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# Lipschitz Shadowing Property for 1-Dimensional Subsystems of $\mathbb{Z}^k$-Actions

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**Abstract** In this paper, the shadowing property for 1-dimensional subsystems of $\mathbb{Z}^k$-actions is investigated. The concepts of pseudo orbit and shadowing property for 1-dimensional subsystems of $\mathbb{Z}^k$-actions are introduced in two equivalent ways. For a smooth $\mathbb{Z}^k$-action $T$ on a closed Riemannian manifold, we propose a notion of Anosov direction via the induced nonautonomous dynamical system. Adapting Bowen's geometric method to our case, we show that $T$ has the Lipschitz shadowing property along any Anosov direction.

**Keywords** $\mathbb{Z}^k$-actions; Lipschitz shadowing; 1-dimensional subsystem

**MR(2020) Subject Classification** 37C50; 37D20; 37C85

## 1. Introduction

It is well known that the structural stability is a classical theory of smooth dynamical systems (i.e., smooth actions of the groups $\mathbb{Z}$ or $\mathbb{R}$), which has led to the development of shadowing theory. The shadowing property describes the behaviour of pseudo orbits on or near a hyperbolic set. The classical shadowing lemma states that every pseudo orbit lying in a small neighborhood of a hyperbolic set stays uniformly close to some true orbit (with slightly altered initial position). At present, shadowing theory is a well developed branch of the theory of dynamical systems, for the general theory of shadowing theory, we refer to the books [1–3].

Recent years, global qualitative properties of actions of groups more general than $\mathbb{Z}$ and $\mathbb{R}$ have been extensively investigated. For the shadowing theory, there are several recent works for the Abelian group actions in [4–6]. For example, Pilyugin and Tikhomirov [4] considered the characterization of a classical linear $\mathbb{Z}^k$-action $T$ on $\mathbb{C}^m$ generated by pairwise commuting matrices which has the shadowing property [4, Theorem 2]. Precisely, a linear $\mathbb{Z}^k$-action on $\mathbb{C}^m$ has the Lipschitz shadowing property if and only if there exists at least one hyperbolic 1-dimensional rational subspace of $\mathbb{R}^k$, here a 1-dimensional subspace $L$ of $\mathbb{R}^k$ is said to be rational if $L$ passes through some integer lattices except for the origin $\overrightarrow{0}$. When each of the above

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generators $A_i, 1 \le i \le k$, has integer entries and whose determinant is equal to $\pm 1$, we can get, from the above result, a characterization of the induced $\mathbb{Z}^k$-action $T$ on the torus $\mathbb{T}^m$ which has the shadowing property. A natural question is: for more general $\mathbb{Z}^k$-actions on compact metric spaces or Riemannian manifolds, how can we consider the shadowing property along 1-dimensional subspaces, including rational and irrational cases, of $\mathbb{R}^k$?

The main aim of this paper is to answer the above question for certain $\mathbb{Z}^k$-actions. Let $(X, d)$ be a compact metric space and $T$ be a continuous $\mathbb{Z}^k$-action on $X$. Given a nonzero vector $\overrightarrow{v} \in \mathbb{R}^k$, let $L_{\overrightarrow{v}}$ be the 1-dimensional subspace of $\mathbb{R}^k$ in which $\overrightarrow{v}$ lies.

In Section 2, we introduce the definitions of pseudo orbit and shadowing property for $T$ along $L_{\overrightarrow{v}}$ in two ways. One way is based on the "thickening" technique which was introduced by Boyle and Lind [7] to investigate the expansive subdynamics of $T$. For irrational $L_{\overrightarrow{v}}$, this technique makes it "visible" in $\mathbb{Z}^k$ via thickening $L_{\overrightarrow{v}}$ by a positive number $t$, and hence pseudo orbit and shadowing property are defined via these visible elements in $\mathbb{Z}^k$. The other way is to choose a sequence of maps $\{g_n\}_{n\in\mathbb{Z}}$ along $L_{\overrightarrow{v}}$ and define the pseudo orbit and shadowing property via the induced nonautonomous dynamical system. It is shown that the shadowing properties defined in these two ways are equivalent (Theorem 2.6).

In Section 3, we investigate the Lipschitz shadowing property for smooth $\mathbb{Z}^k$-actions. Let $X = M$ be a closed Riemannian manifold and $T$ be a differentiable $\mathbb{Z}^k$-action. It is well known that in the classical theory of smooth $\mathbb{Z}^1$ actions, we often require that the system have certain hyperbolicity when the shadowing property is considered. For a smooth $\mathbb{Z}^k$-action $T$, we propose a notion of hyperbolic (Anosov) direction via the induced nonautonomous dynamical system. Adapting Bowen's geometric method in [8] to our case, we show that $T$ has the Lipschitz shadowing property along any Anosov direction (Theorem 3.4).

## 2. Shadowing for 1-dimensional subsystems of $\mathbb{Z}^k$-actions

Let $(X, d)$ be a compact metric space and denote the group of homeomorphisms on $X$ with $\mathrm{Homeo}(X, X)$. Fix $k > 1$. A continuous $\mathbb{Z}^k$-action $T$ on $X$ is a homomorphism from $\mathbb{Z}^k$ to $\mathrm{Homeo}(X, X)$ satisfying the following properties:

- $T^{\overrightarrow{n}}(\cdot) \in \mathrm{Homeo}(X, X)$ for $\overrightarrow{n} \in \mathbb{Z}^k$;
- $T^{\overrightarrow{0}}(x) = x$ for $x \in X$;
- $T^{\overrightarrow{n}+\overrightarrow{m}}(\cdot) = T^{\overrightarrow{n}}(T^{\overrightarrow{m}}(\cdot))$ for $\overrightarrow{n}, \overrightarrow{m} \in \mathbb{Z}^k$.

Note that $T$ is generated by $k$ pairwise commuting homeomorphisms, we denote the collection of generators by

$$\mathcal{G} = \{f_i = T(\overrightarrow{e_i}) = T^{\overrightarrow{e_i}} : 1 \le i \le k\}, \tag{2.1}$$

where $\overrightarrow{e_i} = (0, \ldots, 1^{(i)}, \ldots, 0)$ is the standard $i$-th generator of $\mathbb{Z}^k$. A Borel probability measure $\mu$ on $X$ is said to be *$T$-invariant*, if $\mu$ is $f_i$-invariant for $1 \le i \le k$. We say a subset $\Gamma \subset X$ is $T$-invariant if $f_i(\Gamma) = \Gamma$ for each $i$.

In [4], Pilyugin and Tikhomirov introduced the notions of pseudo orbits and shadowing property for $T$. Here we state them in the following equivalent forms. Let $\delta > 0$ and $\varepsilon > 0$. A set of points $\xi = \{x_{\overrightarrow{n}} : \overrightarrow{n} \in \mathbb{Z}^k\}$ is called a *$\delta$-pseudo orbit of $T$ on* $\Gamma$ if $\xi \subset \Gamma$ and

$$\sup_{\overrightarrow{n} \in \mathbb{Z}^k} \max_{1 \le i \le k} d(x_{\overrightarrow{n}+\overrightarrow{e}_i}, f_i(x_{\overrightarrow{n}})) \le \delta.$$

A point $x \in X$ is called $\varepsilon$-shadows the above pseudo orbit $\xi$ if

$$\sup_{\overrightarrow{n} \in \mathbb{Z}^k} d(x_{\overrightarrow{n}}, T^{\overrightarrow{n}}(x)) \le \varepsilon.$$

**Definition 2.1** *Let $T$ be a $\mathbb{Z}^k$-action on $X$ and $\Gamma$ be a $T$-invariant set.*

*(1) We say that $T$ has the shadowing property on $\Gamma$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo orbit for $T$ in $\Gamma$ can be $\varepsilon$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $T$ has the shadowing property.*

*(2) We say that $T$ has the Lipschitz shadowing property for $T$ on $\Gamma$ provided there exist $\delta_0, \widehat{L} > 0$ such that any $\delta$-pseudo orbit of $T$ in $\Gamma$ with $\delta \le \delta_0$ can be $\widehat{L}\delta$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $T$ has the Lipschitz shadowing property for $T$.*

Combining with another important property "expansiveness", Pilyugin and Tikhomirov [4] showed that if there exists $\overrightarrow{n} \in \mathbb{Z}^k$ such that the homeomorphism $T^{\overrightarrow{n}}$ has the shadowing property (in another word, $T$ has the shadowing property along a 1-dimensional rational subspace of $\mathbb{R}^k$), then $T$ has the shadowing property [4, Theorem 1]. And then they gave a characterization of a classical linear $\mathbb{Z}^k$-action $T$ on $\mathbb{C}^m$ which has the Lipschitz shadowing property, as we mentioned in the Introduction section. Their work inspires us to consider how to introduce the definitions of pseudo orbits and shadowing property for $T$ along irrational 1-dimensional subspaces of $\mathbb{R}^k$ and use them to investigate the shadowing property for general smooth $\mathbb{Z}^k$-actions.

Given a nonzero vector $\overrightarrow{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$, and denote $L_{\overrightarrow{v}} = \{\overrightarrow{v'} : \overrightarrow{v'} = k\overrightarrow{v}, \overrightarrow{v} \in \mathbb{R}^k, k \in \mathbb{R}\}$, which is a 1-dimensional subspace of $\mathbb{R}^k$. Here, let the direction of $L_{\overrightarrow{v}}$ be the direction of vector $\overrightarrow{v}$. We call $L_{\overrightarrow{v}}$ (or $\overrightarrow{v}$) is rational if $L_{\overrightarrow{v}} \cap \mathbb{Z}^k \backslash \{\overrightarrow{0}\} \ne \emptyset$, i.e., the 1-dimensional subspace $L_{\overrightarrow{v}}$ passes through some integer lattices except for $\overrightarrow{0}$. Otherwise, we say $L_{\overrightarrow{v}}$ (or $\overrightarrow{v}$) is irrational. Since a subspace $L_{\overrightarrow{v}}$ of $\mathbb{R}^k$ may be "invisible" within $\mathbb{Z}^k$, we use the technique of thickening $L_{\overrightarrow{v}}$ by a positive number $t$ to make $L_{\overrightarrow{v}}$ be "visible". The following notations concerning thickening are derived from [7].

For a subspace $L_{\overrightarrow{v}}$ of $\mathbb{R}^k$, let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^k$ and $\pi_{L_{\overrightarrow{v}}}$ denote orthogonal projection to $L_{\overrightarrow{v}}$ along its orthogonal complement $L_{\overrightarrow{v}}^{\perp}$, so that $\pi_{L_{\overrightarrow{v}}} + \pi_{L_{\overrightarrow{v}}^{\perp}} = id$. Then the set

$$L_{\overrightarrow{v}}^t = \{\overrightarrow{v'} \in \mathbb{R}^k : |\pi_{L_{\overrightarrow{v}}^{\perp}}(\overrightarrow{v'})| \le t\}$$

is the result of thickening $L_{\overrightarrow{v}}$ by $t$.

For any $L_{\overrightarrow{v}}$, we can select $t > 0$ such that $L_{\overrightarrow{v}}^t \cap \mathbb{Z}^k \backslash \{\overrightarrow{0}\} \ne \emptyset$. Given $\overrightarrow{n} = (n_1, \ldots, n_k) \in L_{\overrightarrow{v}}^t \cap \mathbb{Z}^k$, for each $1 \le i \le k$, let $\overrightarrow{n}_{i_+}$ (resp., $\overrightarrow{n}_{i_-}$) be the element of $L_{\overrightarrow{v}}^t \cap \mathbb{Z}^k$ which is nearest to $\overrightarrow{n}$ in the $i$-th positive (resp., negative) direction, and if no such $\overrightarrow{n}_{i_+}$ (resp., $\overrightarrow{n}_{i_-}$) exists, then let $\overrightarrow{n}_{i_+} = \overrightarrow{n}$ (resp., $\overrightarrow{n}_{i_-} = \overrightarrow{n}$). Hence, the set

$$\{\overrightarrow{n}\} \cup \{\overrightarrow{n}_{i_+}, \overrightarrow{n}_{i_-} : 1 \le i \le k\}$$

is a "small" neighborhood of $\overrightarrow{n}$ in $L_{\overrightarrow{v}}^t \cap \mathbb{Z}^k$ consisting of $\overrightarrow{n}$ and its adjacent elements along the axis directions.

Let $\Gamma$ be a $T$-invariant set. Fix $\delta > 0$ and $\varepsilon > 0$. A set of points $\xi = \{x_{\overrightarrow{n}} : \overrightarrow{n} \in L^t_{\overrightarrow{v}} \cap \mathbb{Z}^k\}$ is called a $\delta$-pseudo orbit of $L^t_{\overrightarrow{v}}$ for $T$ on $\Gamma$ if $\xi \subset \Gamma$ and

$$\sup_{\overrightarrow{n} \in L^t_{\overrightarrow{v}} \cap \mathbb{Z}^k} \max_{1 \le i \le k} \{d(x_{\overrightarrow{n}}, T^{\overrightarrow{n} - \overrightarrow{n}_{i_-}}(x_{\overrightarrow{n}_{i_-}})), d(x_{\overrightarrow{n}_{i_+}}, T^{\overrightarrow{n}_{i_+} - \overrightarrow{n}}(x_{\overrightarrow{n}}))\} \le \delta.$$

A point $x \in X$ is called $\varepsilon$-*shadows* the above pseudo orbit $\xi$ if

$$\sup_{\overrightarrow{n} \in L^t_{\overrightarrow{v}} \cap \mathbb{Z}^k} d(x_{\overrightarrow{n}}, T^{\overrightarrow{n}}(x)) \le \varepsilon.$$

**Definition 2.2** *Let $T$ be a $\mathbb{Z}^k$-action on $X$ and $\Gamma$ be a $T$-invariant set.*

*(1) We say that $L_{\overrightarrow{v}}$ has the shadowing property for $T$ on $\Gamma$ provided there exists $t > 0$ satisfying the following property: for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo orbit of $L^t_{\overrightarrow{v}}$ for $T$ in $\Gamma$ can be $\varepsilon$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $L_{\overrightarrow{v}}$ has the shadowing property for $T$.*

*(2) We say that $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property for $T$ on $\Gamma$ provided there exist $t, \delta_0, \widehat{L} > 0$ such that any $\delta$-pseudo orbit of $L^t_{\overrightarrow{v}}$ for $T$ in $\Gamma$ with $\delta \le \delta_0$ can be $\widehat{L}\delta$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property for $T$.*

Via the thickening technique, we have given the definitions of pseudo orbits and shadowing property for $T$ along 1-dimensional subspaces, especially the irrational cases, of $\mathbb{R}^k$. Now we redefine these notions via the nonautonomous dynamical systems along $L_{\overrightarrow{v}}$ and discuss the equivalence of these two kinds of definitions.

Given $\overrightarrow{v} = (v_1, \ldots, v_k) \ne \overrightarrow{0}$, we define the nonautonomous dynamical systems along $L_{\overrightarrow{v}}$. Firstly define a sequence of $\{\overrightarrow{m}_n\}_{n \in \mathbb{Z}} \subset \mathbb{Z}^k$ as follows: choose $\overrightarrow{m}_n$ to be any integer vector in

$$\{\overrightarrow{m} : |\overrightarrow{m} - n\frac{\overrightarrow{v}}{|\overrightarrow{v}|}| = \min_{\overrightarrow{l} \in \mathbb{Z}^k} |\overrightarrow{l} - n\frac{\overrightarrow{v}}{|\overrightarrow{v}|}|\}$$

with the smallest norm, where $|\overrightarrow{v}| = \sqrt{v_1^2 + \cdots + v_k^2}$. Obviously, $\overrightarrow{m}_0 = (0, \ldots, 0)$ and for any $\overrightarrow{m}_n = (m_{n,1}, \ldots, m_{n,k})$, $\overrightarrow{m}_{n'} = (m_{n',1}, \ldots, m_{n',k})$ and $1 \le i \le k$, if $n < n'$ then $m_{n,i} \le m_{n',i}$ for $v_i \ge 0$ (or $m_{n,i} \ge m_{n',i}$ for $v_i \le 0$).

**Definition 2.3** *Let $g_n = T^{\overrightarrow{m}_{n+1} - \overrightarrow{m}_n}$ for $n \in \mathbb{Z}$. Note that $g_n = id$ when $\overrightarrow{m}_{n+1} = \overrightarrow{m}_n$. Then we call $g^{\overrightarrow{v}}_{-\infty,+\infty} = \{g_n\}_{n \in \mathbb{Z}}$ is a nonautonomous dynamical system along $L_{\overrightarrow{v}}$.*

Clearly, the sequence $\{\overrightarrow{m}_n\}_{n \in \mathbb{Z}}$ and hence the induced nonautonomous dynamical system $g^{\overrightarrow{v}}_{-\infty,+\infty}$ along $L_{\overrightarrow{v}}$, may not be unique. However, we can see that for any such $\{\overrightarrow{m}_n\}_{n \in \mathbb{Z}}$, we have

$$\lim_{n \to \infty} \Big(\frac{m_{n,1}}{\sqrt{\sum_{i=1}^k (m_{n,i})^2}}, \ldots, \frac{m_{n,k}}{\sqrt{\sum_{i=1}^k (m_{n,i})^2}}\Big) = \frac{1}{|\overrightarrow{v}|}(v_1, \ldots, v_k). \tag{2.2}$$

**Remark 2.4** Fix a nonautonomous dynamical system $g^{\overrightarrow{v}}_{-\infty,\infty} = \{g_n\}_{n \in \mathbb{Z}}$ along $L_{\overrightarrow{v}}$ as above. Note that if $\overrightarrow{m}_{n+1} = \overrightarrow{m}_n$, then $g_n = id$. By the definition of $g^{\overrightarrow{v}}_{-\infty,\infty}$, there are at most $[\sqrt{k}]$ adjacent identity mappings between any two non-identity mappings in $g^{\overrightarrow{v}}_{-\infty,\infty}$. In the long run, these identity mappings do not affect the dynamic behavior of $g^{\overrightarrow{v}}_{-\infty,\infty}$, particularly the shadowing property of $g^{\overrightarrow{v}}_{-\infty,\infty}$ discussed below. Take such $g_n = id$ away successively from $g^{\overrightarrow{v}}_{-\infty,\infty}$, we get a modified nonautonomous dynamical system $(g^{\overrightarrow{v}}_{-\infty,\infty})'$ along $L_{\overrightarrow{v}}$. Without loss of generality, we always assume in the rest of this paper that no element in $g^{\overrightarrow{v}}_{-\infty,\infty} = \{g_n\}_{n\in\mathbb{Z}}$ is equal to $id$.

Let $\alpha, \varepsilon > 0$. A sequence of points $\xi = \{x_n\}_{n=-\infty}^{+\infty}$ is said to be an *$\alpha$-pseudo orbit* for $g^{\overrightarrow{v}}_{-\infty,+\infty} = \{g_n\}_{n\in\mathbb{Z}}$ on $\Gamma$ if $\xi \subset \Gamma$ and

$$\sup_{n\in\mathbb{Z}} d(g_n(x_n), x_{n+1}) \le \alpha.$$

We say that $x \in X$ $\varepsilon$-shadows an $\alpha$-pseudo orbit $\xi = \{x_n\}_{n=-\infty}^{+\infty}$ if

$$\max\{\sup_{n\ge 0} d(g_0^n(x), x_n),\ \sup_{n<0} d((g_n^{-1})^{-1}(x), x_n)\} \le \varepsilon,$$

where $g_0^n = g_{n-1} \circ \cdots \circ g_0$ for $n \ge 1$, $g_0^n = id$ for $n = 0$ and $g_n^{-1} = g_{-1} \circ \cdots \circ g_n$ for $n \le -1$.

**Definition 2.5** *Let $T$ be a $\mathbb{Z}^k$-action on $X$ and $\Gamma$ be a $T$-invariant set.*

*(1) We say that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property on $\Gamma$ provided that for any $\varepsilon > 0$ there exists $\alpha > 0$ such that every $\alpha$-pseudo orbit for $g^{\overrightarrow{v}}_{-\infty,+\infty}$ in $\Gamma$ can be $\varepsilon$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property.*

*(2) We say that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property on $\Gamma$ provided there exist $\alpha_0, L > 0$ such that any $\alpha$-pseudo orbit (with $\alpha \le \alpha_0$) for $g^{\overrightarrow{v}}_{-\infty,+\infty}$ in $\Gamma$ can be $L\alpha$-shadowed by some point $x \in X$. In particular, when $\Gamma = X$, we say that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property.*

The sequence of the generators $\{f_i : 1 \le i \le k\}$ is said to be equi-Lipschitz continuous if there exists a constant $L_1'$ such that $d(f_i(x), f_i(y)) \le L_1' d(x,y)$ for any $x, y \in X, 1 \le i \le k$.

**Theorem 2.6** *Let $T$ be a continuous $\mathbb{Z}^k$-action on $X$ and $\Gamma$ be a $T$-invariant set. Fix a nonzero $\overrightarrow{v} \in \mathbb{R}^k$ and let $g^{\overrightarrow{v}}_{-\infty,+\infty} = \{g_n\}_{n\in\mathbb{Z}}$ be a nonautonomous dynamical system along $L_{\overrightarrow{v}}$. Then the following statements are equivalent:*

*(1) $L_{\overrightarrow{v}}$ has the shadowing property for $T$ on $\Gamma$.*

*(2) $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property on $\Gamma$.*

*Moreover, if the family of generators $\{f_i : 1 \le i \le k\}$ is equi-Lipschitz continuous on $X$, then for any $\overrightarrow{v} \in \mathbb{R}^k$ the following statements are equivalent:*

*(1') $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property for $T$ on $\Gamma$.*

*(2') $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property on $\Gamma$.*

**Proof** By Definition 2.3, there is a sequence of $\{\overrightarrow{m}_n\}_{n\in\mathbb{Z}} \subset \mathbb{Z}^k$ such that $g_n = T^{\overrightarrow{m}_{n+1} - \overrightarrow{m}_n}$.

(1)$\Rightarrow$(2). Suppose $L_{\overrightarrow{v}}$ has the shadowing property for $T$ on $\Gamma$, i.e., there exists $t > 0$ satisfying the following property: for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\text{every } \delta(\varepsilon)\text{-pseudo orbit of } L^t_{\overrightarrow{v}} \text{ in } \Gamma \text{ can be } \varepsilon\text{-shadowed by some point } x \in X. \tag{2.3}$$

Here $t$ may be less than $\sqrt{k}$ or greater than or equal to $\sqrt{k}$. When $t \ge \sqrt{k}$, then clearly $\{\overrightarrow{m}_n\}_{n\in\mathbb{Z}} \subset L^t_{\overrightarrow{v}} \cap \mathbb{Z}^k$. Otherwise, we can choose $t_1 > \sqrt{k} > t$ and show in the following that $L_{\overrightarrow{v}}$

has the shadowing property for $T$ on $\Gamma$ with the thickness $t_1$. Let $\xi=\{x_{\overrightarrow{n}}:\overrightarrow{n}\in L^{t_1}_{\overrightarrow{v}}\cap\mathbb{Z}^k\}$ be a $\delta$-pseudo orbit of $L^{t_1}_{\overrightarrow{v}}$ on $\Gamma$ for some $\delta>0$, then clearly its subset $\xi'=\{x_{\overrightarrow{n}}:\overrightarrow{n}\in L^{t}_{\overrightarrow{v}}\cap\mathbb{Z}^k\}$ is also a $\delta$-pseudo orbit of $L^{t}_{\overrightarrow{v}}$. By equicontinuity of $\{f_i:1\le i\le k\}$, for any $\varepsilon>0$ there exists $\varepsilon'>0$ such that if $\xi'$ is $\varepsilon'$ shadowed by a point $x$, then $\xi$ can be $\varepsilon$ shadowed by the same point. For $\varepsilon'$, take $\delta(\varepsilon')>0$ such that (2.3) holds. So any $\delta(\varepsilon')$-pseudo orbit $\xi=\{x_{\overrightarrow{n}}:\overrightarrow{n}\in L^{t_1}_{\overrightarrow{v}}\cap\mathbb{Z}^k\}$ on $\Gamma$ can be $\varepsilon$ shadowed by some point, and hence we get that $L_{\overrightarrow{v}}$ has the shadowing property for $T$ on $\Gamma$ with the thickness $t_1$. For simplicity of notations, we therefore assume that $t\ge\sqrt{k}$ and then

$$\{\overrightarrow{m}_n\}_{n\in\mathbb{Z}}\subset L^{t}_{\overrightarrow{v}}\cap\mathbb{Z}^k. \tag{2.4}$$

Now we show that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property on $\Gamma$. Given $\varepsilon>0$, take $\delta(\varepsilon)>0$ such that (2.3) holds. By equicontinuity of $\{f_i:1\le i\le k\}$ and (2.4), for the above $\delta(\varepsilon)$ there exists $\alpha>0$ such any $\alpha$-pseudo orbit $\xi$ for $g^{\overrightarrow{v}}_{-\infty,+\infty}$ can be extended to a $\delta(\varepsilon)$-pseudo orbit $\eta(\xi)$ of $L^{t}_{\overrightarrow{v}}$. Therefore, by (2.3) $\eta(\xi)$ can be $\varepsilon$-shadowed by some point $x\in X$ with respect to $L^{t}_{\overrightarrow{v}}$, and hence clearly $\xi$ can be $\varepsilon$-shadowed by $x$ with respect to $g^{\overrightarrow{v}}_{-\infty,+\infty}$. This shows that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property on $\Gamma$.

(2)$\Rightarrow$(1). We choose $t>\sqrt{k}$ such that (2.4) holds, and show in the following that $L_{\overrightarrow{v}}$ has the shadowing property for $T$ on $\Gamma$ with the thickness $t$.

Let $\xi=\{x_{\overrightarrow{n}}:\overrightarrow{n}\in L^{t}_{\overrightarrow{v}}\cap\mathbb{Z}^k\}$ be a pseudo orbit of $L^{t}_{\overrightarrow{v}}$ and its subsequence $\eta=\{x_{\overrightarrow{m}_n}\}^{+\infty}_{n=-\infty}$ be a pseudo orbit for $g^{\overrightarrow{v}}_{-\infty,+\infty}$. By equicontinuous of $\{f_i:1\le i\le k\}$, for any $\varepsilon>0$ there exists $\varepsilon_0>0$ such that if $\eta$ is $\varepsilon_0$-shadowed by a point $x\in X$ then $\xi$ can be $\varepsilon$-shadowed by the same point $x$. Suppose $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the shadowing property on $\Gamma$, then for $\varepsilon_0>0$ there exists $\alpha>0$ such that any $\alpha$-pseudo orbit $\eta$ for $g^{\overrightarrow{v}}_{-\infty,+\infty}$ can be $\varepsilon_0$-shadowed by some point $x\in X$. By equicontinuous of $\{f_i:1\le i\le k\}$ and (2.4), for $\alpha>0$ there exists $\delta>0$ such that if $\xi$ is a $\delta$-pseudo orbit of $L^{t}_{\overrightarrow{v}}$, then its subsequence $\eta$ is an $\alpha$-pseudo orbit for $g^{\overrightarrow{v}}_{-\infty,+\infty}$. So any $\delta$-pseudo orbit of $L^{t}_{\overrightarrow{v}}$ can be $\varepsilon$-shadowed by some point $x\in X$, and hence we get that $L_{\overrightarrow{v}}$ has the shadowing property on $\Gamma$.

Now suppose that the family of generators $\{f_i:1\le i\le k\}$ is equi-Lipschitz continuous on $X$. Then it is clear that $g^{\overrightarrow{v}}_{-\infty,\infty}$ is also equi-Lipschitz continuous.

(1')$\Rightarrow$(2'). Suppose $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property for $T$ on $\Gamma$, then there exist $t>0$, $\delta_0,\widehat{L}>0$ such that any $\delta$-pseudo orbit of $L^{t}_{\overrightarrow{v}}$ for $T$ in $\Gamma$ with $\delta\le\delta_0$ can be $\widehat{L}\delta$-shadowed by some point $x\in X$. By a similar discussion as in (1)$\Rightarrow$(2), we can get $\widehat{L}$ increases while as $t$ increases. Without loss of generality, we assume that $t\ge\sqrt{k}$ and then (2.4) holds.

Now we show that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property on $\Gamma$. By equi-Lipschitz continuity of $\{f_i:1\le i\le k\}$ and (2.4), there exists $K>0$ such that any $\alpha$-pseudo orbit $\xi$ for $g^{\overrightarrow{v}}_{-\infty,+\infty}$ can be extended to a $K\alpha$-pseudo orbit $\eta(\xi)$ of $L^{t}_{\overrightarrow{v}}$ for $T$. Therefore, by Lipschitz shadowing property of $L_{\overrightarrow{v}}$, the $K\alpha$-pseudo orbit($K\alpha\le\delta_0$) $\eta(\xi)$ can be $\widehat{L}K\alpha$-shadowed by some point $x\in X$ with respect to $L^{t}_{\overrightarrow{v}}$, and hence clearly the $\alpha$-pseudo orbit($\alpha\le\alpha_0$) $\xi$ can be $L\alpha$-shadowed by $x$ with respect to $g^{\overrightarrow{v}}_{-\infty,+\infty}$, where $\alpha_0=\frac{\delta_0}{K}$ and $L=\widehat{L}K$. This shows that $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property on $\Gamma$.

(2')⇒(1'). We choose $t > \sqrt{k}$ such that (2.4) holds, and show in the following that $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property for $T$ on $\Gamma$ with the thickness $t$. Let $\xi = \{x_{\overrightarrow{n}} : \overrightarrow{n} \in L^t_{\overrightarrow{v}} \cap \mathbb{Z}^k\}$ be a $\delta$-pseudo orbit of $L^t_{\overrightarrow{v}}$ for $T$ on $\Gamma$, by equi-Lipschitz continuity of $\{f_i : 1 \le i \le k\}$ and (2.4), there exists $K'(L'_1) > 0$ such that its subsequence $\eta = \{x_{\overrightarrow{m}_n}\}_{n=-\infty}^{+\infty}$ is a $K'\delta$-pseudo orbit for $g^{\overrightarrow{v}}_{-\infty,+\infty}$. Suppose $g^{\overrightarrow{v}}_{-\infty,+\infty}$ has the Lipschitz shadowing property, then $\eta$ can be $LK'\delta$-shadowed by some point $x \in X$ if $K'\delta \le \alpha_0$. By equi-Lipschitz continuity of $\{f_i : 1 \le i \le k\}$ again, there exists $\widehat{L}(L'_1, L)$ such that $\xi$ is $\widehat{L}\delta$-shadowed by the same point. Thus $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property on $\Gamma$ with constants $\delta_0 = \frac{\alpha_0}{K'}, \widehat{L} > 0$. □

## 3. Shadowing along an Anosov direction for smooth $\mathbb{Z}^k$-actions

In this section, we investigate the shadowing property of 1-dimensional subsystems for smooth $\mathbb{Z}^k$-actions. Let $M$ be an $m$-dimensional closed Riemannian manifold. We denote by $\|\cdot\|$ and $d(\cdot,\cdot)$, respectively, the norm on $TM$ and the metric on $M$ induced by the Riemmanian metric.

Let $T : \mathbb{Z}^k \to \mathrm{Diff}^r(M, M), r \ge 1$, be a $C^r$ $\mathbb{Z}^k$-action on $M$, where $\mathrm{Diff}^r(M, M)$ is the space of $C^r$ diffeomorphisms equipped with the $C^r$-topology. We still denote the generators of $T$ by $f_i$, $1 \le i \le k$. A Borel probability measure $\mu$ on $M$ is said to be $T$-invariant (resp., ergodic), if $\mu$ is $f_i$-invariant (resp., ergodic) for each $i$.

In the classical theory of smooth $\mathbb{Z}^1$ actions, we often require that the system has certain hyperbolicity when the shadowing property is considered [2, 3]. For example, any Anosov diffeomorphism has the Lipschitz shadowing property and, more generally, any diffeomorphism has the Lipschitz shadowing property on its hyperbolic sets.

**Definition 3.1** *Let $T$ be a smooth $\mathbb{Z}^k$-action on $M$ and $\Gamma$ be a $T$-invariant set. We say that $T$ has a hyperbolic direction on $\Gamma$ provided there exists a nonzero $\overrightarrow{v} \in \mathbb{R}^k$ such that any induced nonautonomous dynamical system $g^{\overrightarrow{v}}_{-\infty,+\infty} = \{g_n\}_{n\in\mathbb{Z}}$ is uniformly hyperbolic on $\Gamma$ in the following sense: there exist an invariant splitting $T_\Gamma M = E^s \bigoplus E^u$ and constants $0 < \lambda < 1, N > 0$ such that for any $x \in \Gamma, t \in \mathbb{Z}$ we have*

$$\|D(g_{t+N-1} \circ \cdots \circ g_t)(x)v\| \le \lambda\|v\|, v \in E^s(x),$$

$$\|D(g_{t+N-1} \circ \cdots \circ g_t)^{-1}(x)v\| \le \lambda\|v\|, v \in E^u(x).$$

*Particularly, when $\Gamma = M$, we say that $T$ has an Anosov direction.*

**Example 3.2** Let $T$ be the $\mathbb{Z}^2$-action on the torus $\mathbb{T}^2$ with the generators induced by the matrices $\{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_2 = A_1^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Clearly, they are both hyperbolic. The eigenvalues of $A_1$ are $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$, and let $E_1$ and $E_2$ be the corresponding eigenspaces in $\mathbb{R}^2$. Since $A_2$ is the inverse of $A_1$, $A_2$ has eigenvalues $\mu_1 = \lambda_2$ and $\mu_2 = \lambda_1$, with eigenspaces $F_1 = E_2$ and $F_2 = E_1$. Denote by $\mathbb{G}_1$ the set of all 1-dimensional subspaces (or 1-planes) of $\mathbb{R}^2$. Let $L_1$ be the line in $\mathbb{G}_1$ with slope 1. We can see that any direction of $L_{\overrightarrow{v}} \in \mathbb{G}_1 \setminus \{L_1\}$ is an Anosov direction.

**Example 3.3** Let $T$ be a $C^r, r \geq 1, \mathbb{Z}^k$-action on $M$ with the generators $\{f_i, 1 \leq i \leq k\}$ and $\mu$ a $T$-ergodic measure. By the Multiplicative Ergodic Theorem for $T$ (see [9]), there exist a measurable $T$-invariant set $\Gamma \in M$ with $\mu(\Gamma) = 1$, an invariant splitting $T_\Gamma M = \bigoplus_{j=1}^s E_j$, and numbers $\lambda_{i,j}, 1 \leq i \leq k, 1 \leq j \leq s$, satisfying the following properties:

(1) for $0 \neq u \in E_j, 1 \leq j \leq s$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_i^n(x)u\| = \lambda_{i,j}, \quad x \in \Gamma. \tag{3.1}$$

(2) for each $\overrightarrow{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, each $1 \leq j \leq s$ and any $0 \neq u \in E_j$, we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|D(f_1^{n_1} \circ \cdots \circ f_k^{n_k})^t(x)u\| = \sum_{i=1}^k n_i \lambda_{i,j}, \quad x \in \Gamma. \tag{3.2}$$

For a nonzero vector $\overrightarrow{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ and a nonautonomous dynamical system $g_{-\infty,\infty}^{\overrightarrow{v}} = \{g_n\}_{n\in\mathbb{Z}}$ along $L_{\overrightarrow{v}}$, by the proof of [10, Theorem 3.3], we can see

$$\lim_{n\to\infty} \frac{1}{n} \log \|D(g_{n+t-1} \circ \cdots \circ g_t)(x)u\| = \sum_{i=1}^k \frac{v_i}{|\overrightarrow{v}|} \lambda_{ij}, \ t \in \mathbb{Z}, x \in \Gamma, u \in E_j.$$

If the limit in (3.1) is uniform in $x \in \Gamma$, then we can see that any nonzero vector $\overrightarrow{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ with $\sum_{i=1}^k v_i \lambda_{i,j} \neq 0$, $1 \leq j \leq s$, can determine a hyperbolic direction on $\Gamma$.

Now we give the main result of this paper.

**Theorem 3.4** *Let $T$ be a $C^r, r \geq 1, \mathbb{Z}^k$-action on $M$. If $T$ has an Anosov direction, then there exists $L_{\overrightarrow{v}}$ which has the Lipschitz shadowing property for $T$ on $M$.*

Since $M$ is compact, $\|Df_i\|$ is bounded for any $i \in \{1, \ldots, k\}$, then the family $\{f_i : 1 \leq i \leq k\}$ is equi-Lipschitz continuous. By Theorem 2.6, we only need to prove that any nonautonomous dynamical system $g_{-\infty,+\infty}^{\overrightarrow{v}}$ along $L_{\overrightarrow{v}}$ has the Lipschitz shadowing property on $M$.

**Lemma 3.5** *For a fixed $N > 0$, there exists a constant $L_1^*$ such that if a sequence of points $\xi = \{x_k\}_{k=-\infty}^{+\infty}$ is an $\alpha$-pseudo orbit for $g_{-\infty,\infty}^{\overrightarrow{v}} = \{g_n\}_{n\in\mathbb{Z}}$, then the subsequence of points $\xi' = \{x_{kN}\}_{k=-\infty}^{+\infty}$ is a $L_1^*\alpha$-pseudo orbit for $\{g_{(k+1)N-1} \circ \cdots \circ g_{kN}\}_{k\in\mathbb{Z}}$.*

**Proof** Clearly, the sequence $g_{-\infty,+\infty}^{\overrightarrow{v}} = \{g_n\}_{n\in\mathbb{Z}}$ of maps is equi-Lipschitz continuous, i.e., there exists a constant $L_1$ such that $d(g_n(x), g_n(y)) \leq L_1 d(x, y)$ for any $x, y \in \Gamma, n \in \mathbb{Z}$.

Since $\xi = \{x_n\}_{n=-\infty}^{+\infty}$ is an $\alpha$-pseudo orbit, we have $d(g_n(x_n), x_{n+1}) \leq \alpha$. By Lipschitz continuity, $d(g_0(x_0), x_1) \leq \alpha$ implies $d(g_1 \circ g_0(x_0), g_1(x_1)) \leq L_1 d(g_0(x_0), x_1) \leq L_1\alpha$, and $d(g_1(x_1), x_2) \leq \alpha$, then we have $d(g_1 \circ g_0(x_0), x_2) \leq L_1\alpha + \alpha$. Inductively, we can get

$$d(g_{N-1} \circ \cdots \circ g_0(x_0), x_N) \leq L_1^{N-1}\alpha + L_1^{N-2}\alpha + \cdots + L_1\alpha + \alpha = \frac{L_1^N - 1}{L_1 - 1} \cdot \alpha.$$

Denote $L_1^* = \frac{L_1^N - 1}{L_1 - 1}$, then we have $d(g_{N-1} \circ \cdots \circ g_0(x_0), x_N) \leq L_1^*\alpha$.

The same reasoning as above shows that

$$d(g_{(k+1)N-1}\circ\cdots\circ g_{kN}(x_{kN}), x_{(k+1)N}) \le L_1^*\alpha \quad k\in\mathbb{Z}.$$

Hence, the subsequence of points $\xi' = \{x_{kN}\}_{k=-\infty}^{+\infty}$ is a $L_1^*\alpha$-pseudo orbit. □

**Proof of Theorem 3.4** Suppose $T$ has an Anosov direction. Then we can choose a uniform hyperbolic nonautonomous dynamical system $g^{\overrightarrow{v}}_{-\infty,+\infty} = \{g_n\}_{n\in\mathbb{Z}}$ induced by a nonzero $\overrightarrow{v}\in\mathbb{R}^k$. By Proposition 3.6, there exists $N>0$ such that $\{g_{(k+1)N-1}\circ\cdots\circ g_{kN}\}_{k\in\mathbb{Z}}$ has the Lipschitz shadowing property.

Let $\xi = \{x_n\}_{n=-\infty}^{+\infty}$ be an $\alpha$-pseudo orbit for $g^{\overrightarrow{v}}_{-\infty,\infty} = \{g_n\}_{n\in\mathbb{Z}}$. Then by Lemma 3.5, its subsequence $\{x_{kN}\}_{k=-\infty}^{+\infty}$ is an $L_1^*\alpha$-pseudo orbit for $\{g_{(k+1)N-1}\circ\cdots\circ g_{kN}\}_{k\in\mathbb{Z}}$. Since $\{g_{(k+1)N-1}\circ\cdots\circ g_{kN}\}_{k\in\mathbb{Z}}$ has the Lipschitz shadowing property, there exist numbers $\delta_0>0, L_2>0$ such that any $\delta$-pseudo orbit (with $\delta\le\delta_0$) $\{x_{kN}\}_{k=-\infty}^{+\infty}$ for $\{g_{(k+1)N-1}\circ\cdots\circ g_{kN}\}_{k\in\mathbb{Z}}$ can be $L_2\delta$-shadowed by a point $x\in M$, i.e.,

$$d(x,x_0)\le L_2\delta, \tag{3.3}$$

$$d(g_{(k+1)N-1}\circ\cdots\circ g_0(x), x_{(k+1)N}) \le L_2\delta, \quad k\ge 0,$$

and

$$d((g_{-1}\circ\cdots\circ g_{kN})^{-1}(x), x_{kN}) \le L_2\delta, \quad k\le -1.$$

Take $\delta = L_1^*\alpha$. By equi-Lipschitz continuity and (3.3), we have

$$d(g_0(x), g_0(x_0)) \le L_1 d(x,x_0) \le L_1L_2\delta = L_1L_2L_1^*\alpha,$$

then we get

$$d(g_0(x), x_1) \le d(g_0(x), g_0(x_0)) + d(g_0(x_0), x_1) = L_1L_2L_1^*\alpha + \alpha$$

by triangle inequality. By the same ways as above, we have

$$\begin{aligned} d(g_1g_0(x), x_2) &\le L_1^2L_2L_1^*\alpha + L_1\alpha + \alpha, \\ &\cdots \\ d(g_{N-2}\circ\cdots\circ g_0(x), x_{N-1}) &\le L_1^{N-1}L_2L_1^*\alpha + L_1^{N-2}\alpha + \cdots + L_1\alpha + \alpha \\ &= L_1^{N-1}L_2L_1^*\alpha + (L_1^* - L_1^{N-1})\alpha \\ &\le L_1^*(L_1^{N-1}L_2 + 1)\alpha. \end{aligned}$$

Inductively, for any $k\ge 1$, we can get

$$d(g_{(k+1)N-1}\circ\cdots\circ g_0(x), x_{(k+1)N}) \le L_2L_1^*\alpha,$$

$$d(g_{(k+1)N}\circ g_{(k+1)N-1}\circ\cdots\circ g_0(x), x_{(k+1)N+1}) \le L_1L_2L_1^*\alpha + \alpha,$$

$$d(g_{(k+1)N+1}\circ g_{(k+1)N}\circ g_{(k+1)N-1}\circ\cdots\circ g_0(x), x_{(k+1)N+2}) \le L_1^2L_2L_1^*\alpha + L_1\alpha + \alpha,$$

$$\cdots$$

$$d(g_{(k+2)N-2}\circ\cdots\circ g_{(k+1)N-1}\circ\cdots\circ g_0(x), x_{(k+2)N-1}) \le L_1^*(L_1^{N-1}L_2 + 1)\alpha.$$

By the same discussion, for any $k \le -1$, we can get

$$d(g_{kN}^{-1} \circ \cdots \circ g_{-1}^{-1}(x), x_{kN}) \le L_2 L_1^* \alpha,$$

$$d(g_{kN+1}^{-1} \circ \cdots \circ g_{-1}^{-1}(x), x_{kN+1}) \le L_1 L_2 L_1^* \alpha + \alpha,$$

$$d(g_{kN+2}^{-1} \circ \cdots \circ g_{-1}^{-1}(x), x_{kN+2}) \le L_1^2 L_2 L_1^* \alpha + L_1 \alpha + \alpha,$$

$$\cdots$$

$$d(g_{kN+N-1}^{-1} \circ \cdots \circ g_{-1}^{-1}(x), x_{kN+N-1}) \le L_1^* (L_1^{N-1} L_2 + 1)\alpha.$$

Denote $L = L_1^*(L_1^{N-1}L_2 + 1)$, $\alpha_0 = \frac{\delta_0}{L_1^*}$, and we can see that $x \in M$ $L\alpha$-shadows the $\alpha$-pseudo orbit $\xi = \{x_n\}_{n=-\infty}^{+\infty}$. Hence, $g_{-\infty,+\infty}^{\overrightarrow{v}}$ has the Lipschitz shadowing property on $\Gamma$.

The proof of the theorem is completed. □

Now, we give the following Proposition, which is crucial to Theorem 3.4.

**Proposition 3.6** *Let $T$ be a $C^r, r \ge 1, \mathbb{Z}^k$-action on $M$ and $g_{-\infty,+\infty}^{\overrightarrow{v}} = \{g_n\}_{n\in\mathbb{Z}}$ be uniformly hyperbolic on $M$ (see Definition 3.1). Then there exists $N > 0$ such that the nonautonomous dynamical system $\{g_{(k+1)N-1} \circ \cdots \circ g_{kN}\}_{k\in\mathbb{Z}}$ has the Lipschitz shadowing property in the following sense: there exist numbers $\delta_0 > 0, L_2 > 0$ such that for any $\delta$-pseudo orbit $\xi = \{x_{kN}\}_{k=-\infty}^{+\infty}$, $\delta \le \delta_0$, $x_{kN} \in M$, there exists a point $x \in M$ that $L_2\delta$-shadows $\xi$.*

**Proof** Here, we shall adapt Bowen's method as in [8] to prove this Proposition. Since $g_{-\infty,+\infty}^{\overrightarrow{v}} = \{g_n\}_{n\in\mathbb{Z}}$ is uniformly hyperbolic on $M$, there exists $N > 0$ such that the nonautonomous dynamical $\{g_{(k+1)N-1} \circ \cdots \circ g_{kN}\}_{k\in\mathbb{Z}}$ is uniformly hyperbolic on $M$ in the following sense: there exist an invariant splitting $TM = E^s \bigoplus E^u$ and constant $0 < \lambda < 1$ such that for any $x \in \Gamma, k \in \mathbb{Z}$ we have

$$\|D(g_{(k+1)N-1} \circ \cdots \circ g_{kN})(x)v\| \le \lambda \|v\|, \quad v \in E^s(x),$$

$$\|D(g_{(k+1)N-1} \circ \cdots \circ g_{kN})^{-1}(x)v\| \le \lambda \|v\|, \quad v \in E^u(x).$$

Let $\xi = \{x_{kN}\}_{k=-\infty}^{+\infty}$ be a $\delta$-pseudo orbit for $\{g_{(k+1)N-1} \circ \cdots \circ g_{kN}\}_{k\in\mathbb{Z}}$ in $M$. To simplify the notation during the proof, denote $h_k = g_{(k+1)N-1} \circ \cdots \circ g_{kN}$ and $y_k = x_{kN}$.

Using the standard graph transform method, we can establish the stable manifold theorem for $\{h_k\}_{k\in\mathbb{Z}}$ on $M$. For any $x \in M$, denote the local stable manifold

$$W_\rho^s(x) := \{y \in M : \sup_{k\ge 0} d(h_0^k(x), h_0^k(y)) \le \rho\}$$

and the local unstable manifold

$$W_\rho^u(x) := \{y \in M : \sup_{k\le 0} d((h_k^{-1})^{-1}(x), (h_k^{-1})^{-1}(y)) \le \rho\},$$

where $h_0^k = h_{k-1} \circ \cdots \circ h_0$ for $k \ge 1$, $h_0^k = id$ for $k = 0$ and $h_k^{-1} = h_{-1} \circ \cdots \circ k_k$ for $k \le -1$. Clearly, the local stable manifolds and unstable manifolds have the local transversal intersection property, i.e., there exist $H > 0$ and $\rho_M > 0$ such that for $x, y \in M$ with $d(x, y) \le \rho \le \rho_M$ the intersection $W_{H\rho}^s(x) \cap W_{H\rho}^u(y)$ consists of a single point in $M$ which is denoted by $[x, y]$, and similarly let $W_{H\rho}^s(y) \cap W_{H\rho}^u(x) = [y, x]$.

Take a $\delta_1 > 0$ such that

$$\begin{aligned} &d(h_k(x), h_k(y)) < \lambda d(x,y) \ for\ y \in W^s_{\delta_1}(x), k \in \mathbb{Z}, \\ &d(h_k^{-1}(x), h_k^{-1}(y)) < \lambda d(x,y) \ for\ y \in W^u_{\delta_1}(x), k \in \mathbb{Z}. \end{aligned} \tag{3.4}$$

We can assume the hyperbolicity constant $\lambda$ is small enough to meet our needs. (Otherwise, we can find a $P \in \mathbb{N}$ such that $\lambda^P$ is small enough to meet our needs, and transform the problem of $\{h_k\}_{k\in\mathbb{Z}}$ into the problem of $\{h_{(t+1)P-1} \circ \cdots \circ h_{tP}\}_{t\in\mathbb{Z}}$. It is easy to prove that $\{h_k\}_{k\in\mathbb{Z}}$ has the Lipschitz shadowing property if and only if $\{h_{(t+1)P-1} \circ \cdots \circ h_{tP}\}_{t\in\mathbb{Z}}$ so does.)

Let $\delta_0 = \min\{\frac{\delta_1}{2H+1}, \frac{\delta_1}{4H}, \frac{\rho_M}{4}\}$, and take $\lambda$ small enough satisfying

$$2\lambda < 1, \quad 2\lambda H < 1. \tag{3.5}$$

Note that $\xi = \{y_k\}_{k=-\infty}^{+\infty}$ is a $\delta$-pseudo orbit($\delta \le \delta_0$) for $\{h_k\}_{k\in\mathbb{Z}}$ in $M$. We will find a sequence $\{y_k^*\}_{k=-\infty}^{+\infty}$ which $L_2\delta$-shadows $\xi$ in three steps.

Step 1. Find a sequence $\{y_k^u\}_{k=0}^{+\infty}$, which $(2H+2)\delta$-shadows the positive half sequence $\{y_k\}_{k=0}^{+\infty}$.

Firstly, we consider a finite piece $\{y_k\}_{k=0}^{n}$ of $\xi$ for $n \ge 1$. In the following, we will define two sequences $\{z_k\}_{k=1}^{n}$, and $\{y'_k\}_{k=1}^{n-1}$ successively. The existence and uniqueness of these points are ensured by the property of local transversal intersection. The first sequence $\{z_k\}_{k=1}^{n}$ is defined as follows. Since $d(h_0(y_0), y_1) < \delta$, take

$$\{z_1\} = [y_1, h_0(y_0)] \in W^s_{H\delta}(y_1).$$

Note that $z_1 \in W^s_{H\delta}(y_1)$, by the invariance of local stable manifolds we obtain that $h_1(z_1) \in W^s_{\lambda H\delta}(h_1(y_1))$. As $h_1(z_1), y_2 \in M$ and $d(y_2, h_1(y_1)) < \delta$, then by (3.4) and (3.5), we get

$$d(h_1(z_1), y_2) \le d(h_1(z_1), h_1(y_1)) + d(h_1(y_1), y_2) < \lambda H\delta + \delta < 2\delta.$$

It follows from the property of local transversal intersection that

$$\{z_2\} = [y_2, h_1(z_1)] \in W^s_{2H\delta}(y_2).$$

Now assume that for any $2 \le k \le n-1$,

$$\{z_k\} = [y_k, h_{k-1}(z_{k-1})] \in W^s_{2H\delta}(y_k).$$

Note that $z_k \in W^s_{2H\delta}(y_k)$, by the invariance of local stable manifolds, we obtain that $h_k(z_k) \in W^s_{2\lambda H\delta}(h_k(y_k))$. As $h_k(z_k), y_{k+1} \in M$ and $d(y_{k+1}, h_k(y_k)) < \delta$, then by (3.4) and (3.5), we get

$$d(h_k(z_k), y_{k+1}) \le d(h_k(z_k), h_k(y_k)) + d(h_k(y_k), y_{k+1}) < 2\lambda H\delta + \delta < 2\delta.$$

It follows from the property of local transversal intersection that

$$\{z_{k+1}\} = [y_{k+1}, h_k(z_k)] \in W^s_{2H\delta}(y_{k+1}).$$

Now we define the last sequence $\{y'_k\}_{k=1}^{n-1}$ as follows. Let $y'_{n-1} = h_{n-1}^{-1}(z_n)$. Since $z_n \in W^u_{2H\delta}(h_{n-1}(z_{n-1}))$, by hyperbolicity, we have $y'_{n-1} \in W^u_{2H\delta}(z_{n-1})$. Assume for any $2 \le k \le n-1$, $y'_k$ is defined. Then take $y'_{k-1} = h_{k-1}^{-1}(y'_k)$. Finally let $y'_0 = h_0^{-1}(y'_1)$. Hyperbolicity and (3.5) are the guarantee of the above steps. From the definition of these two sequences, we can see that each element in the family

$$\{\{y_0, y_0'\}, \{h_0(y_0), z_1, y_1'\}, \{h_1(z_1), z_2, y_2'\}, \ldots, \{h_{n-2}(z_{n-2}), z_{n-1}, y_{n-1}'\}, \{h_{n-1}(z_{n-1}), z_n\}\}$$

lies in a local unstable manifold. Moreover, $\{y_k'\}_{k=0}^{n-1}$ shadows $\{y_k\}_{k=0}^{n-1}$. Now we estimate, for $1 \leq k \leq n-1$, the distance between $y_k$ and $y_k'$. Since $z_n \in W^u_{2H\delta}(h_{n-1}(z_{n-1}))$, and $y_{n-1}' = h_{n-1}^{-1}(z_n)$, by hyperbolicity, we have $y_{n-1}' \in W^u_{2\lambda H\delta}(z_{n-1})$, that is to say

$$d(z_{n-1}, y_{n-1}') < 2\lambda H\delta < \delta \quad (\text{by } (3.5)).$$

So

$$d(y_{n-1}, y_{n-1}') \leq d(y_{n-1}, z_{n-1}) + d(z_{n-1}, y_{n-1}') < 2H\delta + \delta.$$

Then

$$d(h_{n-2}(z_{n-2}), y_{n-1}') \leq d(h_{n-2}(z_{n-2}), z_{n-1}) + d(z_{n-1}, y_{n-1}') < 2H\delta + \delta.$$

Hence,

$$\begin{aligned} d(z_{n-2}, y_{n-2}') &< \lambda d(h_{n-2}(z_{n-2}), y_{n-1}') < \lambda(2H\delta + \delta) \\ &< \delta + \lambda\delta < 2\delta, \quad (\text{by } (3.4) \text{ and } (3.5)) \end{aligned}$$

so we have

$$d(y_{n-2}, y_{n-2}') \leq d(y_{n-2}, z_{n-2}) + d(z_{n-2}, y_{n-2}') < 2H\delta + 2\delta.$$

Now assume that for $3 \leq i \leq n-3$, $d(z_{n-i}, y_{n-i}') < 2\delta$, so we have $d(x_{n-i}, y_{n-i}) < 2H\delta + 2\delta$. Then

$$d(h_{n-i-1}(z_{n-i-1}), y_{n-i}') \leq d(h_{n-i-1}(z_{n-i-1}), z_{n-i}) + d(z_{n-i}, y_{n-i}') < 2H\delta + 2\delta.$$

Hence,

$$\begin{aligned} d(z_{n-i-1}, y_{n-i-1}') &< \lambda d(h_{n-i-1}(z_{n-i-1}), y_{n-i}') \\ &< \lambda(2H\delta + 2\delta) < \delta + 2\lambda\delta < 2\delta \quad (\text{by } (3.4),\ (3.5)), \end{aligned}$$

so we have

$$d(y_{n-i-1}, y_{n-i-1}') \leq d(y_{n-i-1}, z_{n-i-1}) + d(z_{n-i-1}, y_{n-i-1}') < 2H\delta + 2\delta.$$

Note that $d(z_1, y_1') < 2\delta$, so we have $d(y_1, y_1') < d(y_1, z_1) + d(z_1, y_1') < H\delta + 2\delta$. Then

$$d(h_0(y_0), y_1') < d(h_0(y_0), z_1) + d(z_1, y_1') < H\delta + 2\delta,$$

and

$$d(y_0, y_0') < \lambda(d(h_0(y_0), y_1')) < \lambda(H\delta + 2\delta) < 2\delta.$$

Therefore, we prove that $\{y_k'\}_{k=0}^{n-1}$ $(2H+2)\delta$-shadows $\{y_k\}_{k=0}^{n-1}$. In fact, by the construction, we can see that the sequence $\{y_k'\}_{k=0}^{n-1}$ is uniquely determined by $\{y_k\}_{k=0}^{n-1}$, and $y_0' \in W^u_{2\delta}(y_0)$. For convenience, we relabel $\{y_k'\}_{k=1}^{n-1}$ by $\{y_{k,n}'\}_{k=1}^{n-1}$ to indicate its dependence on $n$. Let $y_0^u$ be one limit point of $\{y_{0,n}'\}_{n=0}^{+\infty}$. Obviously, $y_0^u \in W^u_{2\delta}(x_0)$. Now we define sequences $\{y_k^u\}_{k=0}^{+\infty}$ successively as follows. Let $y_1^u = h_0(y_0^u)$. Inductively define $y_{k+1}^u = h_k(y_k^u)$ for $k \geq 2$. It is easy to see that

$y_k^u \in W_{2\delta}^u(z_k)$, $z_k \in W_{2H\delta}^s(y_k)$ for any $k \geq 1$ and the sequence $\{y_k^u\}_{k=0}^{+\infty}$ $(2H+2)\delta$-shadows $\{y_k\}_{k=0}^{+\infty}$.

Step 2. Find a sequence $\{y_k^s\}_{k=-\infty}^{0}$ which $(2H+2)\delta$-shadows the negative half sequence $\{y_k\}_{k=-\infty}^{0}$.

Since the strategy in this step is similar to that in Step 1 except for the type of local manifold in which the shadowing sequence lies, we only give the outline of the construction of $\{y_k^s\}_{k=-\infty}^{0}$. For simplicity, we assume that for any $k \leq 0$, $d(h_{k-1}^{-1}(y_k), y_{k-1}) < \delta$ (otherwise, we can take $0 < \delta' < \delta$ such that for any $\delta'$-pseudo orbit $\{y_k\}_{k=-\infty}^{+\infty}$, we have $d(y_k, h_{k-1}(y_{k-1})) < \delta'$ for $k \leq 0$ and show that $\{y_k\}_{k=-\infty}^{+\infty}$ can be $L_2\delta'$-shadowed by some sequence of points). Firstly, for any finite piece $\{y_k\}_{k=n}^{0} (n \leq -1)$ of $\xi$, we define two sequences $\{z_i\}_{i=n}^{-1}$ and $\{y_i'\}_{i=n+1}^{-1}$ successively as follows by the property of local transversal intersection, hyperbolicity, (3.5). Let

$$\{z_{-1}\} = [(h_{-1}^{-1}(y_0)), y_{-1}] \in W_{H\delta}^u(y_{-1}),$$

$$\{z_{-2}\} = [(h_{-1}^{-2}(z_{-1})), y_{-2}] \in W_{2H\delta}^u(y_{-2}),$$

$$\cdots$$

$$\{z_n\} = [(h_{-1}^{n}(z_{n+1})), y_n] \in W_{2H\delta}^u(y_n).$$

Let $y_{n+1}' = h_n(z_n)$. Inductively define $y_k' = h_{k-1}(y_{k-1}')$ for any $n+2 \leq k \leq -1$, and let $y_0' = h_{-1}(y_{-1}')$. From the construction, we can see that each element in the family

$$\{\{y_0, y_0'\}, \{h_{-1}^{-1}(y_0), z_{-1}, y_{-1}'\}, \{h_{-2}^{-1}(z_{-1}), z_{-2}, y_{-2}'\},$$

$$\ldots, \{h_{n+1}^{-1}(z_{n+2}), z_{n+1}, y_{n+1}'\}, \{h_n^{-1}(z_{n+1}), z_n\}\}$$

lies in a local stable manifold. Moreover $\{y_k\}_{k=n+1}^{0}$ shadows $\{y_k\}_{k=n+1}^{0}$. Now we estimate the distance between $y_k$ and $y_k'$ for any $n+1 \leq k \leq 0$. Since $z_n \in W_{2H\delta}^s(h_n^{-1}(z_{n+1}))$, we have $d(h_n^{-1}(z_{n+1}), z_n) < 2H\delta$. By hyperbolicity, we get

$$d(z_{n+1}, y_{n+1}') < \lambda d(h_n^{-1}(z_{n+1}), z_n) < 2\lambda H\delta < \delta.$$

And then

$$d(y_{n+1}, y_{n+1}') \leq d(y_{n+1}, z_{n+1}) + d(z_{n+1}, y_{n+1}') \leq 2H\delta + \delta.$$

Similarly, we have $d(z_k, y_k') < 2\delta$, $d(y_k, y_k') < 2H\delta + 2\delta$ for any $n+2 \leq k \leq -2$. Moreover, $d(z_{-1}, y_{-1}') < 2\delta$, $d(y_{-1}, y_{-1}') < H\delta + 2\delta$,

$$d(y_0, y_0') < \lambda d(h_{-1}^{-1}(y_0), y_{-1}') \leq \lambda[d(h_{-1}^{-1}(y_0), z_{-1}) + d(z_{-1} + y_{-1}')] < \lambda(H\delta + 2\delta) < 2\delta.$$

Therefore, we prove that $\{y_k'\}_{k=n+1}^{0}$ $(2H+2)\delta$-shadows $\{y_k\}_{k=n+1}^{0}$.

Relabel $\{y_k'\}_{k=n+1}^{0}$ by $\{y_{k,n}'\}_{k=n+1}^{0}$ and let $y_0^s$ be one limit point of $\{y_{0,n}'\}_{n=-\infty}^{0}$. Obviously, $y_0^s \in W_{2\delta}^s(y_0)$. Now define the sequence $\{y_k^s\}_{k=-\infty}^{0}$ as follows. Let $y_{-1}^s = h_{-1}^{-1}(y_0^s)$. Inductively define $y_k^s = h_k^{-1}(y_{k+1}^s)$ for any $k \leq -2$. Clearly, $y_k^s \in W_{2\delta}^s(z_k)$ and $z_k \in W_{2H\delta}^u(y_k)$ for any $k \leq -1$, and $\{y_k^s)\}_{k=-\infty}^{0}(2H+2)\delta$-shadows $\{y_k\}_{k=-\infty}^{0}$.

Step 3. Construct the desired sequence $\{y_k^*\}_{k=-\infty}^{+\infty}$.

Note that

$$y_0^s \in W_{2\delta}^s(y_0),\ y_0^u \in W_{2\delta}^u(y_0),$$

so we have $d(y_0^s, y_0^u) \le d(y_0^s, y_0) + d(y_0, y_0^u) < 4\delta$. By the property of local transversal intersection, we can take

$$y_0^* = [y_0^u, (y_0^s)] \in W_{4H\delta}^s(y_0^u).$$

We now define the sequence $\{y_k^*\}_{k=-\infty}^{+\infty}$ as follows. For the positive direction, inductively define $y_k^* = h_{k-1}(y_{k-1}^*)$ for $k \ge 1$. By hyperbolicity, $d(y_k^u, y_k^*) < \lambda d(y_{k-1}^u, y_{k-1}^*)$ for $k \ge 1$, i.e., $d(y_0^u, y_0^*) < 4H\delta$ and $d(y_k^u, y_k^*) < 4\lambda^k H\delta$ for $k \ge 1$. Therefore, $d(y_0^*, y_0) \le d(y_0^u, y_0^*) + d(y_0, y_0^u) < 4H\delta + 2\delta$ and for $k \ge 1$,

$$d(y_k^*, y_k) \le d(y_k^u, y_k^*) + d(y_k, y_k^u) < 4\lambda^k H\delta + 2H\delta + 2\delta < 4H\delta + 2\delta. \tag{3.6}$$

For the negative direction, inductively define $y_k^* = h_k^{-1}(y_{k+1}^*)$ for $k \le -1$. By hyperbolicity, $d(y_k^s, y_k^*) < \lambda d(y_{k+1}^s, y_{k+1}^*)$ for $k \le -1$, i.e., $d(y_0^s, y_0^*) < 4H\delta$ and $d(y_k^s, y_k^*) < 4\lambda^{-k} H\delta$ for $k \le -1$. Hence, $d(y_0^*, y_0) \le d(y_0^s, y_0^*) + d(y_0, y_0^s) < 4H\delta + 2\delta$ and for $k \le -1$,

$$d(y_k^*, y_k) \le d(y_k^s, y_k^*) + d(y_k, y_k^s) < 4\lambda^{-k} H\delta + 2H\delta + 2\delta < 4H\delta + 2\delta. \tag{3.7}$$

Also note

$$d(y_0, y_0^*) < 4H\delta + 2\delta. \tag{3.8}$$

Denote $L_2 = 4H + 2$. By (3.6)–(3.8), we conclude that the sequence $\{y_k^*\}_{k=-\infty}^{+\infty}$ $L_2\delta$-shadows the $\delta$-pseudo orbit $\{y_k\}_{k=-\infty}^{+\infty}$. $\square$

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