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Large Deviations for a Test of Symmetry Based on Kernel Density Estimator of Directional Data

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Abstract Assume that f_n is the nonparametric kernel density estimator of directional data based on a kernel function K and a sequence of independent and identically distributed random variables taking values in d-dimensional unit sphere $\mathbb{S}^{d-1}$. We established that the large deviation principle for $\{\sup_{x\in\mathbb{S}^{d-1}}|f_n(x)-f_n(-x)|, n\geq 1\}$ holds if the kernel function is a function with bounded variation, and the density function f of the random variables is continuous and symmetric.

Keywords symmetry test; kernel density estimator; directional data; large deviations

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1. Introduction

Suppose that $\{X_i, i\geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random vectors taking values on a d-dimensional unit sphere $\mathbb{S}^{d-1}:=\{x\in\mathbb{R}^d, |x|=1\}$, $d\geq 2$ with density function f such that

$$\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\Theta(x)=1,$$

where Θ is the Lebesgue measure on $\mathbb{S}^{d-1}$, i.e., $\{X_i\}$ is a set of directional data. Such data occurs in many fields, such as geology and medicine [1].

Bai et al. [2] obtained uniform strong consistency and L_1-consistency of the following kernel estimator

$$f_n(x):=(nr^{d-1}(n))^{-1}C(r(n))\sum_{i=1}^{n}K(\frac{1-x'X_i}{r^2(n)}),\quad x\in\mathbb{S}^{d-1}, \tag{1.1}$$

where $\{r(n), n\geq 1\}$ is a bandsequence, which is a sequence of positive numbers fulfilling

$$r(n)\to 0, nr^{d-1}(n)\to+\infty, \text{as } n\to\infty. \tag{1.2}$$

Let K be a non-negative function defined on $\mathbb{R}$ such that $\forall x<0$, $K(x)=0$, and

$$0<\int_0^\infty K(v)v^{(d-3)/2}\mathrm{d}v<\infty, \tag{1.3}$$

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and $C(r(n))$ is a positive number satisfying

$$(C(r(n)))^{-1}=\frac{1}{r^{d-1}(n)}\int_{\mathbb{S}^{d-1}}K(\frac{1-x'y}{r^2(n)})\Theta(\mathrm{d}y). \tag{1.4}$$

Notice that [3, 4]

$$\int_{\mathbb{S}^{d-1}}g(a'x)\Theta(\mathrm{d}x)=\frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}\int_{-1}^{1}g(|a|v)(1-v^2)^{(d-3)/2}\mathrm{d}v, \tag{1.5}$$

where g is a nonnegative measurable function and $a\in\mathbb{R}^d\backslash\{0\}$. $C(r(n))$ is independent of x, and

$$M(K):=\lim_{n\to\infty}(C(r(n)))^{-1}=\frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)}\int_0^{\infty}K(v)v^{(d-3)/2}\mathrm{d}v, \tag{1.6}$$

Zhao and Wu [5] proved a central limit theorem for integrated square error of f_n under some mild conditions. Gao and Li [4] obtained large deviations and moderate deviations for kernel density estimators of directional data. Li and Gao [6] studied rates of strong uniform consistency for kernel density estimators of directional data. He and Gao [7] proved moderate deviations and large deviations for a test of symmetry based on kernel density estimator. Xu and Zhou [8, 9] extended the results of He and Gao [7] to the multidimensional case. Li [10] established moderate deviations for a test of symmetry based on kernel density estimator of directional data. Inspired by the above-mentioned results, in this paper, we try to establish uniform large deviations for a test of symmetry based on kernel density estimator of directional data by the empirical process approach [11–13], which is motivated by Xu and Zhou [8, 9], Xu et al. [14], He and Gao [7], Gao [15], and Gao and Li [4]. Our results complement that in Xu and Zhou [8, 9], Xu et al. [14], He and Gao [7], Gao [15], Gao and Li [4], Li [10], Xu et al. [14].

We suppose that f and K fulfill the following conditions:

(A1) f is continuous and symmetric.

(A2) K is a bounded variations function satisfying (1.3), and $\exists\,\gamma>\frac{d-1}{4}$ such that

$$\lim_{z\to+\infty}z^{\gamma}K(z)<\infty. \tag{1.7}$$

We denote by $\|g\|=\sup_x|g(x)|$ the supremum norm.

As in [4, Remark 1.1], if K is bounded, then by (1.3), $\forall\,t\in\mathbb{R}$,

$$\varphi(t):=\int_0^{\infty}z^{(d-3)/2}\exp(tK(z)-1)\mathrm{d}z<\infty.$$

The following theorem is the main result in this paper.

Theorem 1.1 *Let* $\{r(n),n\ge1\}$ *satisfy*

$$r(n)\to0,nr^{d-1}(n)\to+\infty,\frac{\log(r(n))^{-1}}{nr^{d-1}(n)}\to0, as\ n\to+\infty. \tag{1.8}$$

Suppose that (A1) and (A2) hold. Then $\forall\lambda>0$,

$$\lim_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}=-J(\lambda), \tag{1.9}$$

where

$$J(\lambda) = \inf\Big\{ \inf_{x\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}}\{t\lambda - f(x)(\psi(t)+\psi(-t))\}, \inf_{x\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}}\{-t\lambda - f(x)(\psi(t)+\psi(-t))\}\Big\}$$
$$= \inf_{x\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}}\{t\lambda - f(x)(\psi(t)+\psi(-t))\}, \tag{1.10}$$

and

$$\psi(t) := \frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^\infty (\exp\{\frac{tK(v)}{M(K)}\}-1)v^{\frac{d-3}{2}}\mathrm{d}v, \quad t\in\mathbb{R}. \tag{1.11}$$

2. Large deviations

Similar to that of Gao and Li [4], in this section we present a pointwise large deviations (Proposition 2.3) and establish Theorem 1.1. We conclude Theorem 1.1 by the pointwise large deviation and a comparison lemma (Lemma 2.5).

We first cite Lemma 2.1 of Gao and Li [4].

Lemma 2.1 *Suppose that $\{r(n), n\geq 1\}$ satisfies (1.2). Assume that K is a bounded function satisfying (1.3) and f is continuous. Then*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{S}^{d-1}} \big|E(\frac{1}{r^{d-1}(n)}K(\frac{1-x'X_1}{r^2(n)})) - f(x)M(K)\big| = 0. \tag{2.1}$$

Especially,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{S}^{d-1}} \big|E(\frac{1}{r^{d-1}(n)}K^2(\frac{1-x'X_1}{r^2(n)})) - f(x)M(K^2)\big| = 0, \tag{2.2}$$

and

$$\lim_{n\to\infty}\sup_{x\in\mathbb{S}^{d-1}} \Big|\int_{\mathbb{S}^{d-1}} (\exp\{tC(r(n))K(\frac{1-x'X_1}{r^2(n)})\}-1)f(y)\Theta(\mathrm{d}y) - f(x)\psi(t)\Big| = 0. \tag{2.3}$$

Lemma 2.2 *Let $\{r(n), n\geq 1\}$ satisfy (1.2). Suppose that (A1) and (A2) hold. Write*

$$\Psi_x^{(n)}(t) := E\{\exp\{tnr^{d-1}(n)(f_n(x)-f_n(-x))\}\}. \tag{2.4}$$

Then

$$\lim_{n\to\infty}\sup_{x\in\mathbb{S}^{d-1}} |\frac{1}{nr^{d-1}(n)}\log\Psi_x^{(n)}(t) - f(x)(\psi(t)+\psi(-t))| = 0. \tag{2.5}$$

Proof From the fact that $\{X_i, i\geq 1\}$ are i.i.d., we could deduce that

$$\Psi_x^{(n)}(t) = E\Big(\exp\Big\{tC(r(n))\sum_{i=1}^n (K(\frac{1-x'X_i}{r^2(n)}) - K(\frac{1+x'X_i}{r^2(n)}))\Big\}\Big)$$
$$= (E(\exp\{tC(r(n))(K(\frac{1-x'X_1}{r^2(n)}) - K(\frac{1+x'X_1}{r^2(n)}))\}))^n$$
$$= \Big(\int_{\mathbb{S}^{d-1}} \exp\{tC(r(n))(K(\frac{1-x'y}{r^2(n)}) - K(\frac{1+x'y}{r^2(n)}))\}f(y)\Theta(\mathrm{d}y)\Big)^n.$$

First, we suppose that K has a bounded support. Because $r(n)\to 0$, as $n\to\infty$, the support of $K(\frac{1-x'y}{r^2(n)})$ and $K(\frac{1+x'y}{r^2(n)})$ have an empty intersection for n large sufficiently. Therefore,

$$\frac{1}{nr^{d-1}(n)}\log\Psi_x^{(n)}(t)$$

$$= \frac{1}{r^{d-1}(n)} \log\Big(1 + \int_{\mathbb{S}^{d-1}} (\exp\{tC(r(n))K(\frac{1-x'y}{r^2(n)})\} - 1 + \exp\{-tC(r(n))K(\frac{1+x'y}{r^2(n)})\} - 1) f(y)\Theta(\mathrm{d}y)\Big),$$

and so from (2.3), we conclude (2.5). Now, we drop the assumption of bounded support,

$$\Psi_x^{(n)}(t) = \Big[1 + \int_{\mathbb{S}^{d-1}} (\exp\{tC(r(n))K(\frac{1-x'y}{r^2(n)})\} - 1) f(y)\Theta(\mathrm{d}y) + \int_{\mathbb{S}^{d-1}} (\exp\{-tC(r(n))K(\frac{1+x'y}{r^2(n)})\} - 1) f(y)\Theta(\mathrm{d}y) + r^{d-1}(n)\alpha\Big]^n$$
$$= \Big[1 + \int_{\mathbb{S}^{d-1}} (\exp\{tC(r(n))K(\frac{1-x'y}{r^2(n)})\} + \exp\{-tC(r(n))K(\frac{1-x'y}{r^2(n)})\} - 2) f(y)\Theta(\mathrm{d}y) + r^{d-1}(n)\alpha\Big]^n,$$

where

$$\alpha = \Big[\int_{\mathbb{S}^{d-1}} \exp\{tC(r(n))(K(\frac{1-x'y}{r^2(n)}) - K(\frac{1+x'y}{r^2(n)}))\} f(y)\Theta(\mathrm{d}y) - 1 - \int_{\mathbb{S}^{d-1}} (\exp\{tC(r(n))K(\frac{1-x'y}{r^2(n)})\} - 1) f(y)\Theta(\mathrm{d}y) - \int_{\mathbb{S}^{d-1}} (\exp\{-tC(r(n))K(\frac{1+x'y}{r^2(n)})\} - 1) f(y)\Theta(\mathrm{d}y)\Big] \Big/ r^{d-1}(n).$$

By the assumptions of the theorem, for any $\varepsilon > 0$, for n large sufficiently, $|\alpha| \le M\varepsilon(2\exp\{tCK_0\} + 4)$, where $M = \|f\|$, $K_0 = \sup_z K(z)$, C is some constant. Hence $\alpha = o(1)$ as $n \to \infty$, it is uniform with respect to x and t. Therefore, we have

$$\Psi_x(t) := \lim_{n\to\infty} \frac{1}{nr^{d-1}(n)} \log \Psi_x^{(n)}(t) = f(x)(\psi(t) + \psi(-t)). \quad \square$$

As $\Psi_x(t)$ is differentiable with respect to $t \in \mathbb{R}$, therefore, by Gärtner-Ellis theorem, we obtian the following pointwise large deviation.

Proposition 2.3 *Suppose $\{r(n), n \ge 1\}$ satisfies (1.2). Let (A1) and (A2) hold. Then $\forall x \in \mathbb{S}^{d-1}$, for every closed set $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{nr^{d-1}(n)} \log P\{(f_n(x) - f_n(-x)) \in F\} \le -\inf_{\lambda\in F} J_x(\lambda), \tag{2.6}$$

and for every open set $G \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{nr^{d-1}(n)} \log P\{(f_n(x) - f_n(-x)) \in G\} \ge -\inf_{\lambda\in G} J_x(\lambda), \tag{2.7}$$

where

$$J_x(\lambda) = \sup_{t\in\mathbb{R}} \{t\lambda - f(x)(\psi(t) + \psi(-t))\}.$$

Suppose that (S, φ) is a measurable space and $\mathscr{F}$ is a uniformly bounded collection of measurable functions on it. We call $\mathscr{F}$ a bounded measurable VC (Vapnik-Červonenkis) class of functions if $\mathscr{F}$ is separable and if $\exists A > 0$, $v > 0$ such that, for any probability μ on (S, φ) and any $0 < \tau < 1$,

$$N(\mathscr{F}, \|\cdot\|_{L_2(\mu)}, \tau\|F\|_{L_2(\mu)}) \leq (\frac{A}{\tau})^v,$$

where $F = \sup\{|g|, g \in \mathscr{F}\}$ and we denote the τ-covering number of the metric space ($\mathscr{F}$, $\|\cdot\|_{L_2(\mu)}$) by $N(\mathscr{F}, \|\cdot\|_{L_2(\mu)}, \tau)$, which is the smallest number of balls of radius not larger than τ and centers in $\mathscr{F}$ needed to cover $\mathscr{F}$. We call (A, v) the characteristic of the class $\mathscr{F}$. Let K be a bounded variation function. Then by [16, Lemma 22],

$$\mathscr{F} = \{y \mapsto K(\frac{1 - x'y}{r^2}) - K(\frac{1 + x'y}{r^2}); x \in \mathbb{S}^{d-1}, r \in \mathbb{R}_+\}$$

is a bounded measurable VC class of functions.

The following deviation inequality of the VC class plays a crucial role in the proof of Theorem 1.1. As in [4], we give and establish a comparison lemma between the pointwise large deviations and the uniform large deviations by the deviation inequality.

Theorem 2.4 ([13]) *Suppose $\mathscr{F}$ is a uniformly bounded measurable VC class of functions, and σ^2 and U are any numbers satisfying $\sigma^2 \geq \sup_{g\in\mathscr{F}} E((g(X_1) - Eg(X_1))^2)$, $U \geq \sup_{g\in\mathscr{F}} \|g\|$ and $0 < \sigma \leq U/2$. Then $\exists C$, L depending only on the characteristic (A, v) of the class $\mathscr{F}$, such that the inequality*

$$P\Big(\Big\|\sum_{i=1}^n (g(X_i) - Eg(X_i))\Big\|_{\mathscr{F}} > t\Big)$$
$$\leq L\exp\{-\frac{t}{LU}\log(1 + \frac{tU}{L(\sqrt{n}\sigma + U\sqrt{\log\frac{U}{\sigma}})^2})\}$$

is valid for all

$$t \geq C(U\log\frac{U}{\sigma} + \sqrt{n}\sigma\sqrt{\log\frac{U}{\sigma}}),$$

where for any map Φ from $\mathscr{F}$ to $\mathbb{R}$, denote $\|\Phi\|_{\mathscr{F}} = \sup\{|\Phi(g)|; g \in \mathscr{F}\}$.

We below cite Lemmas 2.3 and 2.5 in [4].

Lemma 2.5 *Write*

$$S_\gamma(x) = \{y \in \mathbb{S}^{d-1}; |x - y| \leq \gamma\}, \quad x \in \mathbb{S}^{d-1}, \ \gamma > 0.$$

Then for any $n \geq 1$, $\exists l_n \leq B(\delta)r^{2(1-d)}(n)$, $x_1, \ldots, x_{l_n} \in \mathbb{S}^{d-1}$, such that

$$\mathbb{S}^{d-1} = \cup_{i=1}^{l_n} S_{\delta r^2(n)}(x_i),$$

where $B(\delta)$ is a constant independent of n.

Lemma 2.6 *Suppose (A2) holds. Then*

$$\lim_{\delta\to 0}\int_0^\infty \sup_{|x|<\delta} |K(z) - K(z + x)|^2 z^{\frac{d-3}{2}} \mathrm{d}z = 0.$$

Lemma 2.7 *Suppose that $\{r(n), n \geq 1\}$ satisfies (1.8) and (A1) and (A2) hold, for all $0 < \delta < 1$, let $B_{n,k}$, $k = 1, \ldots, l_n$, be l_n balls with $|x - y| \leq \delta r^2(n)$, $x, y \in B_{n,k}$, such that $\{B_{n,k}, k =$*

$1,\dots,l_n\}$ is a covering of $\mathbb{S}^{d-1}$ and $l_n\le B(\delta)r^{2(1-d)}(n)$. Take $z_{n,k}\in B_{n,k}$, $k=1,\dots,l_n$, $n\ge 1$. Then $\forall\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\sup_{1\le k\le l_n}\sup_{x\in B_{n,k}}|f_{n,k}(x)-f_{n,k}(-x)|\ge\varepsilon\}=-\infty, \tag{2.8}$$

where $f_{n,k}(x)=f_n(x)-f_n(z_{n,k})$, $f_{n,k}(-x)=f_n(-x)-f_n(-z_{n,k})$.

Proof Since $\mathscr{F}=\{y\mapsto K(\frac{1-x'y}{r^2});x\in\mathbb{S}^{d-1},r\in\mathbb{R}_+\}$ is a bounded measurable VC class of functions,

$$\begin{aligned}\mathscr{F}_{n,k}=&\{K(\frac{1-x'\cdot}{r^2})-K(\frac{1-z'_{n,k}\cdot}{r^2})-(K(\frac{1+x'\cdot}{r^2})-K(\frac{1+z'_{n,k}\cdot}{r^2}));\\ &x\in\mathbb{S}^{d-1},r\in\mathbb{R}_+\},\quad k=1,\dots,l_n;\ n\ge 1\end{aligned}$$

are measurable VC classes of functions. Moreover, there is a common VC characteristic (A,v) that does not depend on k and n. Because for any $x\in B_{n,k}$,

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}}(K(\frac{1-x'y}{r^2(n)})-K(\frac{1-z'_{n,k}y}{r^2(n)})-K(\frac{1+x'y}{r^2(n)})+K(\frac{1+z'_{n,k}y}{r^2(n)}))^2f(y)\Theta(\mathrm{d}y)\\ &\le 2\|f\|\int_{\mathbb{S}^{d-1}}(K(\frac{1-x'y}{r^2(n)})-K(\frac{1-x'y}{r^2(n)}-\frac{(z_{n,k}-x)'y}{r^2(n)}))^2+\\ &\quad(K(\frac{1+x'y}{r^2(n)})-K(\frac{1+x'y}{r^2(n)}+\frac{(z_{n,k}-x)'y}{r^2(n)}))^2\mathrm{d}y\\ &\le 2\|f\|\frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}r^{d-1}(n)\int_0^\infty\sup_{|y|<\delta}2|K(z)-K(z+y)|^2z^{\frac{d-3}{2}}\mathrm{d}z,\end{aligned}$$

by Lemma 2.6, for any $\eta\in(0,\varepsilon)$, there exists $\delta_0>0$ satisfying for any $\delta\le\delta_0$ and any $x\in B_{n,k}$,

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}}(K(\frac{1-x'y}{r^2(n)})-K(\frac{1-z'_{n,k}y}{r^2(n)})-K(\frac{1+x'y}{r^2(n)})+K(\frac{1+z'_{n,k}y}{r^2(n)}))^2f(y)\Theta(\mathrm{d}y)\\ &\le 4\|f\|\eta r^{d-1}(n).\end{aligned}$$

Take $U=4\|K\|$ and $\sigma^2=4\|f\|\eta r^{d-1}(n)$. By (A1) and (1.5), we see that $E(f_{n,k}(x)-f_{n,k}(-x))=0$. Then by Lemma 2.5, we deduce that for any n large sufficiently,

$$\begin{aligned}&P\{\sup_{1\le k\le l_n}\sup_{x\in B_{n,k}}|f_{n,k}(x)-f_{n,k}(-x)|\ge\varepsilon\}\\ &\quad\le Ll_n\exp\{-\frac{nr^{d-1}(n)C(r(n))^{-1}\varepsilon}{4L\|K\|}\log(1+\frac{C(r(n))^{-1}\|K\|\varepsilon}{4L\|f\|\eta})\}.\end{aligned}$$

Hence, by (1.8)

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\sup_{1\le k\le l_n}\sup_{x\in B_{n,k}}|f_{n,k}(x)-f_{n,k}(-x)|\ge\varepsilon\}\\ &\quad\le-\frac{nr^{d-1}(n)M(K)\varepsilon}{4L\|K\|}\log(1+\frac{M(K)\|K\|\varepsilon}{4L\|f\|\eta}),\end{aligned}$$

which concludes (2.8) by letting $\eta\to 0$. □

Lemma 2.8 *Suppose (A1) and (A2) hold. Then* $\forall\lambda \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}} \inf_{x\in\mathbb{S}^{d-1}} \{t\lambda - \Psi_x(t)\} = \inf_{x\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}} \{t\lambda - \Psi_x(t)\}. \tag{2.9}$$

Proof Denote $M = \|f\|$ and write

$$g(t,y) = t\lambda - y(\psi(t) + \psi(-t)).$$

Then, for t fixed, $g(t,y)$ is convex as a function of y, and for y fixed, $g(t,y)$ is concave as a function of t. By the minimax theorem [17] we see that

$$\inf_{y\in[0,M]} \sup_{t\in\mathbb{R}} g(t,y) = \sup_{t\in\mathbb{R}} \inf_{y\in[0,M]} g(t,y),$$

which concludes (2.9). □

Lemma 2.9 *Suppose (A1) and (A2) hold. Write*

$$h(t) = \frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^\infty \frac{K(v)}{M(K)} (\exp\{\frac{tK(v)}{M(K)}\} - \exp\{\frac{-tK(v)}{M(K)}\}) v^{\frac{d-3}{2}} \mathrm{d}v$$

Then $\forall\lambda \in \mathbb{R}$,

$$\begin{aligned}\tilde{J}(\lambda) :=& \inf_{x\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}} \{t\lambda - f(x)(\psi(t) + \psi(-t))\} \\ =&\lambda h^{-1}(\lambda/M) - M \frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^\infty (\exp\{\frac{h^{-1}(\lambda/M)K(v)}{M(K)}\} + \\ &\exp\{\frac{-h^{-1}(\lambda/M)K(v)}{M(K)}\} - 2) v^{\frac{d-3}{2}} \mathrm{d}v,\end{aligned}$$

where $M = \|f\|$ *and* h^{-1} *denotes the inverse of* h. *Particularly,* J *is continuous on* $[0,\infty)$.

Proof Obiviouly, h is strictly increasing on $(-\infty,\infty)$ and $\lim_{t\to-\infty} h(t) = -\infty$, $\lim_{t\to\infty} h(t) = \infty$, thereforeb h^{-1} exists, and it is strictly increasing and continuous on $(-\infty,\infty)$. Write

$$\begin{aligned}G(t,y) =&t\lambda - y \frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^\infty (\exp\{\frac{tK(v)}{M(K)}\} + \exp\{\frac{-tK(v)}{M(K)}\} - 2) v^{\frac{d-3}{2}} \mathrm{d}v, \\ &t\in\mathbb{R}, \quad y \in [0,M].\end{aligned}$$

Then $\frac{\partial G(t,y)}{\partial t} = \lambda - yh(t)$, and so

$$\sup_{t\in\mathbb{R}} G(t,y) = \begin{cases} G(h^{-1}(\lambda/y), y), & \text{if } y \neq 0, \\ +\infty, & \text{if } y = 0. \end{cases}$$

Because

$$\begin{aligned}G(h^{-1}(\lambda/y), y) =& \sup_{t\in\mathbb{R}} G(t,y) \\ =& \sup_{t\in\mathbb{R}} \Big\{ t\lambda - y \frac{(2\pi)^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^\infty (\exp\{\frac{tK(v)}{M(K)}\} + \exp\{\frac{-tK(v)}{M(K)}\} - 2) v^{\frac{d-3}{2}} \mathrm{d}v \Big\}\end{aligned}$$

is decreasing with respect to $y \in [0,M]$, we obtain

$$J(\lambda) = \inf_{y\in[0,M]} \sup_{t\in\mathbb{R}} G(t,y) = G(h^{-1}(\lambda/M), M).$$

Particularly, J is continuous on $[0,\infty)$. □

Proof of Theorem 1.1 For any $x \in \mathbb{S}^{d-1}$, by Proposition 2.3, we deduce that

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\\ &\ge \liminf_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{|f_n(x)-f_n(-x)|>\lambda\}\\ &\ge -\inf\{J_x(\lambda),J_x(-\lambda)\}.\end{aligned}$$

Hence

$$\liminf_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\ge -J(\lambda).$$

To obtain the reverse inequality, we note

$$\|f_n(\cdot)-f_n(-\cdot)\|=\sup_{x\in\mathbb{S}^{d-1}}|f_n(x)-f_n(-x)|,$$

and

$$\sup_{x\in\mathbb{S}^{d-1}}|f_n(x)-f_n(-x)|\le \max_{1\le k\le l_n}\{\sup_{x\in B_{n,k}}|f_{n,k}(x)-f_{n,k}(-x)|+|f_n(z_{n,k})-f_n(-z_{n,k})|\}.$$

By Lemma 2.7, we deduce that for any $0<\varepsilon<\lambda/2$,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\\ &=\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\sup_{x\in\mathbb{S}^{d-1}}|f_n(x)-f_n(-x)|>\lambda\}\\ &\le \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log(P(\max_{1\le k\le l_n}\sup_{x\in B_{n,k}}|f_{n,k}(x)-f_{n,k}(-x)|\ge\varepsilon)+\\ &\quad P(\max_{1\le k\le l_n}|f_n(z_{n,k})-f_n(-z_{n,k})|\ge\lambda-\varepsilon))\\ &=\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P(\max_{1\le k\le l_n}|f_n(z_{n,k})-f_n(-z_{n,k})|\ge\lambda-\varepsilon).\end{aligned}$$

On the other hand, by Chebyshev inequality, for any $s\ge 0$, $t\ge 0$,

$$\begin{aligned}&P(\max_{1\le k\le l_n}|f_n(z_{n,k})-f_n(-z_{n,k})|\ge\lambda-\varepsilon)\\ &\le l_n\max_{1\le k\le l_n}\{\exp\{-nr^{d-1}(n)(\lambda-\varepsilon)t\}\Psi^n_{z_{n,k}}(t),\exp\{-nr^{d-1}(n)(\lambda-\varepsilon)s\}\Psi^n_{z_{n,k}}(-s)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\\ &\le -\inf\{\sup_{t\ge 0}\{(\lambda-\varepsilon)t-\sup_{x\in\mathbb{S}^{d-1}}\Psi_x(t)\},\sup_{s\ge 0}\{(\lambda-\varepsilon)s-\sup_{x\in\mathbb{S}^{d-1}}\Psi_x(-s)\}\}.\end{aligned}$$

Then, by Lemma 2.8,

$$\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\le -J(\lambda-\varepsilon).$$

Finally, by Lemma 2.9, the rate function J is continuous, therefore

$$\limsup_{n\to\infty}\frac{1}{nr^{d-1}(n)}\log P\{\|f_n(\cdot)-f_n(-\cdot)\|>\lambda\}\le -J(\lambda).\quad\square$$

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