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Bounds on the A_α-Spectral Radius of a C_3-Free Graph

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Abstract Let G be a simple undirected graph. For any real number $\alpha \in [0,1]$, Nikiforov defined the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ in 2017, where $A(G)$ and $D(G)$ are the adjacency matrix and the degree diagonal matrix of G, respectively. In this paper, we obtain a lower bound on the A_α-spectral radius of a C_3-free graph for $\alpha \in [0,1)$ and a sharp upper bound on the A_α-spectral radius of a C_3-free k-cycle graph for $\alpha \in [1/2,1)$.

Keywords C_3-free graph; k-cycle graph; A_α-spectral radius; bound

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1. Introduction

All graphs considered here are simple and undirected. For a graph G, $A(G)$ is its adjacency matrix and $D(G)$ is the diagonal matrix of its degrees. The matrix $Q(G) = D(G)+A(G)$ is called the signless Laplacian matrix of G. The largest eigenvalue of $A(G)$ is called the spectral radius of G, and the largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G. For any real number $\alpha \in [0,1]$, Nikiforov [1] defined the A_α-matrix of G as $A_\alpha(G) = \alpha D(G)+(1-\alpha)A(G)$, which can be regarded as a common generalization of $A(G)$ and $Q(G)$. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-spectral radius of G, denoted by $\rho_\alpha(G)$.

The investigation on the spectral radius and the signless Laplacian spectral radius of a graph is an important topic in the theory of graph spectra. Much work has been done concerning the bounds on the spectral radius and the signless Laplacian spectral radius of a graph. For related results, one may refer to [2–4] and the references therein. The matrix $A_\alpha(G)$ can not only underpin a unified theory of $A(G)$ and $Q(G)$, but also bring many new interesting problems. For example, the A_α-spectral radius $\rho_\alpha(G)$ of a graph G has been studied widely, and many lower bounds and upper bounds on $\rho_\alpha(G)$ has been obtained. For related reference, one may see [5–15] and the references therein. In particular, Nikiforov [1] gave the following lower bound based on the maximum degree $\Delta = \Delta(G)$:

$$\rho_\alpha(G) \geq \frac{1}{2}(\alpha(\Delta+1) + \sqrt{\alpha^2(\Delta+1)^2 + 4\Delta(1-2\alpha)}). \tag{1.1}$$

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If $\alpha \in [0,1)$ and G is connected, the equality holds if and only if $G = K_{1,\,\Delta}$. Following from this lower bound, Nikiforov [1] obtained a simpler lower bound: $\rho_\alpha(G) \geq \alpha(\Delta+1)$ for $\alpha \in [0,1/2]$ and $\rho_\alpha(G) \geq \alpha\Delta + (1-\alpha)^2/\alpha$ for $\alpha \in [1/2,1)$.

In this paper, we obtain a lower bound on the A_α-spectral radius of a C_3-free graph for $\alpha \in [0,1)$, which is better than (1.1).

A k-cyclic graph G is a connected graph in which the number of edges equals the number of vertices plus $k-1$. In particular, G is called a tree, a unicyclic graph, a bicyclic graph or a tricyclic graph if $k = 0,1,2,3$, respectively. For $\alpha \in [0,1]$ and a tree T of order n, Nikiforov et al. [10] proved that

$$\rho_\alpha(T) \leq \frac{n\alpha + \sqrt{n^2\alpha^2 + 4(n-1)(1-2\alpha)}}{2}. \tag{1.2}$$

The equality holds if and only if T is the star $K_{1,n-1}$.

In this paper, we generalize this result, and give a sharp upper bound on the A_α-spectral radius of a C_3-free k-cycle graph for $\alpha \in [1/2,1)$.

The rest of the paper is organized as follows. In Section 2, we recall some useful notions and lemmas used further, and prove a new lemma. In Section 3, we obtain a lower bound on the A_α-spectral radius of a C_3-free graph for $\alpha \in [0,1)$. In Section 4, we give sharp upper bound on the A_α-spectral radius of a C_3-free k-cycle graph for $\alpha \in [1/2,1)$.

2. Preliminaries

Denote by C_n and $K_{1,n-1}$ the cycle and the star, respectively, each on n vertices. Let G be a simple undirected graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, $d_G(v)$ or $d(v)$ denotes the degree of v, and $N_G(v)$ or $N(v)$ denotes the set of all neighbors of v in G. The average 2-degree m_u of a vertex u of G is the average degree of the adjacent vertices of u, that is $m_u = \sum_{v\in N(u)} d(v)/d(u)$. Given a Hermitian matrix A of order n, we index its eigenvalues as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

In order to complete the proofs of our main results, we need the following lemmas.

Lemma 2.1 (Weyl's inequalities [16]) *Let A and B be Hermitian matrices of order n, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then*

$$\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B), \text{ if } \ i+j \geq n+1,$$

$$\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A+B), \text{ if } \ i+j \leq n+1.$$

Lemma 2.2 ([1]) *If G is a graph with no isolated vertices, then*

$$\rho_\alpha(G) \leq \max_{u\in V(G)} \Big\{\alpha d(u) + \frac{1-\alpha}{d(u)} \sum_{uv\in E(G)} d(v)\Big\}.$$

We say that the vertices u and v are equivalent in G, if there exists an automorphism $p: G \to G$ such that $p(u) = v$.

Lemma 2.3 ([1]) *Let G be a connected graph of order n, and let u and v be equivalent vertices in G. If $X = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector to $\rho_\alpha(G)$, then $x_u = x_v$.*

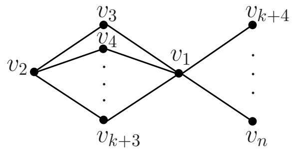

Figure 1 G_n^k

Lemma 2.4 *Let $\alpha \in [0, 1)$, $0 \le k \le n-3$, and G_n^k be a C_3-free k-cyclic graph with $\Delta(G_n^k) = n-2$ (shown in Figure 1). Then $\rho_\alpha(G_n^k)$ is the largest root of the following equation*

$$\begin{aligned}&x^4 - \alpha(k+n+2)x^3 + [2\alpha(k+n-1) + \alpha^2(kn+3n-2) - k - n + 1]x^2 +\\ &[2\alpha(kn+2n-2k-4) + 4\alpha^2(2k-2n+4-kn) - \alpha^3(k+1)n]x +\\ &4\alpha[k(k-n+4) - n + 3] + \alpha^2(3kn - 5k^2 - 16k + 3n - 11) +\\ &2\alpha^3(kn + k^2 + n - 1) - k(k-n+4) + n - 3 = 0.\end{aligned}$$

Proof Let $V(G_n^k) = \{v_1, v_2, \ldots, v_n\}$, $\rho = \rho_\alpha(G_n^k)$, $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to ρ, where x_i corresponds to the vertex v_i $(1 \le i \le n)$. By Lemma 2.3, we have

$$x_3 = x_4 = \cdots = x_{k+3},\ x_{k+4} = \cdots = x_n.$$

From $A_\alpha(G_n^k)X = \rho X$, we have

$$\begin{aligned}&(\rho - (n-2)\alpha)x_1 = (1-\alpha)(k+1)x_3 + (1-\alpha)(n-k-3)x_n,\\ &(\rho - (k+1)\alpha)x_2 = (1-\alpha)(k+1)x_3,\\ &(\rho - 2\alpha)x_3 = (1-\alpha)x_1 + (1-\alpha)x_2,\\ &(\rho - \alpha)x_n = (1-\alpha)x_1.\end{aligned}$$

Since $X = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector corresponding to ρ, it follows that $X \neq 0$. This implies that

$$\begin{vmatrix} \rho-(n-2)\alpha & 0 & -(k+1)(1-\alpha) & (k-n+3)(1-\alpha) \\ 0 & \rho-(k+1)\alpha & -(k+1)(1-\alpha) & 0 \\ -(1-\alpha) & -(1-\alpha) & \rho-2\alpha & 0 \\ -(1-\alpha) & 0 & 0 & \rho-\alpha \end{vmatrix} = 0.$$

Hence ρ is the largest root of the following equation

$$\begin{vmatrix} x-(n-2)\alpha & 0 & -(k+1)(1-\alpha) & (k-n+3)(1-\alpha) \\ 0 & x-(k+1)\alpha & -(k+1)(1-\alpha) & 0 \\ -(1-\alpha) & -(1-\alpha) & x-2\alpha & 0 \\ -(1-\alpha) & 0 & 0 & x-\alpha \end{vmatrix} = 0.$$

By computation, we have ρ is the largest root of the following equation

$$x^4 - \alpha(k+n+2)x^3 + [2\alpha(k+n-1) + \alpha^2(kn+3n-2) - k - n + 1]x^2 +$$

$$[2\alpha(kn+2n-2k-4)+4\alpha^2(2k-2n+4-kn)-\alpha^3(k+1)n]x+$$
$$4\alpha[k(k-n+4)-n+3]+\alpha^2(3kn-5k^2-16k+3n-11)+$$
$$2\alpha^3(kn+k^2+n-1)-k(k-n+4)+n-3=0.$$

This completes the proof.

3. A lower bound on $\rho_\alpha(G)$ of a C_3-free graph G

In this section, we give a lower bound on the A_α-spectral radius of a C_3-free graph.

Theorem 3.1 *Let $\alpha \in [0, 1)$, and G be a C_3-free graph. If d_u and m_u are the degree and the average 2-degree of a vertex u of G, respectively, then*

$$\rho_\alpha(G) \geq \max_{u\in V(G)} \left\{ \frac{\alpha(d_u+m_u)+\sqrt{\alpha^2(d_u-m_u)^2+4(1-\alpha)^2 d_u}}{2} \right\}. \tag{3.1}$$

Moreover, if G is the star $K_{1,n-1}$, then G satisfies the above equality.

Proof Denoted by $N(u)=\{v_1,\dots,v_k\}$ the set of all neighbors of a vertex u of G, where $k=d_u$. Let $N[u]=\{u,v_1,\dots,v_k\}$, $A_\alpha(N[u])$ be the principal submatrix of $A_\alpha(G)$ correponding to $N[u]$. Since G is C_3-free, we have

$$A_\alpha(N[u]) = \begin{pmatrix} \alpha d_u & 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & \alpha d_1 & 0 & \cdots & 0 \\ 1-\alpha & 0 & \alpha d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1-\alpha & 0 & 0 & \cdots & \alpha d_k \end{pmatrix},$$

where $d_i=d(v_i)$ for $i=1,2,\dots,k$. By Lemma 2.1, we have

$$\lambda_1(A_\alpha(N[u]))>\alpha d_u,\quad \lambda_1(A_\alpha(N[u]))>\alpha d_i$$

for each $i=1,2,\dots,k$, and

$$\rho_\alpha(G)=\lambda_1(A_\alpha(G))\geq\lambda_1(A_\alpha(N[u])). \tag{3.2}$$

By elementary calculations, we have the characteristic polynomial of $A_\alpha(N[u])$ is

$$\det(\lambda E-A_\alpha(N[u]))=\Big(\lambda-\alpha d_u-\sum_{i=1}^{k}\frac{(1-\alpha)^2}{\lambda-\alpha d_i}\Big)\prod_{i=1}^{k}(\lambda-\alpha d_i). \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\rho_\alpha(G)-\alpha d_u\geq\sum_{i=1}^{k}\frac{(1-\alpha)^2}{\rho_\alpha(G)-\alpha d_i}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{k}\frac{(1-\alpha)^2}{\rho_\alpha(G)-\alpha d_i}\sum_{i=1}^{k}\frac{\rho_\alpha(G)-\alpha d_i}{(1-\alpha)^2}\geq k^2.$$

Hence

$$\rho_\alpha(G) - \alpha d_u \geq \frac{k^2}{\sum_{i=1}^{k} \frac{\rho_\alpha(G)-\alpha d_i}{(1-\alpha)^2}} = \frac{(1-\alpha)^2 d_u}{\rho_\alpha(G) - \alpha m_u}.$$

This implies that

$$\rho_\alpha(G)^2 - \alpha(d_u + m_u)\rho_\alpha(G) + \alpha^2 d_u m_u - (1-\alpha)^2 d_u \geq 0.$$

Hence

$$\rho_\alpha(G) \geq \frac{\alpha(d_u + m_u) + \sqrt{\alpha^2(d_u - m_u)^2 + 4(1-\alpha)^2 d_u}}{2},$$

which proves the required inequality.

Moreover, if G is the star $K_{1,n-1}$, then we can easily verify that the equality holds.

Remark 3.2 Our lower bound (3.1) is better than that (1.1). For convenience, we denote the right side of (3.1) by A. For any C_3-free graph G, let $u \in V(G)$ with $d(u) = \Delta$. Then we have

$$A \geq \frac{\alpha(\Delta + m_u) + \sqrt{\alpha^2(\Delta - m_u)^2 + 4(1-\alpha)^2\Delta}}{2}.$$

Let

$$f(x) = \frac{\alpha(\Delta + x) + \sqrt{\alpha^2(\Delta - x)^2 + 4(1-\alpha)^2\Delta}}{2}.$$

By derivative, we know that $f(x)$ is a strictly increasing function. Noting that $m_u \geq 1$, we have

$$f(m_u) \geq f(1) = \frac{1}{2}\left(\alpha(\Delta+1) + \sqrt{\alpha^2(\Delta+1)^2 + 4\Delta(1-2\alpha)}\right).$$

Therefore,

$$A \geq \frac{1}{2}\left(\alpha(\Delta+1) + \sqrt{\alpha^2(\Delta+1)^2 + 4\Delta(1-2\alpha)}\right).$$

If the above equality holds, then $m_u = 1$. This implies that $G = K_{1,n-1}$. On the other hand, it is easy to check that the above equality holds for $G = K_{1,\,n-1}$.

4. An upper bound on $\rho_\alpha(G)$ of a C_3-free k-cycle graph G

In this section, we give sharp upper bound on the A_α-spectral radius of a C_3-free k-cycle graph for $\alpha \in [1/2, 1)$.

Theorem 4.1 *Let $k \geq 1$, $n \geq k+5$, and G be a C_3-free k-cycle graph of order n. If $\alpha \in [\frac{1}{2}, 1)$, then $\rho_\alpha(G) \leq \rho_\alpha(G_n^k)$, and the equality holds if and only if $G = G_n^k$ (shown in Fig. 2.1), where $\rho_\alpha(G_n^k)$ is the largest root of the following equation*

$$\begin{aligned} &x^4 - \alpha(k+n+2)x^3 + [2\alpha(k+n-1) + \alpha^2(kn+3n-2) - k - n + 1]x^2 + \\ &[2\alpha(kn+2n-2k-4) + 4\alpha^2(2k-2n+4-kn) - \alpha^3(k+1)n]x + \\ &4\alpha[k(k-n+4) - n + 3] + \alpha^2(3kn - 5k^2 - 16k + 3n - 11) + \\ &2\alpha^3(kn + k^2 + n - 1) - k(k-n+4) + n - 3 = 0. \end{aligned}$$

Proof Since G is a C_3-free k-cycle graph and G is not a tree, it follows that $\Delta(G) \leq n-2$. When $\Delta(G) = n-2$, it is easy to see that $G = G_n^k$ $(1 \leq k \leq n-3)$. Since $G_n^k \neq K_{1,\,n-1}$, by

(1.1), we have

$$\rho_\alpha(G_n^k) > \alpha\Delta(G_n^k) + \frac{(1-\alpha)^2}{\alpha} = \alpha(n-2) + \frac{(1-\alpha)^2}{\alpha}.$$

In the case when $\Delta(G) \le n-3$, we have $G \ne G_n^k$ $(1 \le k \le n-3)$. By Lemma 2.2, we have

$$\rho_\alpha(G) \le \max_{u\in V(G)} \Big\{\alpha d(u) + \frac{1-\alpha}{d(u)} \sum_{uv\in E(G)} d(v)\Big\}.$$

Let w be a vertex of G such that

$$\alpha d(w) + \frac{1-\alpha}{d(w)} \sum_{wv\in E(G)} d(v) = \max_{u\in V(G)} \Big\{\alpha d(u) + \frac{1-\alpha}{d(u)} \sum_{uv\in E(G)} d(v)\Big\}.$$

Then $1 \le d(w) \le \Delta(G) \le n-3$. Since G is C_3-free, there is no edge between the neighbors of w. It follows that

$$\sum_{wv\in E(G)} d(v) \le |E(G)| = n+k-1$$

and

$$\rho_\alpha(G) \le \alpha d(w) + \frac{1-\alpha}{d(w)} \sum_{wv\in E(G)} d(v) \le \alpha d(w) + \frac{1-\alpha}{d(w)}(n+k-1).$$

If $d(w) = 1$, we have

$$\begin{aligned}\rho_\alpha(G) &\le \alpha d(w) + \frac{1-\alpha}{d(w)} \sum_{wv\in E(G)} d(v)\\ &\le \alpha + (1-\alpha)\Delta(G) \le \alpha + (1-\alpha)(n-3).\end{aligned}$$

It is easy to verify that

$$\alpha(n-2) + \frac{(1-\alpha)^2}{\alpha} - \alpha - (1-\alpha)(n-3) = \frac{(2n-5)\alpha^2 - n\alpha + 1 + \alpha}{\alpha} > 0$$

for $n \ge 6$ and $\alpha \in [\frac{1}{2}, 1)$. This implies that

$$\rho_\alpha(G) \le \alpha(n-2) + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(G_n^k).$$

Let $f(x) = \alpha x + \frac{1-\alpha}{x}(n+k-1)$. Since $1/2 \le \alpha < 1$, it is easy to see that the function $f(x)$ is convex for $x > 0$ and its maximum in any closed interval is attained at one of the ends of this interval. In the case when $2 \le d(w) \le n-3$, noting that $n \ge k+5$, we have

$$\begin{aligned}\rho_\alpha(G) &\le \alpha d(w) + \frac{1-\alpha}{d(w)}(n+k-1)\\ &\le \max\left\{2\alpha + \frac{1-\alpha}{2}(n+k-1),\ (n-3)\alpha + \frac{1-\alpha}{n-3}(n+k-1)\right\}.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}&\alpha(n-2) + \frac{(1-\alpha)^2}{\alpha} - 2\alpha - \frac{1-\alpha}{2}(n+k-1)\\ &= \frac{(3n+k-7)\alpha^2 - (n+k+3)\alpha + 2}{2\alpha} \ge 0\end{aligned}$$

and

$$\alpha(n-2)+\frac{(1-\alpha)^2}{\alpha}-(n-3)\alpha-\frac{1-\alpha}{n-3}(n+k-1)$$
$$=\frac{(3n+k-7)\alpha^2-(3n+k-7)\alpha+n-3}{(n-3)\alpha}\geq 0$$

for $n\geq 6$, $k\leq n-5$ and $1/2\leq\alpha<1$. These imply that $\rho_\alpha(G)\leq\alpha(n-2)+\frac{(1-\alpha)^2}{\alpha}<\rho_\alpha(G_n^k)$.

Combining the above arguments, we have $\rho_\alpha(G)\leq\rho_\alpha(G_n^k)$ for $1/2\leq\alpha<1$, and the equality holds if and only if $G=G_n^k$. By Lemma 2.4, we obtain the proof.

Remark 4.2 For $\alpha<1/2$, by similar reason as the proof of Theorem 4.1, we can prove that Theorem 4.1 also holds for $\alpha\in[\max\{\frac{n-3}{2n-5},\frac{n+k-1}{3n+k-7}\},\ 1/2)$.

For $\alpha\in[0,\max\{\frac{n-3}{2n-5},\frac{n+k-1}{3n+k-7}\})$, we propose the following conjecture for further research.

Conjecture 4.3 *Let $k\geq 1$, $n\geq k+5$, and G be a C_3-free k-cycle graph of order n. If $\alpha\in[0,\max\{\frac{n-3}{2n-5},\frac{n+k-1}{3n+k-7}\})$, then $\rho_\alpha(G)\leq\rho_\alpha(G_n^k)$, and the equality holds if and only if $G=G_n^k$ (shown in Figure 1).*

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