{d}x.\end{aligned}\tag{3.6}$$

The other nonlinear terms II, III and IV can be split into the following forms:

$$II = \int_{R^3}\sum_{i,j,k=1}^{3}\partial_j(\partial_i d\partial_j d)\partial_{kk}u_i\mathrm{d}x = \int_{R^3}\sum_{i,j,k=1}^{3}\partial_i d\partial_{jj}d\partial_{kk}u_i\mathrm{d}x,\tag{3.7}$$

$$\begin{aligned}III &= -\int_{R^3}\sum_{i,k=1}^{3}(\partial_{kk}u_i\partial_i d + 2\partial_k u_i\partial_k\partial_i d + u_i\partial_i\partial_{kk}d)\cdot\Delta d\mathrm{d}x\\
&= -\int_{R^3}\sum_{i,k=1}^{3}\partial_{kk}u_i\partial_i d\cdot\Delta d\mathrm{d}x - 2\int_{R^3}\sum_{i,k=1}^{3}\partial_k u_i\partial_k\partial_i d\cdot\Delta d\mathrm{d}x,\end{aligned}\tag{3.8}$$

and

$$IV = \int_{R^3}\nabla f(d)\nabla\Delta d\mathrm{d}x = \int_{R^3}\nabla(|d|^2 d)\nabla\Delta d\mathrm{d}x + \|\Delta d\|_{L^2}^2.\tag{3.9}$$

From (3.2)–(3.9), one deduces

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\nabla\Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\\
&= -2\int_{R^3}\sum_{i,k=1}^{3}\partial_k u_i\partial_k\partial_i d\cdot\Delta d\mathrm{d}x + \int_{R^3}\nabla(|d|^2 d)\cdot\nabla\Delta d\mathrm{d}x + \|\Delta d\|_{L^2}^2 +\\
&\quad\int_{R^3}\sum_{i=1}^{2}\sum_{j,k=1}^{3}u_i\partial_i u_j\partial_{kk}u_j\mathrm{d}x + \int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_3\partial_k u_3\partial_k u_j\mathrm{d}x +\\
&\quad\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_k u_3\partial_3\partial_k u_j\mathrm{d}x + \int_{R^3}\sum_{i,j=1}^{2}\sum_{k=1}^{3}u_i\partial_i\partial_k u_j\partial_k u_j\mathrm{d}x -\\
&\quad\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_j\partial_k u_3\partial_k u_3\mathrm{d}x = \sum_{i=1}^{8}J_i.\end{aligned}\tag{3.10}$$

Integrating by parts says

$$\begin{aligned}J_1 &= -2\int_{R^3}\sum_{i=1}^{2}\sum_{k=1}^{3}\partial_k u_i\partial_k\partial_i d\cdot\Delta d\mathrm{d}x - 2\int_{R^3}\sum_{k=1}^{3}\partial_k u_3\partial_k\partial_3 d\cdot\Delta d\mathrm{d}x\\
&= 2\int_{R^3}\sum_{i=1}^{2}\sum_{k=1}^{3}u_i\partial_{kk}\partial_i d\cdot\Delta d\mathrm{d}x + 2\int_{R^3}\sum_{i=1}^{2}\sum_{k=1}^{3}u_i\partial_k\partial_i d\cdot\partial_k\Delta d\mathrm{d}x +\end{aligned}$$

$$
\begin{aligned}
&2\int_{R^3}\sum_{k=1}^{3}\partial_{kk}u_3\partial_3 d\cdot\Delta d\mathrm{d}x+2\int_{R^3}\sum_{k=1}^{3}\partial_k u_3\partial_3 d\cdot\partial_k\Delta d\mathrm{d}x\\
&\leq C\int_{R^3}|u_h||\nabla\Delta d||\Delta d|\mathrm{d}x+C\int_{R^3}|\Delta u||\partial_3 d||\Delta d|\mathrm{d}x+C\int_{R^3}|\nabla u||\partial_3 d||\nabla\Delta d|\mathrm{d}x\\
&=J_{11}+J_{12}+J_{13}.
\end{aligned}
$$

Thanks to Lemma 2.1 and (3.1), we get

$$
\begin{aligned}
J_{11}&\leq\|u_h\|_{L^2}^{\frac{2(p-1)}{3p-2}}\|\partial_3 u_h\|_{L^p}^{\frac{p}{3p-2}}\|\Delta d\|_{L^2}^{\frac{p-2}{3p-2}}\|\nabla\Delta d\|_{L^2}^{\frac{2p}{3p-2}}\|\nabla\Delta d\|_{L^2}\\
&\leq\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+C\|u_h\|_{L^2}^{\frac{4(p-1)}{p-2}}\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\Delta d\|_{L^2}^2\\
&\leq\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+C\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\Delta d\|_{L^2}^2.
\end{aligned}\tag{3.11}
$$

Applying the Hölder inequality and interpolation inequality with $3<p\leq 9$ yields

$$
\begin{aligned}
J_{12}&\leq\|\Delta u\|_{L^2}\|\partial_3 d\|_{L^p}\|\Delta d\|_{L^{\frac{2p}{p-2}}}\\
&\leq C\|\Delta u\|_{L^2}\|\partial_3 d\|_{L^3}^{\frac{9-p}{2p}}\|\partial_3 d\|_{L^9}^{\frac{3p-9}{2p}}\|\Delta d\|_{L^2}^{1-\frac{3}{p}}\|\Delta d\|_{L^6}^{\frac{3}{p}}\\
&\leq C\|\Delta u\|_{L^2}\|\partial_3 d\|_{L^3}^{\frac{9-p}{2p}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{p}}\|\Delta d\|_{L^2}^{1-\frac{3}{p}}\|\nabla\Delta d\|_{L^2}^{\frac{3}{p}}\\
&\leq\frac{1}{16}\|\Delta u\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+C\|\partial_3 d\|_{L^3}^{\frac{9-p}{p-3}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2\|\Delta d\|_{L^2}^2.
\end{aligned}\tag{3.12}
$$

Similarly,

$$
\begin{aligned}
J_{13}&\leq\|\nabla\Delta d\|_{L^2}\|\partial_3 d\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\\
&\leq C\|\nabla\Delta d\|_{L^2}\|\partial_3 d\|_{L^3}^{\frac{9-p}{2p}}\|\partial_3 d\|_{L^9}^{\frac{3p-9}{2p}}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\nabla u\|_{L^6}^{\frac{3}{p}}\\
&\leq C\|\nabla\Delta d\|_{L^2}\|\partial_3 d\|_{L^3}^{\frac{9-p}{2p}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{p}}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\\
&\leq\frac{1}{16}\|\Delta u\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+C\|\partial_3 d\|_{L^3}^{\frac{9-p}{p-3}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.13}
$$

To estimate J_2, integrating by parts and using the Hölder inequality, Sobolev embedding and the Young inequality, one can verify

$$
\begin{aligned}
J_2&=\int_{R^3}|d|^2\nabla d\cdot\nabla\Delta d\mathrm{d}x+\int_{R^3}|d|\nabla dd\cdot\nabla\Delta d\mathrm{d}x\\
&\leq\||d|^2\|_{L^3}\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^2}\leq C\|d\|_{L^6}^2\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^2}\\
&\leq C\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\leq\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}^2.
\end{aligned}\tag{3.14}
$$

Since

$$
|J_4|+|J_5|+|J_6|+|J_7|+|J_8|\leq C\int_{R^3}|u_h||\nabla u||\Delta u|\mathrm{d}x+C\int_{R^3}|u_h||\nabla u||\nabla^2 u|\mathrm{d}x.
$$

Thanks to Lemma 2.1 and the Young inequality, we have

$$
\int_{R^3}|u_h||\nabla u||\Delta u|\mathrm{d}x\leq\|u_h\|_{L^2}^{\frac{2(p-1)}{3p-2}}\|\partial_3 u_h\|_{L^p}^{\frac{p}{3p-2}}\|\nabla u\|_{L^2}^{\frac{p-2}{3p-2}}\|\Delta u\|_{L^2}^{\frac{2p}{3p-2}}\|\Delta u\|_{L^2}
$$

$$\le \frac{1}{16}\|\Delta u\|_{L^2}^2 + C\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\nabla u\|_{L^2}^2. \tag{3.15}$$

Reasoning in the same methods, we can get

$$\begin{aligned}\int_{R^3}|u_h||\nabla u||\nabla^2 u|\mathrm{d}x &\le \frac{1}{16}\|\nabla^2 u\|_{L^2}^2 + C\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\nabla u\|_{L^2}^2\\ &\le \frac{1}{16}\|\Delta u\|_{L^2}^2 + C\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}}\|\nabla u\|_{L^2}^2,\end{aligned} \tag{3.16}$$

which together with (3.10)–(3.15) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\nabla\Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\\ &\quad\le C(\|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}} + 1 + \|\partial_3 d\|_{L^3}^{\frac{9-p}{p-3}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2).\end{aligned}$$

The Gronwall inequality and Lemma 2.2 guarantee

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T(\|\nabla\Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\mathrm{d}\tau\\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2)\exp\Big\{\int_0^T(1 + \|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}} + \|\partial_3 d\|_{L^3}^{\frac{9-p}{p-3}}\|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2)\mathrm{d}\tau\Big\}\\ &\le C(\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2)\exp\Big\{C\int_0^T(1 + \|\partial_3 u_h\|_{L^p}^{\frac{2p}{p-2}} + \|\nabla|\partial_3 d|^{\frac{3}{2}}\|_{L^2}^2)\mathrm{d}\tau\Big\},\end{aligned}$$

which completes the proof of Theorem 1.1. □

Acknowledgements We thank the referees for their time and comments.

References

[1] J. L. ERICKSEN. *Hydrostatic theory of liquid crystals*. Arch. Rational Mech. Anal., 1962, **9**: 371–378.

[2] F. M. LESLIE. *Some constitutive equations for liquid crystals*. Arch. Rational Mech. Anal., 1968, **28**(4): 265–283.

[3] Fanghua LIN. *Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena*. Comm. Pure Appl. Math., 1989, **42**(6): 789–814.

[4] Fanghua LIN, Chun LIU. *Partial regularity of the dynamic system modeling the flow of liquid crystals*. Discrete Contin. Dynam. Systems, 1996, **2**(1): 1–22.

[5] Jishan FAN, Boling GUO. *Regularity criterion to some liquid crystal models and the Landau-Lifshitz equations in* R^3. Science in China Series A: Mathematics, 2008, **51**: 1787–1797.

[6] Qiao LIU, Jihong ZHAO, Shangbin CUI. *A regularity criterion for the three-dimensional nematic liquid crystal flow in terms of one directional derivative of the velocity*. J. Math. Phys., 2011, **52(3)**: 033102, 8 pp.

[7] Ruiying WEI, Yin LI, Zhengan YAO. *Two new regularity criteria for nematic liquid crystal flows*. J. Math. Anal. Appl., 2015, **424**(1): 636–650.

[8] Chenyin QIAN. *Remark on the regularity criterion for the nematic liquid crystal flows in* R^3. Appl. Math. Comput., 2016, **274**: 679-689.

[9] Chenyin QIAN. *A further note on the regularity criterion for the nematic liquid crystal flows in* R^3. Appl. Math. Comput., 2016, **290**: 258–266.

[10] Lingling ZHAO, Wendong WANG, Suyu WANG. *Blow-up criteria for the 3D liquid crystal flows involving two velocity components*. Appl. Math. Lett., 2020, **96**: 75–80.

[11] Baoquan YUAN, Qiang LI. *Note on global regularity solution to the 3D liquid crystal equations*. Appl. Math. Lett., 2020, **109**: 106491.

[12] Chenyin QIAN. *A remark on the global regularity for the 3D Navier-Stokes equations*. Appl. Math. Lett., 2016, **57**: 126–131.

[13] Chongsheng CAO, E. S. TITI. *Global regularity critreion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor*. Arch. Ration. Mech. Anal., 2011, **202**: 919–932.