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Optimal Lagrange Interpolation of a Class of Infinitely Differentiable Functions

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Abstract This paper investigates the optimal Lagrange interpolation of a class F_∞ of infinitely differentiable functions on $[-1,1]$ in $L_\infty[-1,1]$ and weighted spaces $L_{p,\omega}[-1,1]$, $1\le p<\infty$ with ω a continuous integrable weight function in $(-1,1)$. We proved that the Lagrange interpolation polynomials based on the zeros of polynomials with the leading coefficient 1 of the least deviation from zero in $L_{p,\omega}[-1,1]$ are optimal for $1\le p<\infty$. We also give the optimal Lagrange interpolation nodes when the endpoints are included in the nodes.

Keywords worst case setting; optimal Lagrange interpolation; infinitely differentiable function space

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1. Introduction and main results

Let F be a Banach space of functions defined on a compact set D that can be continuously embedded in $C(D)$, BF be the unit ball of F, and $G\,(\supseteqq F)$ be a normed linear space with norm $\|\cdot\|_G$. We want to approximate functions f from F by using a finite number of arbitrary function values $f(t)$ (standard information) for some $t\in D$. We consider only nonadaptive information. For $\mathbf{x}=(\xi_1,\xi_2,\dots,\xi_n)\in D^n$, we use $I_{\mathbf{x}}$ to denote the nonadaptive information operator, i.e.,

$$I_{\mathbf{x}}(f):=(f(\xi_1),f(\xi_2),\dots,f(\xi_n))\in\mathbb{R}^n,\quad f\in F.$$

We say that $A_n=\varphi\circ I_{\mathbf{x}}$ is an algorithm based on the information operator $I_{\mathbf{x}}$, where φ is an arbitrary mapping from $\mathbb{R}^n$ to G. The worst case error of the algorithm A_n for BF in G is defined by

$$e(BF,A_n,G):=\sup_{f\in BF}\|f-A_n(f)\|_G. \tag{1.1}$$

For the construction of algorithms for approximating multivariate functions using function values, the univariate Lagrange interpolation polynomial algorithms play a key role [1–7]. Next we introduce the Lagrange interpolation polynomial algorithms on $[-1,1]$.

Let $\xi_1,\xi_2,\dots,\xi_n$ be n distinct points in $[-1,1]$. Denote $\mathbf{x}=(\xi_1,\xi_2,\dots,\xi_n)$. Then, the Lagrange interpolation polynomial $L_{\mathbf{x}}f$ of a function $f:[-1,1]\to\mathbb{R}$ based on knots $\mathbf{x}=$

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$(\xi_1, \xi_2, \ldots, \xi_n)$ is defined by

$$L_{\mathbf{x}} f \in \mathcal{P}_{n-1} \text{ and } L_{\mathbf{x}} f(\xi_k) = f(\xi_k),\ k = 1, 2, \ldots, n, \tag{1.2}$$

where and in the following $\mathcal{P}_n$ represents the space of all algebraic polynomials of degree at most n. The classical Lagrange interpolation formula gives $L_{\mathbf{x}} f(x) = \sum_{k=1}^{n} f(\xi_k)\ell_k(x)$, where

$$\ell_k(x) = \frac{W_{\mathbf{x}}(x)}{(x-\xi_k)W'_{\mathbf{x}}(\xi_k)},\quad W_{\mathbf{x}}(x) = \prod_{k=1}^{n}(x-\xi_k).$$

Choosing nodes is important for Lagrange interpolation polynomial algorithms. Given a sufficiently smooth function, if nodes are not suitably chosen, then the Lagrange interpolation polynomials do not converge to the function as the number of the nodes tends to infinity. A well-known example is the Runge's phenomenon. Hence the study of optimal Lagrange interpolation nodes becomes a hot research topic, see [8–10] and the references therein. In general, if nodes $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in [-1, 1]^n$ satisfy

$$e(BF, L_{\mathbf{c}}, G) = e(n, BF, G) := \inf_{\mathbf{x}=(\xi_1,\xi_2,\ldots,\xi_n)\in[-1,1]^n} e(BF, L_{\mathbf{x}}, G), \tag{1.3}$$

then we call $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ the nth optimal Lagrange interpolation nodes and $L_{\mathbf{c}}$ the nth optimal Lagrange interpolation algorithm for BF in G. The value $e(BF, L_{\mathbf{c}}, G)$ is called the nth optimal Lagrange interpolation error for BF in G and we denote it as $e(n, BF, G)$, see (1.3).

Let $L_\infty \equiv L_\infty[-1, 1]$ be the space of measurable functions defined on $[-1, 1]$, for which the norm $\|f\|_\infty := \operatorname{ess\,sup}_{x\in[-1,1]}|f(x)|$ is finite. Meanwhile, for $1 \le p < \infty$ and continuous integrable $\omega(x) > 0$ on $(-1, 1)$, let $L_{p,\omega} \equiv L_{p,\omega}[-1, 1]$ be the space of measurable functions defined on $[-1, 1]$, for which the norm $\|f\|_{p,\omega} := (\int_{-1}^{1} |f(x)|^p \omega(x)\mathrm{d}x)^{1/p}$ is finite.

Using $C^r \equiv C^r[-1, 1]$, $r = 0, 1, 2, \ldots$ to denote the spaces of functions with rth order continuous derivative on $[-1, 1]$, respectively. The most important optimal Lagrange interpolation problem is for BC^0 in L_∞ (see [11]). For $n = 3$ and $n = 4$, the results can be found in [12, 13], respectively. For $n \ge 5$, it is still an open problem. For $r \ge 1$, it is well known that the rth optimal Lagrange interpolation nodes are all zeros of the rth Chebyshev polynomial of the first kind ($T_r(x) = \cos(r \arccos x)$) for BC^r in L_∞. Noticed that the approximation of infinitely differentiable multivariate functions has been investigated in [5, 7, 14–19], [20] considered the sampling numbers of the space F_∞ which is defined by

$$F_\infty = \{f \in C^\infty[-1, 1] \,|\, \|f\|_{F_\infty} = \sup_{n\in\mathbb{N}_0} \|f^{(n)}\|_\infty < \infty\}.$$

By [20, Theorem 1.3] we know that the nth optimal Lagrange interpolation nodes are all zeros of the nth Chebyshev polynomial of the first kind for BF_∞ in L_∞. We will give the optimal Lagrange interpolation nodes for BF_∞ in $L_{p,\omega}, 1 \le p < \infty$. First, we set

$$E_{n,p,\omega} := \inf_{g\in\mathcal{P}_{n-1}} \|x^n - g(x)\|_{p,\omega},\quad 1 \le p < \infty, \tag{1.4}$$

where $\mathcal{P}_n$ represents the space of all algebraic polynomials of degree at most n. Furthermore, let $W_{n,p,\omega} \in \mathcal{P}_n$ satisfy

$$W_{n,p,\omega}(x) = x^n + c_1 x^{n-1} + \cdots + c_n \text{ and } \|W_{n,p,\omega}\|_{p,\omega} = E_{n,p,\omega}. \tag{1.5}$$

$W_{n,p,\omega}$ is unique and has exactly n zeros [21, Lemma 2.2]

$$-1 < \xi_{1,p,\omega} < \xi_{2,p,\omega} < \cdots < \xi_{n,p,\omega} < 1. \tag{1.6}$$

Let $L_{n,p,\omega}f$ be the Lagrange interpolation polynomial of a function $f : [-1,1] \to \mathbb{R}$ based on the nodes given by (1.6). Then $L_{n,p,\omega}f$ has the explicit expression

$$L_{n,p,\omega}f(x) = \sum_{k=1}^{n} f(\xi_{k,p,\omega})\ell_{k,p,\omega}(x), \tag{1.7}$$

where

$$\ell_{k,p,\omega}(x) = \frac{W_{n,p,\omega}(x)}{(x-\xi_{k,p,\omega})W'_{n,p,\omega}(\xi_{k,p,\omega})}, \quad k = 1,\ldots,n,$$

and

$$W_{n,p,\omega}(x) = \prod_{k=1}^{n}(x-\xi_{k,p,\omega}). \tag{1.8}$$

First, we obtained the following result.

Theorem 1.1 *Let $1 \le p < \infty$ and assume that $\omega(x) > 0$ is continuous integrable on $(-1,1)$. Then we have*

$$e(n, BF_\infty, L_{p,\omega}) = e(BF_\infty, L_{n,p,\omega}, L_{p,\omega}) = \frac{E_{n,p,\omega}}{n!}, \tag{1.9}$$

where $L_{n,p,\omega}$ and $E_{n,p,\omega}$ are given by (1.7) and (1.4), respectively.

In practice one often wants to have boundary points as interpolation nodes, i.e.,

$$\mathbf{x} = \{-1, \xi_2, \ldots, \xi_{n-1}, 1\}. \tag{1.10}$$

Then the following question arises: for which set of points $-1 < c_2 < c_3 < \cdots < c_{n-1} < 1$, we have

$$e(BF, L_{\mathbf{c}}, G) = \overline{e}(n, BF, G) = \inf_{\mathbf{x}=(-1,\xi_2,\ldots,\xi_{n-1},1)} e(BF, L_{\mathbf{x}}, G). \tag{1.11}$$

Hoang [10] obtained the rth optimal Lagrange interpolation nodes of this problem for BC^r in L_∞. We will consider this problem for BF_∞ in L_∞ and $L_{p,\omega}, 1 \le p < \infty$. We obtain the following results.

Theorem 1.2 *(1) Let $p = \infty$. Then we have*

$$\overline{e}(n, BF_\infty, L_\infty) = e(BF_\infty, L_{\mathbf{c}}, L_\infty) = \frac{1}{(\cos\frac{\pi}{2n})^n 2^{n-1} n!}, \tag{1.12}$$

where

$$\mathbf{c} = (-1, \cos\frac{(2n-3)\pi}{2n} / \cos\frac{\pi}{2n}, \ldots, \cos\frac{3\pi}{2n} / \cos\frac{\pi}{2n}, 1). \tag{1.13}$$

(2) Let $1 \le p < \infty$ and assume that $\omega(x) > 0$ is continuous integrable on $(-1,1)$. Then we have

$$\overline{e}(n, BF_\infty, L_{p,\omega}) = e(BF_\infty, L_{\mathbf{c}}, L_{p,\omega}) = \frac{E_{n-2,p,\overline{\omega}}}{n!}, \tag{1.14}$$

where

$$\overline{\omega}(x) = (1-x^2)^p\omega(x), \mathbf{c} = (-1, \xi_{1,p,\overline{\omega}}, \xi_{2,p,\overline{\omega}}, \ldots, \xi_{n-2,p,\overline{\omega}}, 1), \tag{1.15}$$

and $\xi_{1,p,\varpi}, \xi_{2,p,\varpi}, \ldots, \xi_{n-2,p,\varpi}$ are given by (1.6) with n replaced by $n-2$.

The remainder of this paper is organized as follows. In Section 2, we give proof of Theorems 1.1 and 1.2. In Section 3, we give seven examples to show our results.

2. Proof of Theorems 1.1 and 1.2

To prove Theorem 1.1, we first give a lemma.

Lemma 2.1 ([22]) *Let $f \in C^n$. Then, the remainder $R_{\mathbf{x}}f(x) := f(x) - L_{\mathbf{x}}f(x)$ for the Lagrange interpolation polynomial based on $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [-1,1]^n$ can be represented in the form*

$$R_{\mathbf{x}}f(x) = f(x) - L_{\mathbf{x}}f(x) = \frac{f^{(n)}(\xi)}{n!} W_{\mathbf{x}}(x), \quad x \in [-1,1], \tag{2.1}$$

for some $\xi \in [-1,1]$ depending on x and the knots $x_1, \ldots, x_n$.

Proof of Theorem 1.1 We consider the upper estimate first. Let $\{\xi_{i,p,\omega}\}_{i=1}^n$ and $W_{n,p,\omega}$ be given by (1.6) and (1.5), respectively. Then for $f \in BF_\infty$, we have $\|f^{(n)}\|_\infty \le 1$. Combining this fact with (2.1), we obtain

$$|f(x) - L_{n,p,\omega}f(x)| \le \frac{|W_{n,p,\omega}(x)|}{n!}, \quad x \in [-1,1].$$

It follows that

$$\|f - L_{n,p,\omega}f\|_{p,\omega} \le \frac{\|W_{n,p,\omega}\|_{p,\omega}}{n!} = \frac{E_{n,p,\omega}}{n!}. \tag{2.2}$$

From (1.1) and (2.2) we obtain

$$e(BF_\infty, L_{n,p,\omega}, L_{p,\omega}) \le \frac{E_{n,p,\omega}}{n!}. \tag{2.3}$$

From (1.3) and (2.3) we obtain the upper estimate.

Now we consider the lower estimate. Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are n arbitrary distinct points in $[-1,1]$ and $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n)$. Let $g_0(x) = \frac{x^n}{n!}$. Then $g_0 \in BF_\infty$. From $g_0^{(n)}(x) = 1$ and (2.1) it follows that

$$g_0(x) - L_{\mathbf{x}}g_0(x) = \frac{1}{n!}\prod_{k=1}^n (x - \xi_k) = \frac{g(x)}{n!}, \quad x \in [-1,1], \tag{2.4}$$

where

$$g(x) = \prod_{k=1}^n (x - \xi_k) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n. \tag{2.5}$$

Then, it follows from (1.5) that

$$\|g\|_{p,\omega} \ge E_{n,p,\omega}. \tag{2.6}$$

Hence for any $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n) \in [-1,1]^n$, from (2.4) and (2.6) it follows that

$$e(BF_\infty, L_{\mathbf{x}}, L_{p,\omega}) \ge \|g_0 - L_{\mathbf{x}}g_0\|_{p,\omega} = \frac{\|g\|_{p,\omega}}{n!} \ge \frac{E_{n,p,\omega}}{n!}. \tag{2.7}$$

By (1.3) and (2.7) we obtain the lower estimate. This completes the proof of Theorem 1.1. $\square$

To prove Theorem 1.2, we first introduce a lemma [10, Theorem 2.1].

Lemma 2.2 *Let* $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ *be given by (1.13). Then*

$$\Big\|\prod_{i=1}^{n}(x-c_i)\Big\|_\infty = \inf_{\mathbf{x}=(-1,\xi_2,\ldots,\xi_{n-1},1)}\Big\|\prod_{i=1}^{n}(x-\xi_i)\Big\|_\infty. \tag{2.8}$$

Proof of Theorem 1.2 We consider (1) first. For $f \in BF_\infty$, from (2.1) it follows that

$$|f(x) - L_{\mathbf{c}}f(x)| \le \frac{1}{n!}\Big|\prod_{i=1}^{n}(x - \cos\frac{(2i-1)\pi}{2n}/\cos\frac{\pi}{2n})\Big|, \quad x \in [-1,1]. \tag{2.9}$$

Let $t = x\cos\frac{\pi}{2n}$. Then we have

$$\prod_{i=1}^{n}(x - \cos\frac{(2i-1)\pi}{2n}/\cos\frac{\pi}{2n}) = \frac{T_n(t)}{(\cos\frac{\pi}{2n})^n 2^{n-1}}, \quad t \in [-\cos\frac{\pi}{2n}, \cos\frac{\pi}{2n}]. \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$e(BF_\infty, L_{\mathbf{c}}, L_\infty) \le \frac{1}{(\cos\frac{\pi}{2n})^n 2^{n-1} n!}\sup_{t\in[-\cos\frac{\pi}{2n},\cos\frac{\pi}{2n}]}|T_n(t)| = \frac{1}{(\cos\frac{\pi}{2n})^n 2^{n-1} n!}. \tag{2.11}$$

On the other hand, let $\mathbf{x} = (-1, \xi_2, \ldots, \xi_{n-1}, 1) \in [-1,1]^n$ and $g_0(x) = \frac{x^n}{n!} \in BF_\infty$. Then (2.4) holds. It follows from (2.4), (2.8) and (2.10) that

$$\begin{aligned} e(BF_\infty, L_{\mathbf{x}}, L_\infty) &\ge \|g_0 - L_{\mathbf{x}}g_0\|_\infty = \frac{1}{n!}\Big\|\prod_{k=1}^{n}(x-\xi_k)\Big\|_\infty \\ &\ge \frac{1}{n!}\Big\|\prod_{k=1}^{n}(x - \cos\frac{(2k-1)\pi}{2n}/\cos\frac{\pi}{2n})\Big\|_\infty \\ &= \frac{1}{(\cos\frac{\pi}{2n})^n 2^{n-1} n!}\sup_{t\in[-\cos\frac{\pi}{2n},\cos\frac{\pi}{2n}]}|T_n(t)| = \frac{1}{(\cos\frac{\pi}{2n})^n 2^{n-1} n!}. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) we obtain the result of (1).

Next we consider (2). Let $\overline{\omega}$ and $\mathbf{c}$ be given by (1.15). Then for any $f \in BF_\infty$, from (2.1) it follows that

$$|f(x) - L_{\mathbf{c}}f(x)| \le \frac{(1-x^2)|W_{n-2,p,\overline{\omega}}(x)|}{n!}, \quad x \in [-1,1]. \tag{2.13}$$

From (2.13) it follows that

$$\|f - L_{\mathbf{c}}f\|_{p,\omega} \le \frac{\|W_{n-2,p,\overline{\omega}}\|_{p,\overline{\omega}}}{n!} = \frac{E_{n-2,p,\overline{\omega}}}{n!}. \tag{2.14}$$

From (1.1) and (2.14) we conclude

$$e(BF_\infty, L_{\mathbf{c}}, L_{p,\omega}) \le \frac{E_{n-2,p,\overline{\omega}}}{n!}. \tag{2.15}$$

On the other hand, let $\mathbf{x} = (-1, \xi_2, \ldots, \xi_{n-1}, 1) \in [-1,1]^n$ and $g_0(x) = \frac{x^n}{n!} \in BF_\infty$. Then (2.4) holds. From (2.4), (1.5) and (1.6) it follows that

$$\begin{aligned} e(BF_\infty, L_{\mathbf{x}}, L_{p,\omega}) &\ge \|g_0 - L_{\mathbf{x}}g_0\|_{p,\omega} = \frac{1}{n!}\Big\|\prod_{k=1}^{n}(x-\xi_k)\Big\|_{p,\omega} = \frac{1}{n!}\Big\|\prod_{k=2}^{n-1}(x-\xi_k)\Big\|_{p,\overline{\omega}} \\ &\ge \frac{1}{n!}\Big\|\prod_{k=1}^{n-2}(x-\xi_{k,p,\overline{\omega}})\Big\|_{p,\overline{\omega}} = \frac{E_{n-2,p,\overline{\omega}}}{n!}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we obtain the result of (2). Theorem 1.2 is proved. □

Remark 2.3 From [20, Theorem 1.3] it follows that

$$e(n, BF_\infty, L_\infty) = \frac{1}{2^{n-1}n!}. \tag{2.17}$$

Combining (2.17) with (1.12) and (3.12) gives

$$\lim_{n\to\infty} \frac{\overline{e}(n, BF_\infty, L_p)}{e(n, BF_\infty, L_p)} = 1, \quad p = 2, \infty.$$

This shows that including the boundary points to interpolation nodes does not essentially affect the optimal Lagrange interpolation errors for BF_∞ in L_2 and L_∞.

3. Illustration examples

We will give two examples in the usual L_p spaces and five examples in the weighted L_2 spaces to show our results.

3.1. Two examples

Let $\omega(x) = 1$. Then for $1 \le p < \infty$, we obtain the usual $L_p \equiv L_p[-1, 1]$ spaces.

Example 3.1 For $p = 1$, it follows from [23, pp. 87-88] that

$$E_{n,1,1} = \frac{1}{2^{n-1}}, \; W_{n,1,1}(x) = \frac{U_n(x)}{2^n}, \; \xi_{k,1,1} = \cos\frac{k\pi}{n+1}, \quad k = 1, \ldots, n,$$

where U_n is the nth Chebyshev polynomial of the second kind, i.e., $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$. The nth optimal Lagrange interpolation polynomial $L_{n,1,1}$ for BF_∞ in L_1 is given by

$$L_{n,1,1}f(x) = \sum_{k=1}^{n} f(\xi_{k,1,1})\ell_{k,1,1}(x),$$

where

$$\ell_{k,1,1}(x) = \frac{(-1)^{k+1}(1 - \xi_{k,1,1}^2)U_n(x)}{(n+1)(x - \xi_{k,1,1})}, \quad k = 1, \ldots, n.$$

Furthermore, from Theorem 1.1 it follows that

$$e(n, BF_\infty, L_1) = e(BF_\infty, L_{n,1,1}, L_1) = \frac{E_{n,1,1}}{n!} = \frac{1}{2^{n-1}n!}.$$

Example 3.2 For $p = 2$, we have [22, p. 205]

$$W_{n,2,1}(x) = \frac{2^n(n!)^2}{(2n)!}P_n(x) = \prod_{k=1}^{n}(x - \xi_{k,2,1}), \tag{3.1}$$

where P_n is the nth Legendre polynomial, i.e., $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$. The nth optimal Lagrange interpolation polynomial $L_{n,2,1}$ for BF_∞ in L_2 is given by

$$L_{n,2,1}f(x) = \sum_{k=1}^{n} f(\xi_{k,2,1})\ell_{k,2,1}(x),$$

where

$$\ell_{k,2,1}(x) = \frac{P_n(x)}{(x - \xi_{k,2,1})P_n'(\xi_{k,2,1})}, \quad k = 1, \ldots, n.$$

From [24, p. 57, (4.6.6)] it follows that

$$\|P_n\|_2 = \frac{2^{1/2}}{\sqrt{2n+1}}. \tag{3.2}$$

By Theorem 1.1, (3.1) and (3.2), we conclude that

$$e(n, BF_\infty, L_2) = e(BF_\infty, L_{n,2,1}, L_2) = \frac{E_{n,2,1}}{n!} = \frac{2^{n+1/2} n!}{(2n)!\sqrt{2n+1}}. \tag{3.3}$$

3.2. Five examples

Let $p = 2$. In this case, for any continuous integrable weight function $\omega(x) > 0$ on $(-1, 1)$, there is a unique orthogonal system $\{p_{k,\omega}\}_{k\in\mathbb{Z}_+}$ in $L_{2,\omega}$ which is complete and satisfies the following conditions:

(1) $p_{k,\omega} \in \mathcal{P}_k$ for all $k \in \mathbb{Z}_+$.

(2)

$$\int_{-1}^{1} p_{k,\omega}(x) p_{j,\omega}(x) \omega(x) \mathrm{d}x = \begin{cases} 0, & k \neq j; \\ 1, & k = j. \end{cases} \tag{3.4}$$

(3) The coefficient $C_{k,\omega}$ of the leading term x^k of $p_{k,\omega}$ is positive.

Next we give the relation between $W_{k,2,\omega}$ and $p_{k,\omega}$. For any polynomial

$$p(x) = x^k + a_1 x^{k-1} + \cdots + a_k,$$

we have

$$p(x) = \sum_{j=0}^{k} c_j p_{j,\omega}(x). \tag{3.5}$$

Comparing the coefficients of the leading term x^k in both sides of (3.5), we obtain $c_k = 1/C_{k,\omega}$. Furthermore, from (3.5) and (3.4) it follows that

$$\|p\|_{2,\omega}^2 = c_0^2 + c_1^2 + \cdots + c_{k-1}^2 + 1/C_{k,\omega}^2. \tag{3.6}$$

From (3.6) it follows that

$$\|W_{k,2,\omega}\|_{2,\omega}^2 = \min_{p \text{ with form } (3.5)} \|p\|_{2,\omega}^2 = \min_{c_0, c_1, \dots, c_{k-1} \in \mathbb{R}} (c_0^2 + c_1^2 + \cdots + c_{k-1}^2 + 1/C_{k,\omega}^2)$$

holds if and only if $c_0 = c_1 = \cdots = c_{k-1} = 0$, and $W_{k,2,\omega} = \frac{p_{k,\omega}}{C_{k,\omega}}$. This means

$$E_{k,2,\omega} = \|W_{k,2,\omega}\|_{2,\omega} = 1/C_{k,\omega}. \tag{3.7}$$

Now we let $\omega^{(\alpha,\beta)}$ be the Jacobi weights, i.e., $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha, \beta > -1$ and we denote the corresponding orthogonal system as $\{p_k^{(\alpha,\beta)}\}_{k\in\mathbb{Z}_+}$. It is known that the Jacobi polynomials are given by [24, p. 143]

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+\beta+k+1)} \sum_{j=0}^{k} C_k^j \frac{\Gamma(\alpha+\beta+k+j+1)}{\Gamma(\alpha+j+1)} (\frac{x-1}{2})^j. \tag{3.8}$$

From [24, p. 141] it follows that

$$h_k^{(\alpha,\beta)} = \int_{-1}^{1} (P_k^{(\alpha,\beta)}(x))^2 \omega^{(\alpha,\beta)}(x) \mathrm{d}x = \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2k+1} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{k!\Gamma(\alpha+\beta+k+1)}. \tag{3.9}$$

From (3.8) and $\Gamma(s+1)=s\Gamma(s)$, $s>0$ we conclude that the coefficient of the leading term x^k of $P_k^{(\alpha,\beta)}(x)$ is $\frac{\Gamma(\alpha+\beta+2k+1)}{2^k k!\Gamma(\alpha+\beta+k+1)}$. Combining this fact with (3.9), we obtain that the coefficient of the leading term x^k of $p_k^{(\alpha,\beta)}(x)=P_k^{(\alpha,\beta)}(x)/\sqrt{h_k^{(\alpha,\beta)}}$ is

$$C_{k,\omega^{(\alpha,\beta)}}=\frac{\sqrt{\alpha+\beta+2k+1}\Gamma(\alpha+\beta+2k+1)}{2^{k+(\alpha+\beta+1)/2}\sqrt{k!\Gamma(\alpha+\beta+k+1)\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}}. \tag{3.10}$$

From Theorem 1.1, (3.7) and (3.10), it follows that

$$\begin{aligned} e(n,BF_\infty,L_{2,\omega^{(\alpha,\beta)}})&=e(BF_\infty,L_{n,2,\omega^{(\alpha,\beta)}},L_{2,\omega^{(\alpha,\beta)}})=\frac{1}{n!C_{n,\omega^{(\alpha,\beta)}}}\\ &=\frac{2^{n+(\alpha+\beta+1)/2}\sqrt{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}{\sqrt{n!(\alpha+\beta+2n+1)}\Gamma(\alpha+\beta+2n+1)}. \end{aligned} \tag{3.11}$$

From Theorem 1.2, (3.7), (3.10) and (3.11), it follows that for $n>2$

$$\begin{aligned} \overline{e}(n,BF_\infty,L_{2,\omega^{(\alpha,\beta)}})&=e(BF_\infty,L_{\mathbf{c}},L_{2,\omega^{(\alpha,\beta)}})=\frac{E_{n-2,2,\omega^{(\alpha+2,\beta+2)}}}{n!}=\frac{1}{n!C_{n-2,\omega^{(\alpha+2,\beta+2)}}}\\ &=\frac{2^{n+(\alpha+\beta+1)/2}\sqrt{\Gamma(\alpha+\beta+n+3)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}{n(n-1)\sqrt{(n-2)!(\alpha+\beta+2n+1)}\Gamma(\alpha+\beta+2n+1)}\\ &=\sqrt{\frac{(\alpha+\beta+n+2)(\alpha+\beta+n+1)}{n(n-1)}}e(n,BF_\infty,L_{2,\omega^{(\alpha,\beta)}}), \end{aligned} \tag{3.12}$$

where

$$\mathbf{c}=(-1,\xi_{1,2,\omega^{(\alpha+2,\beta+2)}},\xi_{2,2,\omega^{(\alpha+2,\beta+2)}},\ldots,\xi_{n-2,2,\omega^{(\alpha+2,\beta+2)}},1),$$

and $\xi_{1,2,\omega^{(\alpha+2,\beta+2)}},\xi_{2,2,\omega^{(\alpha+2,\beta+2)}},\ldots,\xi_{n-2,2,\omega^{(\alpha+2,\beta+2)}}$ are given by (1.6) with n replaced by $n-2$.

Next we list five examples.

Example 3.3 For $\alpha=\beta=-1/2$, i.e., $\omega^{(-1/2,-1/2)}(x)=\frac{1}{\sqrt{1-x^2}}$, we know $W_{n,2,\omega^{(-1/2,-1/2)}}(x)=\frac{T_n(x)}{2^{n-1}}$. By a direct computation we obtain

$$E_{n,2,\omega^{(-1/2,-1/2)}}=\|W_{n,2,\omega^{(-1/2,-1/2)}}\|_{2,\omega^{(-1/2,-1/2)}}=\frac{\sqrt{2\pi}}{2^n}.$$

Hence from Theorem 1.1 we conclude that

$$e(n,BF_\infty,L_{2,\omega^{(-1/2,-1/2)}})=e(BF_\infty,L_{\mathbf{x}_{n,\infty}},L_{2,\omega^{(-1/2,-1/2)}})=\frac{E_{n,2,\omega^{(-1/2,-1/2)}}}{n!}=\frac{\sqrt{2\pi}}{2^n n!}, \tag{3.13}$$

where $\mathbf{x}_{n,\infty}=(\cos\frac{(2n-1)\pi}{2n},\cos\frac{(2n-3)\pi}{2n},\ldots,\cos\frac{\pi}{2n})$.

From (3.12) and (3.13) it follows that for $n>2$

$$\overline{e}(n,BF_\infty,L_{2,\omega^{(-1/2,-1/2)}})=\frac{\sqrt{2\pi(n+1)(n-1)}}{2^n(n-1)n!}.$$

Example 3.4 For $\alpha=\beta=1/2$, i.e., $\omega^{(1/2,1/2)}(x)=\sqrt{1-x^2}$, we know $W_{n,2,\omega^{(1/2,1/2)}}(x)=\frac{U_n(x)}{2^n}$. By a direct computation we obtain $E_{n,2,\omega^{(1/2,1/2)}}=\|W_{n,2,\omega^{(1/2,1/2)}}\|_{2,\omega^{(1/2,1/2)}}=\frac{\sqrt{\pi}}{2^{n+1/2}}$. Hence from Theorem 1.1 we conclude that

$$e(n,BF_\infty,L_{2,\omega^{(1/2,1/2)}})=e(BF_\infty,L_{n,1,1},L_{2,\omega^{(1/2,1/2)}})=\frac{E_{n,2,\omega^{(1/2,1/2)}}}{n!}=\frac{\sqrt{\pi}}{2^{n+1/2}n!}. \tag{3.14}$$

From (3.12) and (3.14) it follows that for $n > 2$

$$\overline{e}(n, BF_\infty, L_{2,\omega^{(1/2,1/2)}}) = \frac{\sqrt{\pi(n+3)(n+2)n(n-1)}}{2^{n+1/2}n!n(n-1)}.$$

Example 3.5 For $\alpha = -1/2, \beta = 1/2$, i.e., $\omega^{(-1/2,1/2)}(x) = \sqrt{\frac{1+x}{1-x}}$, we know $W_{n,2,\omega^{(-1/2,1/2)}}(x) = \frac{V_n(x)}{2^n}$, where V_n is the nth Chebyshev polynomial of the third kind, i.e.,

$$V_n(x) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \quad x = \cos\theta.$$

In this case, $\xi_{k,2,\omega^{(-1/2,1/2)}} = \cos\frac{(2k-1)\pi}{2n+1}$, $k = 1, \ldots, n$. By a direct computation we obtain

$$E_{n,2,\omega^{(-1/2,1/2)}} = \|W_{n,2,\omega^{(-1/2,1/2)}}\|_{2,\omega^{(-1/2,1/2)}} = \frac{\sqrt{\pi}}{2^n}.$$

Hence from Theorem 1.1 we conclude that

$$\begin{aligned} e(n, BF_\infty, L_{2,\omega^{(-1/2,1/2)}}) &= e(BF_\infty, L_{n,2,\omega^{(-1/2,1/2)}}, L_{2,\omega^{(-1/2,1/2)}}) \\ &= \frac{E_{n,2,\omega^{(-1/2,1/2)}}}{n!} = \frac{\sqrt{\pi}}{2^n n!}. \end{aligned} \tag{3.15}$$

From (3.12) and (3.15) it follows that for $n > 2$

$$\overline{e}(n, BF_\infty, L_{2,\omega^{(-1/2,1/2)}}) = \frac{\sqrt{\pi(n+2)(n+1)n(n-1)}}{2^n n!n(n-1)}.$$

Example 3.6 For $\alpha = 1/2, \beta = -1/2$, i.e., $\omega^{(1/2,-1/2)}(x) = \sqrt{\frac{1-x}{1+x}}$, we know $W_{n,2,\omega^{(1/2,-1/2)}}(x) = \frac{W_n(x)}{2^n}$, where W_n is the nth Chebyshev polynomial of the fourth kind, i.e.,

$$W_n(x) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}, \quad x = \cos\theta.$$

In this case, $\xi_{k,2,\omega^{(1/2,-1/2)}} = \cos\frac{2k\pi}{2n+1}$, $k = 1, \ldots, n$. By a direct computation we obtain

$$E_{n,2,\omega^{(1/2,-1/2)}} = \|W_{n,2,\omega^{(1/2,-1/2)}}\|_{2,\omega^{(1/2,-1/2)}} = \frac{\sqrt{\pi}}{2^n}.$$

Hence from Theorem 1.1 we conclude that

$$\begin{aligned} e(n, BF_\infty, L_{2,\omega^{(1/2,-1/2)}}) &= e(BF_\infty, L_{n,2,\omega^{(1/2,-1/2)}}, L_{2,\omega^{(1/2,-1/2)}}) \\ &= \frac{E_{n,2,\omega^{(1/2,-1/2)}}}{n!} = \frac{\sqrt{\pi}}{2^n n!}. \end{aligned} \tag{3.16}$$

From (3.12) and (3.16) it follows that for $n > 2$

$$\overline{e}(n, BF_\infty, L_{2,\omega^{(1/2,-1/2)}}) = \frac{\sqrt{\pi(n+2)(n+1)n(n-1)}}{2^n n!n(n-1)}.$$

Example 3.7 For $\alpha = \beta = 0$, i.e., $\omega^{(0,0)}(x) = 1$, it is known that $W_{n,2,1}$ is given by (3.1). In this case, from (3.12) and (3.3) it follows that for $n > 2$

$$\overline{e}(n, BF_\infty, L_2) = \frac{2^{n+1/2}(n+2)!}{(2n)!\sqrt{(n-1)n(n+1)(n+2)(2n+1)}}. \tag{3.17}$$

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